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POSITIVE SOLUTIONS TO $p(x)$-LAPLACIAN–DIRICHLET PROBLEMS WITH SIGN-CHANGING NON-LINEARITIES

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Abstract. Consider the $p(x)$-Laplacian–Dirichlet problem with sign-changing non-linearity of the form

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\,\nabla u) + m(x)\,|u|^{p(x)-2}\,u = \lambda a(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p \in C^0(\overline{\Omega})$ and $\inf_{x\in\overline{\Omega}} p(x) > 1$, $m \in L^\infty(\Omega)$ is non-negative, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(0) > 0$, the coefficient $a \in L^\infty(\Omega)$ is sign-changing in Ω. We give some sufficient conditions to assure the existence of a positive solution to the problem for sufficiently small $\lambda > 0$. Our results extend the corresponding results established in the p-Laplacian case to the $p(x)$-Laplacian case.

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1. Introduction. In this paper, we consider the existence of positive solutions for the following $p(x)$-Laplacian–Dirichlet problem of the form

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\,\nabla u) + m(x)\,|u|^{p(x)-2}\,u = \lambda a(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $\lambda > 0$, the function $a(x)$ is allowed to change sign, p, m and f satisfy the following conditions, respectively:

(P) $p \in C^0(\overline{\Omega})$ and $1 < p_- := \inf_{x\in\overline{\Omega}} p(x) \leq p_+ := \sup_{x\in\overline{\Omega}} p(x) < +\infty$.

(M) $m \in L^\infty(\Omega)$ and $m(x) \geq 0$ for $x \in \Omega$.

(F) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(0) > 0$.

Problem (1.1) involves the variable exponent $p(\cdot)$. The study of various mathematical problems with variable exponent has received considerable attention in recent years. For a survey of this area see [**4**, **7**, **20**, **28**], and for the application background see [**21**, **27**]. The existence and multiplicity of solutions to the $p(x)$-Laplacian equations under various hypotheses were studied by many authors (see e.g. [**3**, **8**, **10–12**, **16**, **23–26**, **29**, **30**]). In this paper, we study the existence of a positive solution to problem (1.1) for sufficiently small $\lambda > 0$.

The existence of positive solutions to problem (1.1) when $p(x) \equiv p$ (a constant) was obtained in [**2**, **5**, **6**, **17**, **18**]. In [**2**, **5**, **6**, **17**] the case that $p = 2$ and $m = 0$ was investigated, where in [**5**] the radially symmetric case was investigated. Hai and Xu [**18**] investigated the case that $p \in (1, \infty)$ and $m \geq 0$. In [**2**, **5**, **6**, **17**, **18**] the authors gave some sufficient conditions on $a(x)$ to assure the existence of a positive solution for small values of λ. We denote by $S_p(a)$ the unique solution of the problem

$$\begin{cases} -div(|\nabla z|^{p-2}\nabla z) + m|z|^{p-2}z = a(x) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

for $a \in L^\infty(\Omega)$. Then the condition given in [**6**, **17**] is

$(A_\varepsilon^{\geq})$ there exists $\varepsilon > 0$ such that $S_2(a^+ - (1+\varepsilon)a^-) \geq 0$ in Ω, where $a^+(x) = \max\{0, a(x)\}$ and $a^-(x) = a^+(x) - a(x)$.

The condition given in [**2**, **18**] is

(A_*) $S_p(a) > 0$ in Ω and $\frac{\partial S_p(a)}{\partial \nu} < 0$ on $\partial\Omega$, where ν denotes the unit outward normal vector.

The $p(x)$-Laplacian is an extension of the p-Laplacian. An essential difference between them is that the p-Laplacian operator is $(p-1)$homogeneous, that is, $\Delta_p(\lambda u) = \lambda^{p-1}\Delta_p u$ for every $\lambda > 0$, but the $p(x)$-Laplacian operator, when $p(x)$ is not a constant, is not homogeneous. Our purpose is to extend the corresponding results established in [**2**, **5**, **6**, **17**, **18**] on the p-Laplacian problems to the $p(x)$-Laplacian case; however, in this respect we face an essential difficulty due to the inhomogeneity of the $p(x)$-Laplacian operator. It is well known that, in the case that $p(x) \equiv p$ (a constant), if z is a positive solution of (1.2), then, by the $(p-1)$homogeneity of the p-Laplacian operator, for any $\lambda > 0$, $\lambda^{\frac{1}{p-1}}z$ is exactly a positive solution of the problem

$$\begin{cases} -div(|\nabla z|^{p-2}\nabla z) + m|z|^{p-2}z = \lambda a(x) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

This fact plays an important role in [**2**, **5**, **6**, **17**, **18**]. It is a pity that, in the $p(x)$-Laplacian case, such fact does not hold. To see this, in Section 2 we give an example which shows that there are $p(x)$ and $a(x)$ such that the corresponding problem (1.2) with $p = p(x)$ has a positive solution, but for sufficiently small $\lambda > 0$, the corresponding problem (1.3) with $p = p(x)$ has no positive solution. Such an example shows that the condition of the same form as $(A_\varepsilon^{\geq})$ or (A_*) is not suitable for the variable exponent problems considered in the present paper. In order to achieve our goal we must find some new conditions which are different from $(A_\varepsilon^{\geq})$ and (A_*) in form, but include $(A_\varepsilon^{\geq})$ and (A_*) as a special case when p is a constant.

