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# A HALL-TYPE CLOSURE PROPERTY FOR CERTAIN FITTING CLASSES

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**Abstract**

A closure operation connected with Hall subgroups is introduced for classes of finite soluble groups, and it is shown that this operation can be used to give a criterion for membership of certain special Fitting classes, including the so-called 'central-socle' classes.

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In this note a closure operation connected with Hall subgroups is introduced for classes of finite soluble groups. It is shown that this operation can be used to give a criterion for membership of certain special Fitting classes, namely the so-called 'central socle' classes $\mathscr{X}_\pi$, and the classes $e_\pi(\mathscr{N}^k)$: see Section 1 for definitions. Thus, for example, let $G$ be a finite soluble group and let $\sigma$ denote the set of primes which divide $|\mathrm{soc}(G)|$; we show (Theorem 2.6) that $G \in \mathscr{X}_\pi$ if and only if the Hall $\tau$-subgroups of $G$ belong to $\mathscr{X}_\pi$ for all sets $\tau$ of the form $\tau = \sigma \cup \{t\}$ where $t$ is a prime.

The paper has three sections. The first consists of preliminaries. In the second, the classes $\mathscr{X}_\pi$ are investigated, while the classes $e_\pi(\mathscr{N}^k)$ form the subject of the third.

## 1. Preliminaries

All groups considered here will belong to the class $\mathscr{S}$ of finite soluble groups: our classes of groups are isomorphism-closed and contain all groups of order 1. A Fitting class is a class of groups closed under the taking of subnormal subgroups and normal products; a background to Fitting class theory can be found in [6, 10].

If $G$ is a group and $\mathscr{F}$ is a Fitting class then $G_{\mathscr{F}}$ denotes the $\mathscr{F}$-radical of $G$,





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while $Z(G)$ denotes the centre of $G$. The set of all primes is denoted by $\mathbf{P}$, $p$ will always denote a prime and $\pi$ will always denote some subset of $\mathbf{P}$. Then $\pi$-soc$(G)$ denotes the product of the minimal normal $\pi$-subgroups of $G$, while soc$(G)$ denotes $\mathbf{P}$-soc$(G)$. Let $\mathscr{F}$ be a Fitting class, and define classes of groups as follows:

$$\mathscr{Z}_\pi = (G \in \mathscr{S} : \pi\text{-soc}(G) \leq Z(G)),$$

$$e_\pi(\mathscr{F}) = (G \in \mathscr{S} : \text{ the } G\text{-chief } \pi\text{-factors below } G_{\mathscr{F}} \text{ are central in } G),$$

$$\mathscr{N} = (G \in \mathscr{S} : G \text{ is nilpotent}),$$

$$\mathscr{S}_\pi = (G \in \mathscr{S} : G \text{ is a } \pi\text{-group}).$$

In addition, we write $\mathscr{Z} = \mathscr{Z}_{\mathbf{P}}$, while $(1)$ denotes the class of groups of order 1.

It is well-known that both $\mathscr{Z}_\pi$ and $e_\pi(\mathscr{F})$ are Fitting classes, and that $\mathscr{Z}_\pi$ is subdirect-product-closed while $e_\pi(\mathscr{F})$ is a Fischer class: see [6] for definitions, and [12] for details. Both these families of classes, especially the former, have been extensively studied and have often been used to furnish examples or counterexamples: see, for example, [2, 4, 5, 7, 12].

Write $\text{Hall}_\pi(G)$ for the set of Hall $\pi$-subgroups of $G$, $\text{Hall}(G)$ for the set of all Hall subgroups of $G$, and $\text{Syl}_p(G)$ for the set of Sylow $p$-subgroups of $G$. Write $C_m$ for the cyclic group of order $m$.

Let $\mathscr{X} \subseteq \mathscr{S}$ be a class of groups and $\mathscr{F}$ be a Fitting class. Define $\mathrm{H}_{\mathscr{F}}\mathscr{X} = (G \in \mathscr{S} : \exists X \in \mathscr{X}$ and $H \in \text{Hall}(X)$ with $H \geq X_{\mathscr{F}}$ such that $G \simeq H)$, and write $\mathrm{H}\mathscr{X}$ for $\mathrm{H}_{(1)}\mathscr{X}$. It is not hard to check that $\mathrm{H}_{\mathscr{F}}$ is a closure operation on classes of groups in the sense that (i) $\mathscr{X} \subseteq \mathrm{H}_{\mathscr{F}}\mathscr{X}$, (ii) $\mathrm{H}_{\mathscr{F}}\mathscr{X} \subseteq \mathrm{H}_{\mathscr{F}}\mathscr{Y}$ if $\mathscr{X} \subseteq \mathscr{Y}$, and (iii) $\mathrm{H}_{\mathscr{F}}\mathscr{X} = \mathrm{H}_{\mathscr{F}}\mathrm{H}_{\mathscr{F}}\mathscr{X}$. If $\mathscr{X} = \mathrm{H}_{\mathscr{F}}\mathscr{X}$, we say that $\mathscr{X}$ is $\mathrm{H}_{\mathscr{F}}$-*closed*, while an H-closed class is called *Hall-closed*: see [1, 2, 3, 8], and the references contained therein, for results related to Hall-closure.

