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SOME REMARKS ON FIXING SUBGRAPHS AND SMOOTHLY EMBEDDABLE SUBGRAPHS

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Abstract

In this note we specify the conditions under which the line graph of a fixing subgraph of a graph G is a smoothly embeddable subgraph of the line graph of G, and vice-versa.

1. Introduction

Throughout this paper the basic graph theoretical notation and terminology used is that of Behzad and Chartrand (1972). All graphs considered will be finite and undirected, and will have no loops or multiple edges. If G is such a graph, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. $L(G)$ denotes the line graph of G. If $u \in V(G)$, the *open neighbourhood* of u is the set $N(u) = \{v \in V(G)\colon u \text{ and } v \text{ are adjacent}\}$, and the *closed neighbourhood* of u is the set $N(u) \cup \{u\}$.

The notion of fixing subgraphs of graphs is introduced in Sheehan (1972a) and that of smoothly embeddable subgraphs of graphs in Sheehan (1972b). If H is a spanning subgraph of G, then H is a *fixing subgraph* of G if every embedding of H into G can be extended to an automorphism of G. We write $\underset{=}{F}(G)$ for the set of fixing subgraphs of G. If U is an induced subgraph of G, then U is a *smoothly embeddable subgraph* of G if every embedding of U into G can be extended to an automorphism of G. We write $\underset{=}{F}_0(G)$ for the set of smoothly embeddable subgraphs of G. Let $\underset{=}{S}(G)$ denote the set of spanning subgraphs of G and $\underset{=}{S}_0(G)$ the set of induced subgraphs of G. If $H \in \underset{=}{S}(G)$, then clearly $L(H) \in \underset{=}{S}_0(L(G))$. Similarly, if $U \in \underset{=}{S}_0(L(G))$, there corresponds a unique $W \in \underset{=}{S}(G)$ such that $L(W) = U$. In view of these facts, it is stated in Sheehan (1972b) that "the relationship between the fixing subgraphs of G and the smoothly embeddable subgraphs of G can be made explicit by a consideration of the line graph of G". What in fact can be made explicit is the relationship between the fixing subgraphs of G and the smoothly embeddable subgraphs of $L(G)$.

We list in Section 2 some basic results to be required later, before proceeding in Section 3 to state some facts, of some interest in their own right, regarding fixing and smoothly embeddable subgraphs of disconnected graphs. These preliminaries allow us to prove in Section 4 the two main theorems of this paper.

2. Preliminary Results

The complications which arise when we attempt to relate $\underline{\underline{F}}(G)$ to $\underline{\underline{F}}_0(L(G))$ are due to the following results of Whitney.

LEMMA 1 (Whitney (1932)). *Let G and H be connected graphs such that $L(G) \cong L(H)$. Then $G \cong H$ unless one of G and H is K_3 and the other is $K_{1,3}$.*

For the next few results, recall that $\Gamma(G)$ denotes the automorphism group of G, $\Gamma_1(G)$ the edge-automorphism group of G and $\Gamma^*(G)$ the subgroup of $\Gamma_1(G)$ whose elements are induced by elements of $\Gamma(G)$. Note that if Δ_1 and Δ_2 are permutation groups, then $\Delta_1 \cong \Delta_2$ means that Δ_1 and Δ_2 are isomorphic as abstract groups, and $\Delta_1 = \Delta_2$ means that Δ_1 and Δ_2 are isomorphic as permutation groups (see Harary (1969)).

LEMMA 2 (*Behzad and Chartrand* (1972)). *Let G be a non-trivial connected graph. Then $\Gamma^*(G) \cong \Gamma(G)$ unless G is K_2.*

COROLLARY. *For a non-trivial graph G, $\Gamma^*(G) \cong \Gamma(G)$ if and only if G has neither K_2 as a component nor two or more isolated vertices.*

LEMMA 3 (Whitney (1932)). *Let G be a non-empty graph. Then $\Gamma_1(G) = \Gamma^*(G)$ if and only if*

(i) *not both G_1 and G_2 (of Figure 1) are components of G, and*
(ii) *none of the graphs G_3, G_4 and G_5 (of Figure 1) is a component of G.*