In Section 2, we give some preliminaries about the $p(x)$-Laplacian and also give an example as mentioned above. In Section 3, we give some sufficient conditions for the existence of a positive solution to problem (1.1) for sufficiently small $\lambda > 0$. Our results are a generalization of the corresponding results established in [**2**, **5**, **6**, **17**, **18**] for the p-Laplacian case to the $p(x)$-Laplacian case.

2. Preliminaries and example. In this paper, if there is no other explanation, it will always be assumed that Ω is a bounded smooth domain in $\mathbb{R}^N$ and p and m satisfy (P) and (M).

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \quad \int_\Omega |u|^{p(x)}\,dx < \infty \right\}$$

with the norm

$$|u|_{L^{p(\cdot)}(\Omega)} = |u|_{p(\cdot)} = \inf\left\{\sigma > 0 \mid \int_\Omega \left|\frac{u}{\sigma}\right|^{p(x)} dx \le 1\right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \mid \ |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}.$$

Denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. $|\nabla u|_{p(\cdot)}$ is an equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$. We refer to [**4**, **7**, **14**, **19**, **22**, **28**] for the elementary properties of the space $W^{1,p(x)}(\Omega)$.

$u \in W_0^{1,p(\cdot)}(\Omega)$ is said to be a (weak) solution of (1.1) if

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + m(x)|u|^{p(x)-2}uv\right)dx = \lambda\int_\Omega a(x)f(u)v\,dx, \forall v \in W_0^{1,p(\cdot)}(\Omega).$$

Define $T = T_{p(\cdot)} : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$ by

$$T(u)v = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + m(x)|u|^{p(x)-2}uv\right)dx, \forall u, v \in W_0^{1,p(\cdot)}(\Omega).$$

PROPOSITION 2.1. ([**12**]) *The mapping* $T : W_0^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$ *is a strictly monotone homeomorphism, and is of type* (S_+), *namely for any sequence* $\{u_n\} \subset W_0^{1,p(\cdot)}(\Omega)$ *for which* $u_n \rightharpoonup u$ *in* $W_0^{1,p(\cdot)}(\Omega)$ *and* $\overline{\lim}_{n\to\infty} T(u_n)(u_n - u) \le 0$, u_n *must converge strongly to u in* $W_0^{1,p(\cdot)}(\Omega)$, *where '$\rightharpoonup$' denotes the weak convergence in* $W_0^{1,p(\cdot)}(\Omega)$.

Denote by $S = S_{p(\cdot)}$ the inverse mapping of T. Then the mapping $S = T^{-1} : (W_0^{1,p(\cdot)}(\Omega))^* \to W_0^{1,p(\cdot)}(\Omega)$ is a strictly monotone homeomorphism. We often view S as the solution operator for the problem

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + m(x)|u|^{p(x)-2}u = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

namely, we denote by $S(h)$ the (unique) solution of (2.1), and according to the different ranges of h and $S(h)$, we may have the different understandings of the mapping S.

PROPOSITION 2.2. (1) *For every* $h \in L^\infty(\Omega)$, (2.1) *has a unique solution* $S(h)$ *and* $S(h) \in L^\infty(\Omega)$.

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(2) (*Comparison principle*) *The mapping* $S : L^\infty(\Omega) \to L^\infty(\Omega)$ *is increasing, that is,* $S(h) \leq S(g)$ *in* Ω *if* $h \leq g$ *in* Ω.

(3) *The mapping* $S : L^\infty(\Omega) \to L^\infty(\Omega)$ *is bounded, and there is a positive constant* C_*, *dependent on* p_+, p_-, N *and* $|\Omega|$, *such that*

$$|S(h)|_{L^\infty(\Omega)} \leq C_* \max\left\{|h|_{L^\infty(\Omega)}^{\frac{1}{p_+-1}}, |h|_{L^\infty(\Omega)}^{\frac{1}{p_--1}}\right\} \text{ for all } h \in L^\infty(\Omega).$$

Proof. For statement (1) see [**13**], and for statement (2) see [**10**]. Here we only prove statement (3). First, let us consider the case that $h(x) \equiv M$ (a constant). By [**10**, Lemma 2.1], there exists a positive constant C_*, dependent on p_+, p_-, N and $|\Omega|$, such that

$$|S(M)|_{L^\infty(\Omega)} \leq C_* \max\left\{|M|^{\frac{1}{p_+-1}}, |M|^{\frac{1}{p_--1}}\right\} \text{ for all } M \in \mathbb{R}.$$