## 2. The central-socle classes

The section begins with Proposition 2.1, to the effect that $\mathscr{Z}_\pi$ is $\mathrm{H}_{\mathscr{N}}$-closed, and this is followed by Examples 2.2 to show that $\mathscr{Z}_\pi$ is not Hall-closed for $\pi \neq \emptyset$. A converse to Proposition 2.1 is proved as Proposition 2.5, and together these results yield a criterion, Theorem 2.6, for membership of $\mathscr{Z}_\pi$. The section ends with a result, not strictly connected with the $\mathrm{H}_{\mathscr{F}}$ operation, in a similar spirit to 2.5.

2.1 PROPOSITION. *Let $\pi \subseteq \mathbf{P}$ and let $G \in \mathscr{Z}_\pi$. Suppose that $H$ is a Hall subgroup of $G$ with $H \geq$ soc$(G)$. Then $H \in \mathscr{Z}_\pi$. Thus $\mathscr{Z}_\pi$ is $\mathrm{H}_{\mathscr{N}}$-closed.*

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PROOF. It is easy to check that $\mathscr{X}_\pi = \bigcap_{p\in\pi} \mathscr{X}_p$, and so we may without loss of generality assume that $\pi = \{p\}$ for some $p \in \mathbf{P}$.

Suppose for a contradiction that $G$ is a group of minimal order subject to

(i) $G \in \mathscr{X}_p$; and
(ii) there exists a Hall subgroup of $G$ which contains $\mathrm{soc}(G)$ but does not belong to $\mathscr{X}_p$.

Let $H$ be a Hall subgroup of $G$ with $H \geq \mathrm{soc}(G)$ but $H \notin \mathscr{X}_p$. Write $\tau = \{t \in \mathbf{P} : t \mid |H|\}$; then $H \in \mathrm{Hall}_\tau(G)$, $F(G) \in \mathscr{S}_\tau$, $F(G) \leq H$, and $O_{\tau'}(G) = 1$. Since $\mathscr{S}_{p'} \subseteq \mathscr{X}_p$, then $p \in \tau$. Let $M \lhd \cdot G$ with $F(G) \leq M$: this is possible because $H < G$. Then $F(M) = F(G) \leq M \cap H \in \mathrm{Hall}_\tau(M)$, and so $M \cap H \in \mathscr{X}_p$ by minimality. In particular, $H \not\leq M$. Thus $G = MH$ and $|G : M| = q \in \tau$. Because $H \notin \mathscr{X}_p$, there exists $L \cdot\trianglelefteq H$ with $L \in \mathscr{S}_p$ and $L \not\leq Z(H)$. Because $F(G) \trianglelefteq H$, then $[F(G), L] \leq F(G) \cap L \trianglelefteq H$. Now $C_G(F(G)) \leq F(G)$, because $G$ is soluble, and so $F(G) \cap L > 1$ because $L > 1$. Since $L \cdot\trianglelefteq H$, it follows that $L \leq F(G)$. In particular, $L \leq M \cap H$. Now $L$ is an irreducible $H$-module. Since $(M \cap H) \trianglelefteq H$, then by Clifford's Theorem, [9, 3.4.1] or [11, V.17.3], we have

$$L|_{(M\cap H)} = U_1 \oplus \cdots \oplus U_n,$$

for some $n \in \mathbf{N}$, where each $U_i$ is an irreducible $(M \cap H)$-module. But this means that, as a normal subgroup of $M \cap H$, $L$ is a direct product of minimal normal subgroups. Thus $L \leq p\text{-}\mathrm{soc}(M \cap H)$. But $M \cap H \in \mathscr{X}_p$ and so

$$L \leq Z(M \cap H). \tag{1}$$

But $L \cdot\trianglelefteq H$ and $L \not\leq Z(H)$; thus

$$H/(M \cap H) \simeq C_q \text{ acts faithfully and irreducibly on } L \in \mathscr{S}_p. \tag{2}$$

In particular, $p \neq q$.

Let $J = \langle L^g : g \in G\rangle$, the normal closure of $L$ in $G$. We have $J \leq F(G) \leq M \cap H$ because $L \leq F(G)$. Then (1) implies that $L \leq Z(J)$. But $Z(J) \trianglelefteq G$ and so $J = Z(J)$ is abelian and must now be a $p$-group, as it is generated by commuting conjugates of $L$.