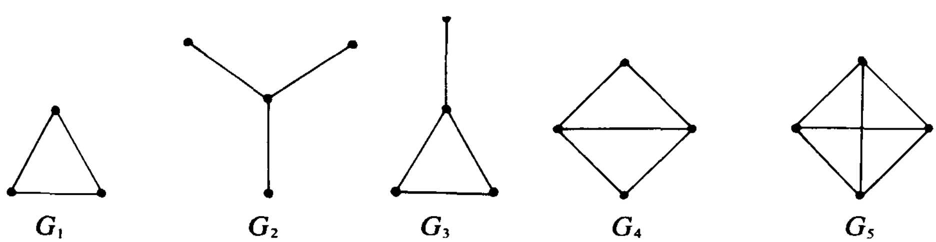

Figure 1

COROLLARY. *Let G be a connected graph with $|V(G)| \geqq 3$. Then $\Gamma_1(G) = \Gamma^*(G) \cong \Gamma(G)$ if and only if G is none of G_3, G_4 and G_5.*

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A trivial consequence of the relevant definitions is

LEMMA 4. $\Gamma(L(G)) = \Gamma_1(G)$.

Because of this last result, the exceptional cases of Lemmas 2 and 3, together with that of Lemma 1, force complications to arise when we consider the relationship between $\underset{=}{F}(G)$ and $\underset{=}{F}_0(L(G))$. Lemma 1 motivates the following definitions.

If M is a graph with a component isomorphic to $K_{1,3}$, let $M^{\perp}$ denote the graph obtained from M by replacing a component isomorphic to $K_{1,3}$ by one component isomorphic to K_1 and one component isomorphic to K_3. If M has no components isomorphic to $K_{1,3}$, let $M^{\perp} = M$. If M has components isomorphic to both K_1 and K_3, let $M^{\top}$ denote the graph obtained from M by replacing two components, one isomorphic to K_1 and the other to K_3 by one component isomorphic to $K_{1,3}$. If M does not have components isomorphic to both K_1 and K_3, let $M^{\top} = M$. Note that by Lemma 1, $L(M) \cong L(M^{\perp}) \cong L(M^{\top})$. If G is a graph with a spanning subgraph isomorphic to H, we say that H *conforms to* G provided either $H = H^{\perp} = H^{\top}$ or if $H \neq H^{\perp}$ then G has no spanning subgraph isomorphic to $H^{\top}$ and if $H \neq H^{\top}$ then G has no spanning subgraph isomorphic to $H^{\perp}$.

To conclude this section, we note some basic results on fixing and smoothly embeddable subgraphs which we shall require. Note that we shall usually make use of these results implicitly in what follows. We first introduce some new notation. If $U \in \underset{=}{S}_0(G)$, then let

$$c(U,G) = |\{V\colon V \in \underset{=}{S}_0(G),\, V \cong U\}|,$$

and let $\Gamma(U,G) = \{g \in \Gamma(G)\colon g^{V(U)} \in \Gamma(U)\}$.

LEMMA 5 (*Sheehan* (1972*a*)). *$H \in \underset{=}{F}(G)$ if and only if G contains exactly $|\Gamma(G)|/|\Gamma(H)|$ distinct copies of H.*

LEMMA 6 (*Sheehan* (1972*b*)). *If $H \in \underset{=}{F}(G)$, then* (i) $\Gamma(H) \leqq \Gamma(G)$ *and* (ii) *if $M \in \underset{=}{S}(G)$, $M \cong H$, then there exists $g \in \Gamma(G)$ such that $M^g = H$.*

LEMMA 7 (*Sheehan* (1974)). *If $U \in \underset{=}{S}_0(G)$, then $U \in \underset{=}{F}_0(G)$ if and only if $c(U,G) = |\Gamma(G)|/|\Gamma(U,G)|$ and $\Gamma(U) \leqq \Gamma(G)$.*

LEMMA 8 (*Sheehan* (1974)). *Let $U \in \underset{=}{F}_0(G)$. Then*

(i) $\Gamma(U) = \{g\colon g = f^{V(U)},\, f \in \Gamma(U,G)\}$ *and*
(ii) *if $V \in \underset{=}{S}_0(G)$, $V \cong U$, then there exists $g \in \Gamma(G)$ such that $V^g = U$.*

3. Fixing and smoothly embeddable subgraphs of disconnected graphs

We note the following results which give the automorphism groups of disconnected graphs in terms of the automorphism groups of their components. Our notation for the sum and wreath product of permutation groups is that of Harary (1969).

LEMMA 9 (*Harary* (1969)). *If G is a disconnected graph which has n_i components isomorphic to G_i for $i = 1, 2, \cdots, m$, then*

$$\Gamma(G) = S_{n_1}[\Gamma(G_1)] + S_{n_2}[\Gamma(G_2)] + \cdots + S_{n_m}[\Gamma(G_m)].$$

(Here S_n is the symmetric group of degree n.)