(Note that Lemma 2.1 in [**10**] was proved for the case that $m = 0$, in fact, the proof of the same result in the case when $m \neq 0$ is similar and the constant C_* is independent of m). Then, for any $h \in L^\infty(\Omega)$, statement (3) follows from the above inequality for the constant function M and the comparison principle (2). □

p is said to be Hölder continuous on $\overline{\Omega}$ if there exist constants $\alpha \in (0, 1)$ and $L > 0$ such that $|p(x) - p(y)| \leq L|x - y|^\alpha$ for all $x, y \in \overline{\Omega}$. p is said to be Log-Hölder continuous on $\overline{\Omega}$ if there exists a positive constant L such that

$$|p(x) - p(y)| \leq \frac{L}{-\ln|x-y|} \quad \text{for all } x, y \in \overline{\Omega} \text{ with } |x-y| \leq \frac{1}{2}.$$

It is obvious that Lipschitz continuity $\Longrightarrow$ Hölder continuity $\Longrightarrow$ Log-Hölder continuity.

PROPOSITION 2.3. (1) ([**1**, **10**, **13**]) *When* p *is Log-Hölder continuous on* $\overline{\Omega}$, *for every* $h \in L^\infty(\Omega)$, $S(h)$ *is Hölder continuous on* $\overline{\Omega}$, *and therefore, the mapping* $S : L^\infty(\Omega) \to C^0(\overline{\Omega})$ *is completely continuous.*

(2) ([**1**, **9**, **10**]) *When* p *is Hölder continuous on* $\overline{\Omega}$, *for every* $h \in L^\infty(\Omega)$, $S(h) \in C^{1,\alpha}(\overline{\Omega})$, *and therefore, the mapping* $S : L^\infty(\Omega) \to C^1(\overline{\Omega})$ *is completely continuous.*

PROPOSITION 2.4. ([**15**]) (*A strong maximum principle*) *Suppose that* p *is Lipschitz continuous on* $\overline{\Omega}$, $h \in L^\infty(\Omega)$, $h(x) \geq 0$ *for* $x \in \Omega$ *and* $h(x) \not\equiv 0$ *in* Ω. *Then* $S(h) \in C^{1,\alpha}(\overline{\Omega})$, $S(h)(x) > 0$ *for* $x \in \Omega$ *and* $\frac{\partial S(h)}{\partial \nu} < 0$ *on* $\partial\Omega$.

Propositions 2.1–2.4 are an extension of the corresponding results established in the case that p is a constant.

An essential difference between the $p(x)$-Laplacian and the p-Laplacian is that the p-Laplacian is homogeneous but the $p(x)$-Laplacian is inhomogeneous. As mentioned in Section 1, in the case that p is a constant, if for a fixed $h \in L^\infty(\Omega)$ there holds $S_p(h)(x) \geq 0$ (resp. $S_p(h)(x) > 0$) for $x \in \Omega$, then for every $\lambda > 0$, there holds also $S_p(\lambda h)(x) \geq 0$ (resp. $S_p(\lambda h)(x) > 0$) for $x \in \Omega$. However, this is not the case when $p(\cdot)$ is not a constant. To see this, we give an example as follows.

EXAMPLE. Let $N = 1$, $\Omega = (-1, 1)$, $m = 0$,

$$p(r) = \begin{cases} 4, & \text{if } |r| \le \frac{1}{4}, \\ -8\left(|r| - \dfrac{1}{2}\right) + 2, & \text{if } \frac{1}{4} \le |r| \le \frac{1}{2}, \\ 2, & \text{if } \frac{1}{2} \le |r| \le 1, \end{cases}$$

$$h_\varepsilon(r) = \begin{cases} -\varepsilon, & \text{if } |r| \le \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < |r| \le 1. \end{cases}$$

where ε is a small positive number.

For this example we have the following

PROPOSITION 2.5. *In the above example, there exists $\varepsilon > 0$ sufficiently small such that $S_{p(\cdot)}(h_\varepsilon) > 0$ in Ω and*

$$\inf_{r \in (-1,1)} S_{p(\cdot)}(\lambda h_\varepsilon)(r) < 0 \quad \textit{for sufficiently small } \lambda > 0. \tag{2.2}$$

Proof. By the definition of $p(r)$, p is Lipschitz continuous on $\overline{\Omega}$. Noting that when $\varepsilon = 0$, $h_0 \ge 0$ and $h_0 \not\equiv 0$ in Ω, by Proposition 2.4, $S(h_0) \in C^1(\overline{\Omega})$, $S(h_0)(x) > 0$ for $x \in \Omega$ and $\frac{\partial S(h_0)}{\partial \nu} < 0$ on $\partial\Omega$. By 2) of Proposition 2.3, for sufficiently small $\varepsilon > 0$, we have $S(h_\varepsilon) \in C^1(\overline{\Omega})$, $S(h_\varepsilon)(x) > 0$ for $x \in \Omega$ and $\frac{\partial S(h_\varepsilon)}{\partial \nu} < 0$ on $\partial\Omega$. Now let $\varepsilon \in (0, 1)$ be small enough. For any $\lambda > 0$, denote $u_\lambda = S(\lambda h_\varepsilon)$. Then, since $p(\cdot)$ and $h_\varepsilon(\cdot)$ are radially symmetric, u_λ is radially symmetric and it is the unique solution of the following problem:

$$\begin{cases} -\left(|u'_\lambda(r)|^{p(r)-2} u'_\lambda(r)\right)' = \lambda h_\varepsilon(r) & \text{in } (0, 1) \\ u_\lambda(1) = 0, \ u'_\lambda(0) = 0. \end{cases} \tag{2.3}$$