Let $S_1 \in \mathrm{Hall}_{\tau'}(G)$. By orders we have $G = HS_1$ and $M \geq S_1$, whence, remembering that $L \trianglelefteq H$, we have

$$J = \langle L^{hs} : h \in H, s \in S_1\rangle = \langle L^s : s \in S_1\rangle.$$

By the Frattini argument, using the conjugacy of Hall subgroups, we have $G = MN_G(S_1)$. But $|G : M| = q$, and so there exists a $q$-element $n_1 \in N_G(S_1)$ such that

$G = M\langle n_1\rangle$. Again by Hall's Theorem, there exists $a \in G$ with $n_1^a \in H$. Write $n = n_1^a \in H\backslash M$ and $S = S_1^a$. Then $n \in N_H(S)$, $G = HS$ and $J = \langle L^s : s \in S\rangle$. It follows that

(3) $L$ is contained in no proper $S$-invariant subgroup of $J$.

We have $S\langle n\rangle \leq G$ because $n \in N_H(S)$ ; also, $S\langle n\rangle \in \mathscr{S}_{p'}$ because $p \in \tau$, $S \in \mathscr{S}_{p'}$ and $|n| = q^\alpha$ with $q \neq p$. Now $J$ is a normal, abelian $p$-subgroup of $G$ and so by [9,5.2.3] we have

(4) $$J = [J, S\langle n\rangle] \times C_J(S\langle n\rangle).$$

Since $J \trianglelefteq G$, there exists $J^0 \cdot\!\lhd G$ with $J^0 \leq J$. Then $J^0 \leq p\text{-soc}(G) \leq Z(G)$ and so $C_J(S\langle n\rangle) \geq J^0 > 1$. Thus $[J, S\langle n\rangle] < J$ by (4). But $[J, S\langle n\rangle]$ is $S\langle n\rangle$-invariant and so $S\langle n\rangle$ centralises the non-trivial group $J/[J, S\langle n\rangle]$. But then any subgroup lying between $[J, S\langle n\rangle]$ and $J$ must be $S$-invariant. By statement (3) above, it follows that $[J, S\langle n\rangle]L = J$. But then

$$1 \neq J/[J, S\langle n\rangle] = [J, S\langle n\rangle]L/[J, S\langle n\rangle] \simeq L/(L \cap [J, S\langle n\rangle]),$$

and since all relevant subgroups here are $\langle n\rangle$-invariant then the isomorphism is an $\langle n\rangle$-isomorphism. But $\langle n\rangle$ centralises $J/[J, S\langle n\rangle]$, and so $\langle n\rangle$ must centralise a non-trivial factor group of $L$. However, $n \in H\backslash M$ whence $H = (M \cap H)\langle n\rangle$ and so by statement (2), $L$ must be a faithful, irreducible module for $\langle n\rangle/\langle n^q\rangle \simeq C_q$, contrary to what we have just seen. This completes the proof.

2.2 EXAMPLES. The main aim of these examples is to show that $\mathscr{X}_\pi$ is not Hall-closed, so that some such condition as '$H \geq \text{soc}(G)$' is needed in 2.1. Examples of classes (i) not $H_{\mathscr{N}}$-closed, and (ii) not $H_{\mathscr{S}_\pi}$-closed, will be given in 3.2.

(i) Suppose that $p$, $q$ and $r$ are distinct primes. It is well-known that there exists a group $G$ with a unique chief series whose factors have orders (reading 'from the top') of the form $p, q^\alpha$ and $r^\beta$, respectively. Then $|\text{soc}(G)| = r^\beta$.

(a) Now suppose that $\pi$ with $\emptyset \subset \pi \subset \mathbf{P}$ (proper inclusions) is given. We show that $\mathscr{X}_\pi$ is not Hall-closed. Choose $q \in \pi$ and $r \in \mathbf{P}\backslash\pi$. Then $G \in \mathscr{X}_\pi$. Let $H \in \text{Hall}_{\{p,q\}}(G)$; then $H$ has a non-central $\pi$-socle of order $q^\alpha$, so $H \notin \mathscr{X}_\pi$.

(b) In Proposition 2.1, it is natural to ask whether the conclusion still holds if the condition '$H \geq \text{soc}(G)$' is replaced by '$H \geq \pi\text{-soc}(G)$'. It need not. For take $\pi = \{p, q\}$. Then $G \in \mathscr{X}_\pi$, while if $H \in \text{Hall}_\pi(G)$ then $H \geq \pi\text{-soc}(G)$ although $H \notin \mathscr{X}_\pi$.