This result will be used implicitly in the rest of the paper.

We have the following two lemmas regarding fixing subgraphs and smoothly embeddable subgraphs of disconnected graphs. The simple proofs of these results are omitted.

LEMMA 10. *Let G be a graph with components $A_1, A_2, \cdots, A_k$. Let H be a spanning subgraph of G and for $i = 1, 2, \cdots, k$, let A'_i be the subgraph of H induced by $V(A_i)$. Then*

(a) *if $H \in \underline{\underline{F}}(G)$, it follows that $A'_i \in \underline{\underline{F}}(A_i)$ for $i = 1, 2, \cdots, k$, and*

(b) *if $H \not\in \underline{\underline{F}}(G)$, it follows that either (i) for some i, with $1 \leqq i \leqq k$, $A'_i \not\in \underline{\underline{F}}(A_i)$ or (ii) for some $j, l \neq j$, with $1 \leqq j, l \leqq k$, there exist components B'_j, B'_l of A'_j and A'_l respectively which are isomorphic and are such that there is no $g \in \Gamma(G)$ which maps B'_j onto B'_l and vice-versa, and fixes $V(G) - (V(B'_j) \cup V(B'_l))$.*

LEMMA 11. (a) *Let $U \in \underline{\underline{F}}_0(G)$. Let K be a component of G such that $V(U) \cap V(K) \neq \phi$. Then if K' is the subgraph of K induced by $V(U) \cap V(K)$, we have $K' \in \underline{\underline{F}}_0(K)$.*

(b) *Suppose $U \not\in \underline{\underline{F}}_0(G)$. Then either*

(i) *there exists a component M of G, with $V(U) \cap V(M) \neq \phi$ such that if M' is the subgraph of M induced by $V(U) \cap V(M)$, then $M' \not\in \underline{\underline{F}}_0(M)$, or*

(ii) *there are components M and N of G, with corresponding subgraphs M' and N' of U respectively such that there are components M'_0 and N'_0 of M' and N' respectively which are isomorphic, and yet, are such that there is no automorphism of G which interchanges M'_0 and N'_0 but fixes all other vertices of $M' \cup N'$.*

4. The relationship between fixing subgraphs and smoothly embeddable subgraphs

Suppose H is a spanning subgraph of G. As remarked earlier, $L(H)$ is an induced subgraph of $L(G)$. We thus ask the question: If $H \in \underline{\underline{F}}(G)$, does

$L(H) \in \underset{=}{F}_0(L(G))$, and vice-versa? The answer to this question is not always yes. For example, if G is the graph shown in Figure 2 and H is the indicated spanning subgraph, then although $H \in \underset{=}{F}(G)$, $\Gamma(L(H)) \not\leqq \Gamma(L(G))$ and so $L(H) \notin \underset{=}{F}_0(L(G))$.

G H

Figure 2

Moreover, if G' is the graph shown in Figure 3 and H' is the indicated spanning subgraph, then although $L(H') \in \underset{=}{F}_0(L(G'))$, $\Gamma(H') \not\leqq \Gamma(G')$ and so $H' \notin \underset{=}{F}(G')$.

G' H'

Figure 3

We now aim to find out just when the answer to our question is yes. The following three results provide the solution.

LEMMA 12. *Let H be a spanning subgraph of G, with $\Gamma(L(G)) = \Gamma_1(G) = \Gamma^*(G) \cong \Gamma(G)$ and $\Gamma(L(H)) = \Gamma_1(H) = \Gamma^*(H) \cong \Gamma(H)$. Then*

(i) *if $H \in \underset{=}{F}(G)$, it follows that $L(H) \in \underset{=}{F}_0(L(G))$ if and only if H conforms to G, and*

(ii) *if $L(H) \in \underset{=}{F}_0(L(G))$, it follows that H conforms to G and $H \in \underset{=}{F}(G)$.*

PROOF. (i) Assume that $H \in \underset{=}{F}(G)$. We show that $|\Gamma(L(H), L(G))| = |\Gamma(H)|$. For let $g \in \Gamma(H)$. As $H \in \underset{=}{F}(G)$, it follows that $\Gamma(H) \leqq \Gamma(G)$, so that $g \in \Gamma(G)$. By our hypothesis on $\Gamma(L(G))$, g induces $g' \in \Gamma(L(G))$. As $g \in \Gamma(H)$, it follows that g' restricts to an automorphism of $L(H)$, which is the identity if and only if g is the identity. We deduce that $|\Gamma(L(H), L(G))| \geqq |\Gamma(H)|$. Conversely, if $h' \in \Gamma(L(H), L(G))$, then $h' \in \Gamma(L(G))$ and is induced by $h \in \Gamma(G)$. However, as h' restricts to an automorphism of $L(H)$, it follows

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that $h \in \Gamma(H)$. We deduce that $|\Gamma(H)| \geqq |\Gamma(L(H), L(G))|$, whence $|\Gamma(H)| = |\Gamma(L(H), L(G))|$.