Indeed, problem (2.3) has a unique solution $u_\lambda(r)$ for $r \in [0, 1]$, which is expressed by formula (2.4). Setting $u_\lambda(r) = u_\lambda(-r)$ for $r \in [-1, 0]$, then the function $u_\lambda(r)$, $r \in [-1, 1]$, is radially symmetric and $u_\lambda = S(\lambda h_\varepsilon)$.

Denote $\Phi(r, \xi) = |\xi|^{p(r)-2}\xi$ for $r \in [-1, 1]$ and $\xi \in \mathbb{R}$. Then for each $r \in [-1, 1]$, $\Phi(r, \cdot) : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Denote by Φ_r^{-1} the inverse mapping of $\Phi(r, \cdot)$, that is

$$\Phi_r^{-1}(\eta) = \begin{cases} \eta^{\frac{1}{p(r)-1}} & \text{if } \eta \ge 0 \\ -|\eta|^{\frac{1}{p(r)-1}} & \text{if } \eta < 0. \end{cases}$$

Then we have

$$u_\lambda(r) = \int_r^1 \Phi_t^{-1}\left(\int_0^t \lambda h_\varepsilon(s)ds\right) dt \quad \text{for } r \in [0, 1]. \tag{2.4}$$

From the definition of h_ε we have

$$\int_0^t h_\varepsilon(s)ds \begin{cases} < 0 & \text{if } 0 < r < \frac{1}{2} + \frac{1}{2}\varepsilon, \\ \ge 0 & \text{if } \frac{1}{2} + \frac{1}{2}\varepsilon \le r \le 1, \end{cases}$$

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and thus by (2.4),

$$\begin{aligned}
u_\lambda(0) &= \int_0^1 \Phi_t^{-1}\left(\int_0^t \lambda h_\varepsilon(s)ds\right)dt \\
&= \int_0^{\frac{1}{2}+\frac{1}{2}\varepsilon} \Phi_t^{-1}\left(\int_0^t \lambda h_\varepsilon(s)ds\right)dt + \int_{\frac{1}{2}+\frac{1}{2}\varepsilon}^1 \Phi_t^{-1}\left(\int_0^t \lambda h_\varepsilon(s)ds\right)dt \\
&< \int_0^{\frac{1}{4}} \Phi_t^{-1}\left(\int_0^t \lambda h_\varepsilon(s)ds\right)dt + \lambda\int_{\frac{1}{2}+\frac{1}{2}\varepsilon}^1 \left(t-\frac{1}{2}-\frac{1}{2}\varepsilon\right)dt \\
&\le -\int_0^{\frac{1}{4}} (\lambda\varepsilon t)^{\frac{1}{3}}\,dt + \lambda\int_{\frac{1}{2}}^1 \left(t-\frac{1}{2}\right)dt \\
&= -\frac{3}{4}\left(\frac{1}{4}\right)^{\frac{4}{3}} \varepsilon^{\frac{1}{3}}\lambda^{\frac{1}{3}} + \frac{1}{8}\lambda.
\end{aligned}$$

This shows that, when $\lambda \le 6^{\frac{3}{2}}(\frac{1}{4})^2\varepsilon^{\frac{1}{2}}$, $u_\lambda(0) < 0$, that is, (2.2) holds.

3. Existence of positive solutions. Let us continue to use the notations as in Sections 1 and 2.

Let

$$\Gamma^{\ge}_{p(\cdot)} = \left\{h \in L^\infty(\Omega) | S_{p(\cdot)}(h)(x) \ge 0 \quad \text{for } x \in \Omega\right\},$$
$$\Gamma^{>}_{p(\cdot)} = \left\{h \in L^\infty(\Omega) | S_{p(\cdot)}(h)(x) > 0 \quad \text{for } x \in \Omega\right\}.$$

It is clear that when $a = 0$, problem (1.1) has only a zero solution, and when $a \ge 0$ and $a(x) \not\equiv 0$ for $x \in \Omega$, using the strong maximum principle, we can easily obtain the existence of a positive solution to (1.1) for small $\lambda > 0$. In this section, we assume that a is sign-changed, that is, a satisfies the following condition:

$$(A^{\pm}_{\infty}) \qquad a \in L^\infty(\Omega),\ a^+ \not\equiv 0 \text{ and } a^- \not\equiv 0.$$

THEOREM 3.1. *Let* (P), (M), (F) *and* $(A^{\pm}_{\infty})$ *hold. Suppose the following condition is satisfied:*