(ii) We now show that $\mathscr{X} = \mathscr{X}_{\mathbf{P}}$ is not Hall-closed: the above example is of no avail for this purpose.

Let $S$ denote the group $SL(2,3)$ and let $Z$ denote $Z(S)$, the centre of $S$. Then $Z = \mathrm{soc}(S)$ has order 2. Let $T$ denote a cyclic group of order 5, and form the regular wreath product $W = S\mathrm{wr}T$ (see [11,§I.15]). We may write $W$ as a semidirect product $W = [S^*]T$, where $S^*$, the 'base group', is a direct product of 5 copies of $S$. Then $Z^* = Z(S^*)$ is the corresponding direct product of the respective centres of the 5 copies of $S$, and has order $2^5$. Now $[Z^*, T]$ has order $2^4$ and is normal in $W$. Write $\overline{W} = W/[Z^*, T]$. Then $\overline{W}$ has a unique minimal normal subgroup, namely $\bar{Z}^* = Z^*/[Z^*, T]$, and $\bar{Z}^* = Z(\overline{W})$. In particular, $\overline{W} \in \mathscr{Z} = \mathscr{Z}_{\mathbf{P}}$. But $\overline{W}$ has a Hall $\{3, 5\}$-subgroup $H$ of order $3^5 5$ and $H \simeq C_3\mathrm{wr}C_5$. Now, $C_3\mathrm{wr}C_5$ has two minimal normal subgroups: a central subgroup of order 3 and a non-central subgroup of order $3^4$. Thus $H \notin \mathscr{Z}$ and so $\mathscr{Z}$ is not Hall-closed.

We next prove some results converse in sense to 2.1.

2.3 LEMMA. *Suppose that $G \in \mathscr{S}$ and that $M \lhd \cdot G$ with $|G : M| = q$ and $M \in \mathscr{Z}_p$ where $p, q \in \mathbf{P}$. Suppose that $p$-$\mathrm{soc}(G) \leq M$. Let $H \in \mathrm{Hall}_\tau(G)$ where $\{p, q\} \subseteq \tau \subseteq \mathbf{P}$. Then if $H \in \mathscr{Z}_p$ it follows that $G \in \mathscr{Z}_p$.*

PROOF. We may suppose that $p$-$\mathrm{soc}(G) \neq 1$. Let $N \cdot\trianglelefteq G$ with $N \in \mathscr{S}_p$. Then $N$ is an irreducible $G$-module; thus by Clifford's Theorem, $N$ is a completely reducible $M$-module. But then $N \leq p$-$\mathrm{soc}(M) \leq Z(M)$. Thus $M \leq C_G(N)$ and so $N$ is an irreducible $G/M$-module. Since $|G : M| = q \in \tau$, then $G = MH$, and so $N$ is an irreducible $H/(M \cap H)$-module. But then $N \leq p$-$\mathrm{soc}(H) \leq Z(H)$. Thus $C_G(N) \geq MH = G$ and the assertion follows.

2.4 NOTATION. If $G \in \mathscr{S}$, write $\sigma_G = \{s \in \mathbf{P} : s \mid |\mathrm{soc}(G)|\}$.

2.5 PROPOSITION. *Let $G \in \mathscr{S}$ and $\pi \in \mathbf{P}$. Suppose that $\mathrm{Hall}_\tau(G) \subseteq \mathscr{Z}_\pi$ for all $\tau \subseteq \mathbf{P}$ of the form $\tau = \sigma_G \cup \{t\}$ where $t \in \mathbf{P}$. Then $G \in \mathscr{Z}_\pi$.*

PROOF. It will suffice to prove that $G \in \mathscr{Z}_p$ for all $p \in \pi \cap \sigma_G$. If $\mathrm{soc}(G) = G$ there is nothing to prove and so we assume that $\mathrm{soc}(G) < G$. Let $M \lhd \cdot G$ with $M \geq \mathrm{soc}(G)$ and write $|G : M| = q \in \mathbf{P}$.

We claim that $\sigma_G = \sigma_M$. For suppose that $s \in \sigma_M$; then there exists $K \cdot\trianglelefteq M$ with $K \in \mathscr{S}_s$. The normal closure $K^G$ satisfies $\mathscr{S}_s \ni K^G \leq M$, and so there exists $L \cdot\trianglelefteq G$ with $L \leq K^G$. Thus $s \in \sigma_G$. Next suppose that $s \in \sigma_G$. Then there exists $K \cdot\trianglelefteq G$ with $K \in \mathscr{S}_s$, and $K \leq M$ because $M \geq \mathrm{soc}(G)$. Thus there exists $L \cdot\trianglelefteq M$ with $L \leq K$, whence $s \in \sigma_M$, and $\sigma_G = \sigma_M$.