Now, because $H \in \underline{\underline{F}}(G)$, G contains exactly $|\Gamma(G)|/|\Gamma(H)|$ distinct copies of H. This number, by hypothesis and the above result, equals $|\Gamma(L(G))|/|\Gamma(L(H), L(G))|$. Now $L(G)$ contains exactly this number of induced subgraphs isomorphic to $L(H)$ if and only if H conforms to G. (Otherwise it contains more.) So

$$c(L(H), L(G)) = |\Gamma(L(G))|/|\Gamma(L(H), L(G))|,$$

and, as our hypothesis on $\Gamma(L(H))$ implies $\Gamma(L(H)) \leqq \Gamma(L(G))$, it follows that $L(H) \in \underline{\underline{F}}_0(L(G))$, if and only if H conforms to G.

(ii) Now suppose $L(H) \in \underline{\underline{F}}_0(L(G))$. We show first of all that H conforms to G. For suppose otherwise. Let K be a spanning subgraph of G isomorphic to $H^{\perp}$ or $H^{\top}$. It follows that $L(K) \cong L(H)$. As $L(H) \in \underline{\underline{F}}_0(L(G))$, there exists $g' \in \Gamma(L(G))$ mapping $L(H)$ onto $L(K)$. By our hypothesis on $\Gamma(L(G))$, g' is induced by $g \in \Gamma(G)$, which must map H onto K. As H and K are non-isomorphic, this is a contradiction. That $H \in \underline{\underline{F}}(G)$ now follows by reversing the argument of (i).

With the aid of the above result, we may deduce the following two important theorems.

THEOREM 1. *Suppose $H \in \underline{\underline{F}}(G)$. Then $L(H) \in \underline{\underline{F}}_0(L(G))$ if and only if none of the following hold.*

(i) *There is a component of H isomorphic to one of the graphs G_3, G_4, G_5 of Figure 1 which is not a component of G;*

(ii) *H has at least one component isomorphic to K_3 and at least one component isomorphic to $K_{1,3}$, not both of which are components of G;*

(iii) *neither (i) nor (ii) holds, and there is a component C of G, such that if C' is the subgraph of H induced by $V(C)$, then C' does not conform to C.*

THEOREM 2. *Suppose $L(H) \in \underline{\underline{F}}_0(L(G))$. Then $H \in \underline{\underline{F}}(G)$ if and only if none of the following hold.*

(i) *There is a component M of G, such that if M' is the subgraph of H induced by $V(M)$, then the ordered pair (M, M') is either (G_3, P_4), (G_4, C_4) or (G_4, P_4);*

(ii) *H has at least two isolated vertices which so not share the same open neighbourhood in G, or has at least one component isomorphic to K_2 whose vertices do not share the same closed neighbourhood in G.*

PROOF OF THEOREM 1. Suppose that $H \in \underline{\underline{F}}(G)$ but that $L(H) \notin \underline{\underline{F}}_0(L(G))$. If $\Gamma(L(G)) = \Gamma_1(G) = \Gamma^*(G) \cong \Gamma(G)$ and $\Gamma(L(H)) = \Gamma_1(H) = \Gamma^*(H) \cong \Gamma(H)$, then it follows by Lemma 12 that (iii) holds. Henceforth assume that these statements about the automorphism groups do not both hold.

Assume first of all that G is connected. Suppose that $\Gamma^*(G) \not\cong \Gamma(G)$. By Lemma 2, G is K_2, whence trivially $L(H) \in \underline{F}_0(L(G))$, contrary to hypothesis. Now suppose that $\Gamma_1(G) \neq \Gamma^*(G)$. By Lemma 3, G is one of the graphs G_3, G_4, G_5. However $\underline{F}(G_3) = \{G_3\}$ and $\underline{F}(G_4) = \{G_4\}$, so that in these cases, $H \in \underline{F}(G)$ implies $L(H) \in \underline{F}_0(L(G))$, contrary to hypothesis. Moreover, as G_5 is K_4 and as all induced subgroups of $L(K_4)$ are smoothly embeddable (see Sheehan (1974)), if G is G_5 we again contradict our hypothesis. We may therefore assume that $\Gamma_1(G) = \Gamma^*(G) \cong \Gamma(G)$. It follows that H does not satisfy $\Gamma_1(H) = \Gamma^*(H) \cong \Gamma(H)$.