$(A^{\ge}_{\varepsilon,\delta})$ *(resp.* $((A^{>}_{\varepsilon,\delta}))$ *There are* $\varepsilon > 0$ *and* $\delta > 0$ *such that*

$$\mu(a^+ - (1+\varepsilon)a^-) \in \Gamma^{\ge}_{p(\cdot)} \left(\textit{resp.} \ \in \Gamma^{>}_{p(\cdot)}\right) \quad \text{for } \mu \in (0, \delta].$$

Then for sufficiently small $\lambda > 0$, *problem* (1.1) *has a non-negative (resp. a positive) solution.*

Proof. We only consider the case of $(A^{>}_{\varepsilon,\delta})$ because the proof for the case of $(A^{\ge}_{\varepsilon,\delta})$ is similar. Let ε and δ be as in condition $(A^{>}_{\varepsilon,\delta})$. Define $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(t) = \begin{cases} f(t) & \text{for } |t| \le 1, \\ f(-1) & \text{for } t < -1, \\ f(1) & \text{for } t > 1. \end{cases}$$

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Consider the following problem:

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + m(x)\,|u|^{p(x)-2}\,u = \lambda a(x)\widetilde{f}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Define $\widetilde{F}(t) = \int_0^t \widetilde{f}(s)ds$ for $t \in \mathbb{R}$ and

$$J_\lambda(u) = \int_\Omega \left(\frac{1}{p(x)}\,|\nabla u|^{p(x)} + \frac{m(x)}{p(x)}\,|u|^{p(x)} - \lambda a(x)\widetilde{F}(u) \right) dx, \ \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

Obviously, there exists a positive constant C such that $|\widetilde{f}(t)| \le C$ for all $t \in \mathbb{R}$, this implies that $|\widetilde{F}(t)| \le C|t|$ for all $t \in \mathbb{R}$. Noting that $p_- > 1$, $m \in L^\infty(\Omega)$, $m(x) \ge 0$ and $a \in L^\infty(\Omega)$, we can see that, for each $\lambda > 0$, the functional $J_\lambda : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ is coercive and sequentially weakly lower semi-continuous, and consequently, J_λ has a global minimizer u_λ which is a weak solution of problem (3.1). Noting that $|\lambda a(x)\widetilde{f}(u_\lambda)|_{L^\infty(\Omega)} \to 0$ as $\lambda \to 0$, by 3) of Proposition 2.2, we have that $|u_\lambda|_{L^\infty(\Omega)} \to 0$ as $\lambda \to 0$. Now we assume that $\lambda > 0$ is small enough such that $|u_\lambda|_{L^\infty(\Omega)} \le 1$. Then $\widetilde{f}(u_\lambda) = f(u_\lambda)$ and so u_λ is a solution of problem (1.1). Set $\gamma = \frac{\varepsilon}{2+\varepsilon}$. Since f is continuous at 0 and $f(0) > 0$, there is $\rho \in (0, 1)$ such that

$$-f(0)\gamma < f(\xi) - f(0) < f(0)\gamma \quad \text{for } |\xi| \le \rho.$$

Take $\lambda_1 > 0$ small enough such that $|u_\lambda|_{L^\infty(\Omega)} \le \rho$ for $\lambda \in (0, \lambda_1]$. Then when $\lambda \in (0, \lambda_1]$,

$$\begin{aligned} \lambda a(x) f(u_\lambda(x)) &= \lambda(a^+(x) - a^-(x)) f(u_\lambda(x)) \\ &= \lambda a^+(x) f(u_\lambda(x)) - \lambda a^-(x) f(u_\lambda(x)) \\ &\ge \lambda a^+(x) f(0)(1-\gamma) - \lambda a^-(x) f(0)(1+\gamma) \\ &= \lambda(1-\gamma) f(0) \left(a^+(x) - \frac{1+\gamma}{1-\gamma} a^-(x) \right) \\ &= \lambda(1-\gamma) f(0) \left(a^+(x) - (1+\varepsilon) a^-(x) \right). \end{aligned} \tag{3.2}$$

Let $\lambda_2 = \frac{\delta}{(1-\gamma)f(0)}$ and $\lambda_3 = \min\{\lambda_1, \lambda_2\}$. Then when $\lambda \in (0, \lambda_3]$, we have that $\lambda(1-\gamma)f(0) \le \delta$, and by condition $(A^>_{\varepsilon,\delta})$,

$$\lambda(1-\gamma)f(0)(a^+(x) - (1+\varepsilon)a^-(x)) \in \Gamma^>_{p(\cdot)}.$$

By (3.2) and the comparison principle, $\lambda a(x) f(u_\lambda(x)) \in \Gamma^>_{p(\cdot)}$, which shows that u_λ is a positive solution of problem (1.1). □

REMARK 3.1. In Section 1, we mentioned condition $(A^{\geq}_\varepsilon)$ which was used in [**6**, **17**] for the case that $p = 2$. We may extend it to the variable exponent case. For given variable exponent $p(\cdot)$, we say that $a \in L^\infty(\Omega)$ satisfies condition $(A^{\geq}_\varepsilon)$ (resp. $(A^>_\varepsilon)$) if the following condition holds:

$(A^{\geq}_\varepsilon)$ (resp. $(A^>_\varepsilon)$) there exists $\varepsilon > 0$ such that

$$(a^+ - (1+\varepsilon)a^-) \in \Gamma^{\geq}_{p(\cdot)} \left(\text{resp.} \in \Gamma^>_{p(\cdot)}\right).$$

Obviously, condition $(A^>_\varepsilon)$ implies condition $(A^{\geq}_\varepsilon)$. In the case when $p = 2$, from the strong comparison principle (i.e. the strong maximum principle) we may see that when $a \in L^\infty(\Omega)\backslash\{0\}$ satisfies condition $(A_\varepsilon^{\geq})$ with some $\varepsilon > 0$, a must satisfy condition $(A_{\varepsilon_1}^{>})$ for every $\varepsilon_1 \in (0, \varepsilon)$. In other words, when $p = 2$, $(A_\varepsilon^{\geq})$ and $(A_\varepsilon^{>})$ are essentially equivalent to each other. However, in the case when $p \neq 2$, because of lack of the general strong comparison principle, in general, condition $(A_\varepsilon^{\geq})$ does not imply condition $(A_{\varepsilon_1}^{>})$ for $\varepsilon_1 \in (0, \varepsilon)$. It is clear that, in the case when p is a constant, $(A_\varepsilon^{\geq})$ and $(A_{\varepsilon,\delta}^{\geq})$ (resp. $(A_\varepsilon^{>})$ and $(A_{\varepsilon,\delta}^{>})$) are essentially equivalent to each other. Thus our Theorem 3.1 is an extension of Theorem 2 in **[6]** and Theorem 1.1 in **[17]** to the $p(x)$-Laplacian case.

For $h \in L^\infty(\Omega)$ and $\varepsilon > 0$, define

$$B_\infty(h, \varepsilon) = \left\{g \in L^\infty(\Omega) \mid |g - h|_{L^\infty(\Omega)} < \varepsilon\right\},$$

and for $\delta > 0$, define

$$K(B_\infty(h, \varepsilon), \delta) = \{\mu g \mid \mu \in (0, \delta] \text{ and } g \in B_\infty(h, \varepsilon)\}.$$

Corollary 3.1. *Let (P), (M), (F) and $(A_\infty^{\pm})$ hold. Suppose the following condition is satisfied:*

$(K_{\varepsilon,\delta}^{\geq})$ (resp. $(K_{\varepsilon,\delta}^{>})$) *There are $\varepsilon > 0$ and $\delta > 0$ such that*

$$K(B_\infty(a, \varepsilon), \delta) \subset \Gamma_{p(\cdot)}^{\geq} \left(\text{resp. } \subset \Gamma_{p(\cdot)}^{>}\right).$$

Then a satisfies $(A_{\varepsilon_1,\delta}^{\geq})$ (resp. $(A_{\varepsilon_1,\delta}^{>})$) for some $\varepsilon_1 > 0$, and consequently, for sufficiently small $\lambda > 0$, problem (1.1) *has a non-negative (resp. a positive) solution.*

Proof. Let a satisfy $(K_{\varepsilon,\delta}^{\geq})$ (resp. $(K_{\varepsilon,\delta}^{>})$). Take $\varepsilon_1 \in (0, \frac{\varepsilon}{|a^-|_{L^\infty(\Omega)}})$. Then

$$|(a^+ - (1 + \varepsilon_1)a^-) - a|_{L^\infty(\Omega)} = \varepsilon_1 |a^-|_{L^\infty(\Omega)} < \varepsilon,$$

which shows $(a^+ - (1 + \varepsilon_1)a^-) \in B_\infty(a, \varepsilon)$. For $\mu \in (0, \delta]$, we have that

$$\mu(a^+ - (1 + \varepsilon_1)a^-) \in K(B_\infty(h, \varepsilon), \delta) \subset \Gamma_{p(\cdot)}^{\geq} \left(\text{resp. } \subset \Gamma_{p(\cdot)}^{>}\right).$$

This shows that $(A_{\varepsilon_1,\delta}^{\geq})$ (resp. $(A_{\varepsilon_1,\delta}^{>})$) is satisfied, and consequently, by Theorem 3.1, problem (1.1) has a non-negative (resp. a positive) solution for sufficiently small $\lambda > 0$. □

Remark 3.2. Let $p \in (1, \infty)$ be a constant and $a \in L^\infty(\Omega)$ satisfy condition (A_*), that is $S_p(a) > 0$ in Ω and $\frac{\partial S_p(a)}{\partial \nu} < 0$ on $\partial\Omega$. Since $S_p : L^\infty(\Omega) \to C^1(\overline{\Omega})$ is continuous, there exists $\varepsilon > 0$ such that $B_\infty(a, \varepsilon) \subset \Gamma_p^{>}$. In this case, for any $\delta > 0$, $K(B_\infty(a, \varepsilon), \delta) \subset \Gamma_p^{>}$ holds. This shows that, when p is a constant, condition (A_*) implies condition $(K_{\varepsilon,\delta}^{>})$ for some $\varepsilon > 0$ and any $\delta > 0$. Hence Theorem 1 of Hai and Xu **[18]** is a special case of Corollary 3.1.