Let $\tau$ be of the form $\tau = \sigma_M \cup \{t\} = \sigma_G \cup \{t\}$, where $t \in \mathbf{P}$. Let $H_1 \in \mathrm{Hall}_\tau(M)$ and let $H \in \mathrm{Hall}_\tau(G)$ with $H_1 = H \cap M$. By hypothesis, $H \in \mathscr{Z}_\pi$ and so $H_1 \in \mathscr{Z}_\pi$. By the minimality of $G$, it follows that $M \in \mathscr{Z}_\pi$.

Now write $\tau_0 = \sigma_G \cup \{q\}$ and fix $H \in \mathrm{Hall}_{\tau_0}(G)$. Let $p \in \pi \cap \sigma_G$ be arbitrary. Then $H \in \mathscr{X}_p$, $M \in \mathscr{X}_p$, and $\{p, q\} \subseteq \tau_0$; it follows from Lemma 2.3 that $G \in \mathscr{X}_p$, and the proof is complete.

Putting together Propositions 2.1 and 2.5, we obtain the promised criterion for membership of the central-socle classes as follows.

2.6 THEOREM. *Let $G \in \mathscr{S}$ and $\pi \subseteq \mathbf{P}$. Then $G \in \mathscr{X}_\pi$ if and only if* $\mathrm{Hall}_\tau(G) \subseteq \mathscr{X}_\pi$ *for all $\tau \subseteq \mathbf{P}$ of the form $\tau = \sigma_G \cup \{t\}$ with $t \in \mathbf{P}$.*

We now give another result in the spirit of 2.5.

2.7 PROPOSITION. *Let $G \in \mathscr{S}$ and $\pi \subseteq \mathbf{P}$. Suppose that* $\mathrm{Hall}_\tau(G) \subseteq \mathscr{X}_\pi$ *for all sets of primes $\tau$ with $|\tau| \leq 2$. Then $G \in \mathscr{X}_\pi$.*

PROOF. Because $\mathscr{X}_\pi = \bigcap_{p\in\pi} \mathscr{X}_p$, we may without loss of generality assume that $\pi = \{p\}$. Suppose for a contradiction that $G$ is a counterexample of minimal order. Then $p\text{-soc}(G) < G$ and there exists $M \trianglelefteq \cdot G$ with $M \geq p\text{-soc}(G)$. If $\tau \subseteq \mathbf{P}$ with $|\tau| = 2$ and if $H \in \mathrm{Hall}_\tau(M)$, then $H = M \cap H_1$ where $H_1 \in \mathrm{Hall}_\tau(G)$, and so $H \in \mathscr{X}_p$. Thus $M \in \mathscr{X}_p$ by minimality. Write $|G : M| = q \in \mathbf{P}$. Now the Hall $\{p, q\}$-subgroups of $G$ belong to $\mathscr{X}_p$ by hypothesis, and the result follows from Lemma 2.3.

## 3. The classes $e_\pi(\mathscr{N}^k)$

This section has a similar structure to Section 2. It is proved in Proposition 3.1 that $e_\pi(\mathscr{N}^k)$ is $\mathrm{H}_{\mathscr{N}}$-closed, and this is followed by some relevant examples (3.2). Proposition 3.3 is a converse to Proposition 3.1, and together these results yield a criterion, Theorem 3.4, for membership of the classes $e_\pi(\mathscr{N}^k)$. Again the section finishes with a result, Proposition 3.5, not strictly connected with the $\mathrm{H}_{\mathscr{F}}$ operation, being an analogue for certain classes $e_\pi(\mathscr{F})$ of Proposition 2.7.

3.1 PROPOSITION. *Let $\pi \subseteq \mathbf{P}$ and $k \in \mathbf{N}, k \geq 0$. Let $G \in e_\pi(\mathscr{N}^k)$. Suppose that $H$ is a Hall subgroup of $G$ with $H \geq G_{\mathscr{N}^k}$. Then $H \in e_\pi(\mathscr{N}^k)$. It follows that $e_\pi(\mathscr{N}^k)$ is $\mathrm{H}_{\mathscr{N}^k}$-closed.*

PROOF. Because $e_\pi(\mathscr{N}^k) = \bigcap_{p\in\pi} e_p(\mathscr{N}^k)$, we may without loss of generality assume that $\pi = \{p\}$ where $p \in \mathbf{P}$.

The proof is by induction on $k$. If $k = 0$ then $\mathscr{N}^k = 1$ and $e_p(1) = \mathscr{S}$; the conclusion clearly holds in this case. We thus suppose that the result holds for all $G_0 \in e_p(\mathscr{N}^{k_0})$ for all $k_0 < k$, and for all $G_1 \in e_p(\mathscr{N}^k)$ with $|G_1| < |G|$.