Suppose that $\Gamma^*(H) \not\cong \Gamma(H)$, but that $\Gamma_1(H) = \Gamma^*(H)$. As in the proof of Lemma 12, we deduce that $|\Gamma(H)| = |\Gamma(L(H), L(G))|$, and hence that H does not conform to G, so that (iii) holds. Now assume that $\Gamma_1(H) \neq \Gamma^*(H)$. By Lemma 3, recalling that $\Gamma_1(G) = \Gamma^*(G)$, we deduce that either (i) or (ii) must hold.

Now suppose that G is not connected. As $L(H) \notin \underline{F}_0(L(G))$, we deduce from Lemma 11 that there is some component A of $L(G)$ such that if A' is the subgraph of $L(H)$ induced by $V(L(H)) \cap V(A)$, then either (a) $A' \notin \underline{F}_0(A)$, or (b) $A' \in \underline{F}_0(A)$, but there is another component B of $L(G)$, with the corresponding subgraph B' of $L(H)$, such that A' and B' have isomorphic components A'_0 and B'_0 respectively, whereas no automorphism of $\Gamma(L(G))$ interchanges A'_0 and B'_0 and fixes all other vertices of $A' \cup B'$. As A is a component of $L(G)$, there is a component K of G such that $A = L(K)$. It follows that there is a subgraph K' of H such that $A' = L(K')$.

Now suppose (a) holds. Thus $L(K') \notin \underline{F}_0(L(K))$. By Lemma 10, as $H \in \underline{F}(G)$ we deduce that $K' \in \underline{F}(K)$. The previous argument for G connected shows that (i), (ii) or (iii) must hold for K, and so either (i), (ii) or (iii) holds for G.

Suppose that (b) holds. Let M and M' correspond to B and B', and K'' and M'' to A'_0 and B'_0, in the same way that K and K' correspond to A and A'. Because $H \in \underline{F}(G)$, if $K'' \cong M''$ it follows that the automorphism g of H which interchanges K'' and M'' and fixes all other vertices of H is in $\Gamma(G)$. But then g induces an automorphism g' of $L(G)$, which contradicts (b). Thus $K'' \not\cong M''$. Because K'' and M'' are connected, and $L(K'') \cong L(M'')$, it follows from Lemma 3 that one of K'', M'' is K_3 and the other is $K_{1,3}$. Thus (ii) must hold for G.

The converse follows by noting first of all that if either (i) or (ii) holds, then by Lemma 3, $\Gamma(L(H)) \not\cong \Gamma(L(G))$, so that $L(H) \notin \underline{F}_0(L(G))$, and secondly that if (iii) holds, then by reversing the relevant arguments in the above proof, $L(C') \notin \underline{F}_0(L(C))$, so that $L(H) \notin \underline{F}_0(L(G))$ by Lemma 11.

PROOF OF THEOREM 2. If (i) holds, then by inspection $H \notin \underset{=}{F}(G)$. If (ii) holds, then $\Gamma(H) \not\leqq \Gamma(G)$ so that again $H \notin \underset{=}{F}(G)$.

Now suppose that $L(H) \in \underset{=}{F}_0(L(G))$ but that $H \notin \underset{=}{F}(G)$. By Lemma 12 we can assume that statements $\Gamma_1(G) = \Gamma^*(G) \cong \Gamma(G)$ and $\Gamma_1(H) = \Gamma^*(H) \cong \Gamma(H)$ do not both hold.

Assume to begin with that G is connected. Suppose that $\Gamma^*(G) \not\cong \Gamma(G)$. By Lemma 2, G is K_2, and, as all spanning subgraphs of K_2 are fixing subgraphs, this contradicts our hypothesis. Now suppose that $\Gamma_1(G) \neq \Gamma^*(G)$. It follows by Lemma 3 that G is G_3, G_4 or G_5. By exhaustively considering all possible graphs H, we deduce that (i) must hold. Henceforth, assume that $\Gamma_1(G) = \Gamma^*(G) \cong \Gamma(G)$, so that $\Gamma_1(H) = \Gamma^*(H) \cong \Gamma(H)$ does not hold.