Now let us consider the radially symmetric case. Suppose that the following condition is satisfied.

(R) $\Omega = B(0, r_0) \subset \mathbb{R}^N$ is a ball, $p(x) = p(|x|) = p(r)$ and $a(x) = a(|x|) = a(r)$ are radially symmetric, and $m = 0$.

In this case, the solution of (1.1) is just the solution of the following problem:

$$\begin{cases} -\left(r^{N-1}|u'(r)|^{p(r)-2}u'(r)\right)' = \lambda r^{N-1}a(r)f(u) & \text{in } (0, r_0), \\ u(r_0) = 0, \quad u'(0) = 0. \end{cases} \tag{3.3}$$

COROLLARY 3.2. *Let* (P), (M), (F), $(A^{\pm}_{\infty})$ *and* (R) *hold. Suppose that a satisfies the following condition*

(I_τ) *there exists* $\tau > 0$ *such that*

$$\int_0^s t^{N-1}a^+(t)dt \geq (1+\tau)\int_0^s t^{N-1}a^-(t)dt \quad \text{for } s \in (0, r_0].$$

Then a satisfies condition $(A^{>}_{\varepsilon,\delta})$ *with* $\varepsilon = \frac{\tau}{2}$ *and any* $\delta > 0$, *and consequently, for sufficiently small* $\lambda > 0$, *problem* (1.1) *has a positive solution.*

Proof. Put $\varepsilon = \frac{\tau}{2}$. Let any $\mu > 0$ be given. Denote $u = S_{p(\cdot)}(\mu(a^+ - (1+\varepsilon)a^-))$. Then

$$\begin{cases} -\left(r^{N-1}|u'(r)|^{p(r)-2}u'(r)\right)' = \mu r^{N-1}(a^+ - (1+\varepsilon)a^-) & \text{in } (0, r_0), \\ u(r_0) = 0, \quad u'(0) = 0. \end{cases}$$

Thus we have, for $r \in (0, r_0]$,

$$\begin{aligned} -\left(r^{N-1}|u'(r)|^{p(r)-2}u'(r)\right) &= \mu\int_0^r t^{N-1}\left(a^+(t) - \left(1+\frac{\tau}{2}\right)a^-(t)\right)dt \\ &\geq \frac{\mu\tau}{2}\int_0^r t^{N-1}a^-(t)dt \geq 0. \end{aligned}$$

This shows that $u'(r) \leq 0$ for all $r \in (0, r_0)$. Noting that $\int_0^{r_0} t^{N-1}a^-(t)dt > 0$, we have $u'(r_0) < 0$, and therefore $u(r) > 0$ for $r \in [0, r_0)$ because $u(r_0) = 0$. This proves that $\mu(a^+ - (1+\varepsilon)a^-) \in \Gamma^{>}_{p(\cdot)}$ for any $\mu > 0$, that is, condition $(A^{>}_{\varepsilon,\delta})$ with $\varepsilon = \frac{\tau}{2}$ and any $\delta > 0$ is satisfied, and consequently, by Theorem 3.1, problem (1.1) has a positive solution for sufficiently small $\lambda > 0$. □

REMARK 3.3. Condition (I_τ) was proposed by Các, Fink and Gatica [**5**] for the case that $p = 2$. Note that condition (I_τ) used in this paper is the same as in [**5**], and it is independent of $p(\cdot)$. The verification of condition (I_τ) is often easy, for example, it is easy to see that, in the radially symmetric case, the function a, defined by

$$a(r) = \begin{cases} 1 & \text{if } |r| \leq \frac{r_0}{2}, \\ -\varepsilon & \text{if } \frac{r_0}{2} < |r| \leq r_0, \end{cases}$$

where $\varepsilon \in (0, \frac{1}{2^N-1})$, satisfies condition (I_τ) with $\tau \in (0, \frac{1}{\varepsilon(2^N-1)} - 1)$. Of course, as was mentioned in [**2**, **6**], (I_τ) is a stronger condition to assure the existence of a positive solution to problem (3.4) for small values of λ.

REMARK 3.4. Let Ω, m, $p(\cdot)$ and $a = h_\varepsilon$ be as in the example given in Section 2, where $\varepsilon > 0$ is sufficiently small, and let $f(t) = 1$ for all $t \in \mathbb{R}$. Proposition 2.5 shows

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that, in this case, condition (A_*) as well as condition $(A^>_{\varepsilon_1})$ with small $\varepsilon_1 > 0$ is satisfied but the corresponding problem (1.1) has no positive solution for sufficiently small $\lambda > 0$.