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Write $\tau = \{q \in \mathbf{P} : q \mid |H|\}$; then $H \in \mathrm{Hall}_\tau(G)$. If $A$ is a group, write $A_j = A_{\mathcal{N}^j}$, the $\mathcal{N}^j$-radical of $A$; then $G_k \in \mathcal{S}_\tau$ and $G_k \leq O_\tau(G) \leq H$, where $O_\tau(G)$ denotes the $\mathcal{S}_\tau$-radical of $G$. Since $\mathcal{S}_{p'} \in e_p(\mathcal{N}^k)$, then $H \in e_p(\mathcal{N}^k)$ if $p \notin \tau$, and so we may without loss assume that $p \in \tau$.

Choose $M \lhd \cdot G$ with $M \geq G_k$ and write $|G : K| = q \in \mathbf{P}$. Then $M \in e_p(\mathcal{N}^k)$, $M \cap H \in \mathrm{Hall}_\tau(M)$ and $M_k = G_k \leq M \cap H$. By the induction hypothesis we have $M \cap H \in e_p(\mathcal{N}^k)$; in particular, $M \cap H \neq H$ and so $G = MH$. Further, all $M \cap H$-chief $p$-factors below $(M \cap H)_k$ are $M \cap H$-central. Since $M \cap H \trianglelefteq H$ then by Clifford's Theorem, any $H$-chief $p$-factor, $X/Y$ say, below $(M \cap H)_k$ is completely reducible as an $M \cap H$-module and, being then a sum of $M \cap H$-trivial modules, must itself be $M \cap H$-trivial. Thus,

(5) The $H$-chief $p$-factors below $(M \cap H)_k$ are $M \cap H$-central.

There are now two cases to consider.

Case (I). Suppose that $H_k \not\leq M$; then $H = (M \cap H)H_k$. Let $X/Y$ be an $H$-chief $p$-factor in $H_k$ in an $H$-chief series which refines $H \geq H_k \geq H_{k-1} \geq 1$. By the Jordan-Hölder theorem, we may restrict attention to a fixed chief series.

We firstly claim that $X/Y$ is trivial as an $M \cap H$-module. If $X \leq (M \cap H)_k = M \cap H_k$, then $X/Y$ is $M \cap H$-central by (1). If $Y \not\leq M$ then $X/Y \simeq_H (X \cap M)/(Y \cap M)$; the latter is still $H$-chief and so again is $M \cap H$-trivial by (1). In the remaining case we have $Y \leq M$, $X \not\leq M$ and $Y = X \cap M$; then we have $[X, M \cap H] \leq X \cap M = Y$ and again $X/Y$ is $M \cap H$-trivial; this justifies our claim.

Suppose that $X/Y$ lies below $H_{k-1}$; then $X/Y$ is $H$-central because $H \in e_p(\mathcal{N}^{k-1})$ by the induction hypothesis and the fact that $e_p(\mathcal{N}^k) \subseteq e_p(\mathcal{N}^{k-1})$. Suppose, on the other hand, that $X/Y$ lies between $H_k$ and $H_{k-1}$. By Clifford's Theorem, $X/Y$ is completely reducible as an $H_k$-module and so must be a sum of $H_k$-trivial submodules because $H_k/H_{k-1}$ is nilpotent; thus $X/Y$ is a trivial $H_k$-module. But $H = (M \cap H)H_k$, and since $X/Y$ is trivial for $M \cap H$, it must be trivial for $H$. It follows that $H \in e_p(\mathcal{N}^k)$, as required.

Case (II). Suppose now that $H_k \leq M$; then $H_k = (M \cap H)_k$. Now $G_k \leq O_\tau(G) \cap H_k \leq (O_\tau(G))_k \leq G_k$, whence $G_k = O_\tau(G) \cap H_k$.

Let $P \in \mathrm{Syl}_p(H_k)$, and write $J = \langle P^g : g \in G \rangle$, the normal closure of $P$ in $G$; note that $J \leq M$. Let $R$ be a Hall $p$-complement in $G_k$; then $\bar{R} = RG_{k-1}/G_{k-1}$ is the unique $p$-complement in $G_k/G_{k-1} \in \mathcal{N}$, and so $\bar{R} \trianglelefteq G/G_{k-1}$. Now $R \in H_k$ and so, since $H_k/H_{k-1} \in \mathcal{N}$, we have $[R, P] \leq H_{k-1}$. But $[R, P] \leq G_k$ because $R \leq G_k \trianglelefteq G$, and so

$$[R, P] \leq G_k \cap H_{k-1} = O_\tau(G) \cap H_k \cap H_{k-1} = G_{k-1}.$$

But then $P \leq C_G(\bar{R}) \trianglelefteq G$ and so $J \leq C_G(\bar{R}) \cap M$. Now let $x \in J$ be a $p'$-element. The $G$-chief $p$-factors between $G_k$ and $G_{k-1}$ are $G$-central because $G \in e_p(\mathcal{N}^k)$, and