Suppose first of all that $\Gamma^*(H) \not\cong \Gamma(H)$, but that $\Gamma_1(H) = \Gamma^*(H)$. By Lemma 2, Corollary, H either has at least two isolated vertices, or at least one component isomorphic to K_2. As $L(H) \in \underset{=}{F}_0(L(G))$, we have $\Gamma(L(H)) \leqq \Gamma(L(G))$, that is, $\Gamma_1(H) \leqq \Gamma_1(G)$. Thus $\Gamma^*(H) \leqq \Gamma^*(G)$. Suppose $\Gamma(H) \not\leqq \Gamma(G)$. The either H has two isolated vertices which do not share the same open neighbourhood in G or has a component isomorphic to K_2 whose vertices do not share the same closed neighbourhood in G. We deduce that (ii) holds. Now suppose $\Gamma(H) \leqq \Gamma(G)$. As $H \notin \underset{=}{F}(G)$, we deduce that G contains more than $|\Gamma(G)|/|\Gamma(H)|$ subgraphs isomorphic to H. Thus $c(L(H), L(G)) > |\Gamma(G)|/|\Gamma(H)| = |\Gamma(L(G))|/|\Gamma(H)|$. As in the proof of Lemma 12, $|\Gamma(H)| = |\Gamma(L(H), L(G))|$, so that

$$c(L(H), L(G)) > |\Gamma(L(G))|/|\Gamma(L(H), L(G))|,$$

which contradicts our assumption that $L(H) \in \underset{=}{F}_0(L(G))$.

Now assume that $\Gamma_1(H) \neq \Gamma^*(H)$. By Lemma 3 either H has a component isomorphic to one of G_3, G_4, G_5 or has components isomorphic to both K_3 and $K_{1,3}$. In each case, $\Gamma(L(H)) \not\leqq \Gamma(L(G))$, so that $L(H) \notin \underset{=}{F}_0(L(G))$, contradicting our hypothesis.

Now suppose that G is not connected. As $H \notin \underset{=}{F}(G)$, we deduce from Lemma 10 that there is some component A of G, such that if A' is the subgraph of H induced by $V(A)$, then either (a) $A' \notin \underset{=}{F}(A)$ or (b) $A' \in \underset{=}{F}(A)$, but there is another component B of G, with the corresponding subgraph B' of H, such that A' and B' have isomorphic components A'_0 and B'_0 respectively, whereas no automorphism of G interchanges A'_0 and B'_0 and fixes all other vertices of $A \cup B$. Clearly we may also suppose that $B' \in \underset{=}{F}(B)$, or we can consider case (a) with B replacing A. As $L(H) \in \underset{=}{F}_0(L(G))$, by Lemma 11 we deduce that $L(A') \in \underset{=}{F}_0(L(A))$.

Suppose that (a) holds. Thus $A' \notin \underset{=}{F}(A)$. As $L(A') \in \underset{=}{F}_0(L(A))$, the previous argument for G connected shows that (i) or (ii) holds for A, and so for G.

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Finally, suppose that (b) holds. As $A'_0 \cong B'_0$, we have $L(A'_0) \cong L(B'_0)$. Let $\psi'_0 \in \Gamma(L(H))$ interchange $L(A'_0)$ and $L(B'_0)$, and fix all other vertices of $L(H)$. As $L(H) \in \underline{F}_0(L(G))$, it follows that ψ'_0 extends to an automorphism ψ' of $L(G)$, such that ψ' interchanges $L(A)$ and $L(B)$ and fixes all other vertices of $L(G)$. As ψ' restricted to $L(H)$ is ψ'_0, it follows that $L(A'_0) = L(A')$ and $L(B'_0) = L(B')$. Consequently $A'_0 = A'$ and $B'_0 = B'$. Now suppose $\psi' \in \Gamma^*(G)$. It follows that ψ' is induced from an automorphism ψ of G which interchanges A'_0 and B'_0 and fixes all other vertices of $A \cup B$. This contradicts statement (b). Thus suppose $\psi' \notin \Gamma^*(G)$. As $L(A)$ and $L(B)$ are connected, it follows by Lemma 3 that one of A, B is K_3 and the other $K_{1,3}$. Now $A' = A'_0 \cong B'_0 = B'$, so that $|V(A')| = |V(B')|$. Thus $|V(A)| = |V(A')| = |V(B')| = |V(B)|$, a contradiction.

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