Finally, we give an example in which the condition $(A^>_{\varepsilon,\delta})$ put in Theorem 3.1 is satisfied but the condition (I_τ) put in Corollary 3.2 is not satisfied.

EXAMPLE 3.5. Let $N = 1$, $\Omega = (-1, 1)$, $m = 0$,

$$p(r) = \begin{cases} 2, & \text{if } |r| \le \frac{1}{2}, \\ 8\left(|r| - \frac{1}{2}\right) + 2, & \text{if } \frac{1}{2} \le |r| \le \frac{3}{4}, \\ 4, & \text{if } \frac{3}{4} \le |r| \le 1, \end{cases}$$

$$a(r) = \begin{cases} -\frac{1}{8}, & \text{if } |r| \le \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < |r| \le 1. \end{cases}$$

Take $\varepsilon = 1$. we will show that there exists $\delta > 0$ such that condition $(A^>_{1,\delta})$ is satisfied, that is, $\mu(a^+ - (1+1)a^-) \in \Gamma^>_{p(\cdot)}$ for $\mu \in (0, \delta]$. Denote $u_\mu = S_{p(\cdot)}(\mu(a^+ - 2a^-))$. Then u_λ is radially symmetric and it is the unique solution of the following problem:

$$\begin{cases} -\left(|u'_\mu(r)|^{p(r)-2}u'_\mu(r)\right)' = \mu(a^+ - 2a^-)(r) & \text{in } (0, 1), \\ u_\mu(1) = 0, \ u'_\mu(0) = 0. \end{cases}$$

Thus, we have

$$u'_\mu(r) = -\Phi_r^{-1}\left(\int_0^r \mu(a^+(s) - 2a^-(s))ds\right) \quad \text{for } r \in (0, 1). \tag{3.4}$$

It is sufficient to prove that $u_\mu(r) > 0$ for sufficiently small $\mu > 0$ and all $r \in [0, 1)$. We may assume $\mu \in (0, 1)$. Noting that when $r \in (0, \frac{1}{2}]$,

$$\int_0^r \mu(a^+(s) - 2a^-(s))ds = \int_0^r -\frac{1}{4}\mu ds = -\frac{1}{4}\mu r < 0,$$

and when $r \in (\frac{1}{2}, 1)$,

$$\begin{aligned} \int_0^r \mu(a^+(s) - 2a^-(s))ds &= \int_0^{\frac{1}{2}} -\frac{1}{4}\mu ds + \int_{\frac{1}{2}}^r \mu ds \\ &= -\frac{1}{8}\mu + \left(r - \frac{1}{2}\right)\mu = \left(r - \frac{5}{8}\right)\mu, \end{aligned}$$

we can see that $u'_\mu(r) > 0$ for $r \in (0, \frac{5}{8})$, $u'_\mu(r) < 0$ for $r \in (\frac{5}{8}, 1)$, and u_μ attains its maximum at $r = \frac{5}{8}$. Since $u_\mu(1) = 0$, we have that $u_\mu(r) > 0$ for $r \in [\frac{5}{8}, 1)$ and

$$\begin{aligned} u_\mu\left(\frac{5}{8}\right) > u_\mu\left(\frac{3}{4}\right) &= -\int_{\frac{3}{4}}^1 u'_\mu(r)dr = \int_{\frac{3}{4}}^1 \Phi_r^{-1}\left(\left(r - \frac{5}{8}\right)\mu\right)dr \\ &\ge \int_{\frac{3}{4}}^1 \left(\frac{1}{8}\mu\right)^{\frac{1}{4-1}} dr = \frac{1}{4}\cdot\frac{1}{2}\mu^{\frac{1}{3}}. \end{aligned}$$

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For $r \in [0, \frac{5}{8})$ we have

$$\begin{aligned} u_\mu(r) &\geq u_\mu(0) = u_\mu\left(\frac{5}{8}\right) - \int_0^{\frac{5}{8}} u'_\mu(r)dr \geq u_\mu\left(\frac{5}{8}\right) - \int_0^{\frac{5}{8}} \Phi_r^{-1}\left(\frac{1}{4}\mu\right)dr \\ &\geq u_\mu\left(\frac{5}{8}\right) - \int_0^{\frac{5}{8}} \left(\frac{1}{4}\mu\right)^{\frac{1}{p(\frac{5}{8})-1}} dr \\ &= \frac{1}{8}\mu^{\frac{1}{3}} - \frac{5}{8}\cdot\left(\frac{1}{4}\mu\right)^{\frac{1}{2}}. \end{aligned}$$

It follows that when $\mu \in (0, (\frac{2}{5})^6)$, $u_\mu(r) > 0$ for all $r \in [0, 1)$. This shows that the condition $(A^{>}_{1,\delta})$ is satisfied for $\delta \in (0, (\frac{2}{5})^6)$. It is obvious that the condition (I_τ) is not satisfied because for any $\tau > 0$ and $s \in (0, \frac{1}{2})$,

$$0 = \int_0^s a^+(t)dt < (1+\tau)\int_0^s a^-(t)dt.$$

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