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so are centralised by $x$. But then $x$, being a $p'$-element, must centralise the Sylow $p$-subgroup of $G_k/G_{k-1}$, by [9,5.3.2]. But $x \in J$ already centralises the $p$-complement $RG_{k-1}/G_{k-1}$ of $G_k/G_{k-1}$, and so $x$ centralises $G_k/G_{k-1}$. But $G_k/G_{k-1}$ is the Fitting subgroup of $G/G_{k-1}$, and so $x \in G_k$ by [9,6.1.3]. But this implies that $JG_k/G_k$ must be a $p$-group. Since $G_k \in \mathscr{S}_\tau$ and $p \in \tau$, it follows that $J \in \mathscr{S}_\pi$. But now $J \leq O_\tau(G)$ and $P \leq O_\tau(G) \cap H_k = G_k$. But then $P \in \mathrm{Syl}_p(G_k)$ and so $p \nmid |H_k : G_k|$.

Let $\mathscr{C}_0$ be a $G$-chief series between $G_k$ and 1, and let $\mathscr{C}$ be an $H$-chief series which refines $H_k \geq G_k \geq 1$ and which refines $\mathscr{C}_0$ below $G_k$. Now all the $G$-chief $p$-factors in $\mathscr{C}_0$ are $G$-central because $G \in e_p(\mathscr{N}^k)$; thus they all have order $p$ and so must be $H$-chief; moreover, they give us all the $p$-factors in $\mathscr{C}$ because $p \nmid |H_k : G_k|$. But now $H \in e_p(\mathscr{N}^k)$, and the proof is complete.

3.2 EXAMPLES. (i) This example is to show that $e_p(\mathscr{N}^2)$ is not $\mathrm{H}_{\mathscr{N}}$-closed. Let $p, q, r$ and $s$ be distinct primes. There exists a group $G$ with a unique chief series whose factors have orders (reading 'from the top') of the form $q$, $p^\alpha$, $r^\beta$ and $s^\gamma$ respectively. Then $G \in e_p(\mathscr{N}^2)$ because $|G_{\mathscr{N}^2}| = s^\gamma r^\beta$. Let $H \in \mathrm{Hall}(G)$ with $|H| = s^\gamma p^\alpha q$. Then $|H_{\mathscr{N}^2}| = s^\gamma p^\alpha$ and $H \notin e_p(\mathscr{N}^2)$. However, $H \geq G_{\mathscr{N}}$, and so $e_p(\mathscr{N}^2)$ is not $\mathrm{H}_{\mathscr{N}}$-closed.

(ii) This example shows that $e_p(\mathscr{S}_\pi)$ is not $\mathrm{H}_{\mathscr{S}_\pi}$-closed when $\pi \subset \mathbf{P}$ with $|\pi| \geq 2$. Let $G$ be the group of Example 2.2(i) with $\{p, q\} \subseteq \pi$, $r \notin \pi$, and $H \in \mathrm{Hall}_\pi(G)$. Then $H \geq O_\pi(G) = 1$. Now $G \in e_q(\mathscr{S}_\pi)$ while $H \notin e_q(\mathscr{S}_\pi)$. Thus 3.1 is not valid if we replace $\mathscr{N}^k$ by an arbitrary Fitting class $\mathscr{F}$.

The next result is an analogue of Proposition 2.5, being converse in sense to 3.1; it is valid for arbitrary $e_\pi(\mathscr{F})$ and not just for the classes $e_\pi(\mathscr{N}^k)$ : as we have just seen, 3.1 is not valid for arbitrary $e_\pi(\mathscr{F})$.

3.3 PROPOSITION. *Let $G \in \mathscr{S}$ and $\pi \subseteq \mathbf{P}$. Let $\mathscr{F}$ be a Fitting class. Suppose that* $\mathrm{Hall}_\tau(G) \subseteq e_\pi(\mathscr{F})$ *for all $\tau \subseteq \mathbf{P}$ of the form $\tau = \rho_G \cup \{t\}$ where $t \in \mathbf{P}$ and $\rho_G = \{s \in \mathbf{P} : s \mid |G_{\mathscr{F}}|\}$. Then $G \in e_\pi(\mathscr{F})$.*

PROOF. Suppose for a contradiction that $G$ is a counterexample of minimal order. Then $G_{\mathscr{F}} < G$ as otherwise $\sigma_G$ contains all primes dividing $|G|$ and so $G \in e_p(\mathscr{F})$ by hypothesis. Let $M \lhd \cdot\, G$ with $M \geq G_{\mathscr{F}}$, and write $|G : M| = q$. Then $M_{\mathscr{F}} = G_{\mathscr{F}}$, and so $\rho_M = \rho_G$. If $H \in \mathrm{Hall}_\tau(M)$ then $H = H_1 \cap M$ for some $H_1 \in \mathrm{Hall}_\tau(G)$ and so $M \in e_\pi(\mathscr{F})$ by minimality. Because $G \notin e_\pi(\mathscr{F})$, there exists a $G$-chief $\pi$-factor $X/Y$ below $G_{\mathscr{F}}$ which is not $G$-central. By Clifford's Theorem, $X/Y$ is completely reducible as an $M$-module, and so $X/Y$ is $M$-central because $M \in e_\pi(\mathscr{F})$. Thus $X/Y$ is faithful and irreducible for $G/M \simeq C_q$. Let $H \in \mathrm{Hall}_\tau(G)$ where $\tau = \rho_G \cup \{q\}$. Then $G = MH$. Thus $X/Y$ is faithful and irreducible for $H/(H \cap M) \simeq G/M$, and

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so is non-trivial for $H$. Now $H \geq H_{\mathscr{F}} \geq G_{\mathscr{F}} \geq X \geq Y$, and so $X/Y$ is $H$-central because $H \in e_\pi(\mathscr{F})$, in contradiction to the preceding statement. The result follows.

Putting together Propositions 3.1 and 3.3, we obtain our criterion for membership of the classes $e_\pi(\mathscr{N}^k)$ as follows.

3.4 THEOREM. *Let $G \in \mathscr{S}$, $\pi \subseteq \mathbf{P}$ and $k \in \mathbf{N}$, $k \geq 0$. Then $G \in e_\pi(\mathscr{N}^k)$ if and only if* $\mathrm{Hall}_\tau(G) \subseteq e_\pi(\mathscr{N}^k)$ *for all $\tau \subseteq \mathbf{P}$ of the form $\tau = \rho_G \cup \{t\}$ where $t \in \mathbf{P}$ and $\rho_G = \{s \in \mathbf{P} : s \mid |G_{\mathscr{N}^k}|\}$.*

The next result is an analogue of Proposition 2.7 for the classes $e_\pi(\mathscr{F})$.

3.5 PROPOSITION. *Let $G \in \mathscr{S}$ and $\pi \subseteq \mathbf{P}$. Let $\mathscr{F}$ be a Hall-closed Fitting class. Suppose that* $\mathrm{Hall}_\tau(G) \subseteq e_\pi(\mathscr{F})$ *for all $\tau \subseteq \mathbf{P}$ with $|\tau| \leq 2$. Then $G \in e_\pi(\mathscr{F})$.*

PROOF. The proof is by induction on $|G|$, the result being trivial if $|G| = 1$. If $M \lhd \cdot G$ and $\tau \subseteq \mathbf{P}$ with $|\tau| \leq 2$ then $\mathrm{Hall}_\tau(M) \subseteq e_\pi(\mathscr{F})$ and so $M \in e_\pi(\mathscr{F})$ by induction. It follows that $G$ contains a unique maximal normal subgroup, which we call $M$; then $M \geq G'$ and $|G : M| = q \in \mathbf{P}$. Let now $X/Y$ be a $G$-chief $\pi$-factor below $G_{\mathscr{F}}$. If $X \not\leq M$ then $X = G$ and $Y = M$ by the unicity of $M \lhd \cdot G$, and then $X/Y$ is certainly $G$-central. Suppose that $X \leq M$. Then $X/Y$ is below $M_{\mathscr{F}}$, and by Clifford's Theorem must be $M$-central. Now $X/Y \in \mathscr{S}_p$ for some $p \in \pi$. Let $H \in \mathrm{Hall}_\tau(G)$ where $\tau = \{p, q\}$. Then $G = MH$ and $X/Y$ is a module for $H/H \cap M \simeq G/M$. But $X \leq YH$ and so $X = X \cap TH = Y(X \cap H)$, whence

$$X/Y \simeq_H (X \cap H)/(Y \cap H).$$

Now $M_{\mathscr{F}} \cap H \in \mathrm{Hall}_\tau(M_{\mathscr{F}}) \subseteq \mathscr{F}$, the final inclusion because $\mathscr{F}$ is Hall-closed, and so $X \cap H \leq M_{\mathscr{F}} \cap H \leq H_{\mathscr{F}}$. But $H \in e_\pi(\mathscr{F})$, and it follows that $X/Y$ is $H$-central and thus $G$-central, as required.

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