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A NOTE ON PERMUTATIONS AND TOPOLOGICAL ENTROPY OF CONTINUOUS MAPS OF THE INTERVAL

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ABSTRACT. Suppose f is a continuous endomorphism of an interval which has a periodic orbit, $p_0 < p_1 < \ldots < p_n$, that defines a permutation σ by $f(p_i) = p_{\sigma(i)}$. If σ is irreducible the topological entropy of f is bounded below by the logarithm of the spectral radius of an $n \times n$ matrix which is induced by σ.

Introduction. In recent years a number of articles have appeared in the literature which have used the existence of periodic points of maps of the interval in order to establish certain results about the topological entropy of the map in question ([3], [4], [5], [10]). For example, if f is a continuous endomorphism of a closed interval which admits a periodic point of period $2^k m$ where m is an odd integer greater than one, it is shown (in [3] for example) that the topological entropy $h(f)$ is bounded below by ${}^{\lambda}m/2^k$, where m is the logarithm of the largest root of the polynomial $X^m - 2X^{m-2} - 1$. This result is the best possible in the sense that there exist maps with points of period $2^k m$ and entropy exactly equal to ${}^{\lambda}m/2^k$ ([9]).

In this note we point out that the above result can be generalized in the following sense. Let σ be a fixed cyclic permutation on the symbols $\{0, 1, \ldots, n\}$ of a type which we shall call irreducible (and define below). Consider any continuous map of the interval, f, which has a periodic orbit; $p_0 < p_1 < \ldots < p_n$; where $f(p_i) = p_{\sigma(i)}$, $i = 0, 1, \ldots, n$. We shall show that there exists a polynomial P_σ of degree $n + 1$ which has a maximal real root λ_σ such that $h(f) \geqslant \log(\lambda_\sigma)$. Again the lower bound is the best possible for this problem. The polynomial P_σ is related to the kneading determinant of Milnor and Thurston ([8]). It is therefore very simple to write down explicitly when the permutation σ can be obtained from the periodic orbit of a unimodal map.

Matrices induced by permutations. Let σ be a cyclic permutation of the set $S = \{0, 1, \ldots, n\}$ and define the $n \times n$ matrix $A_\sigma = (a_{ij})$ by setting $a_{ij} = 1$ if and only if $\sigma(i - 1) < j \leq \sigma(i)$ or $\sigma(i) < j \leq \sigma(i - 1)$ and $a_{ij} = 0$ otherwise.

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For $T \subset S$ let $s(T)$ be the convex hull of $\sigma(T)$ i.e., $i \in s(T)$ if $\sigma(t_1) \leq i \leq \sigma(t_2)$ for some t_1, $t_2 \in T$. Denote the closed interval as usual by $[\alpha, \beta] = \{i \in S : \alpha \leqslant i \leqslant \beta\}$. We shall call a closed interval *maximal* (for σ) if the cardinality of $s^k([\alpha, \beta]) = s(s^{k-1}([\alpha, \beta]))$ is constant for $k = 0, 1, 2, \ldots$

DEFINITION. *The cyclic permutation σ of the set $S = \{0, 1, \ldots, n\}$ is* irreducible *if the only maximal intervals are the trivial ones (singletons and the set S itself).*

LEMMA. *If σ is irreducible then the matrix A_σ is primitive, i.e., $A_\sigma^k > 0$ for some integer $k > 0$.*

PROOF. Let $T = [i-1, i]$. Since σ is injective the cardinalities of the sets $s^k(T)$ form a non-decreasing sequence bounded above by $n + 1$. Thus, we must have $s^k(T) = S$ for $k \geqslant k_0$ because σ is irreducible. We can clearly find a value of k which works for all $i = 1, \ldots, n$. Now for this value of k we have $A^k > 0$ since $a_{ij}^k > 0$ if and only if $\{j-1, j\} \subseteq s^k(\{i-1, i)\}$.

It follows from Perron-Frobenius theory for positive matrices that a primitive matrix has a maximal real eigenvalue $\lambda > 0$ (i.e., every other eigenvalue μ has $|\mu| < \lambda$). Moreover λ has an eigenvector $a = (a_1, \ldots, a_n)$ with $a_i > 0$, $i = 1, \ldots, n$. Choose such an eigenvector with $\Sigma a_i = 1$ and construct a partition $0 = x_0 < x_1 < \ldots, n$; by setting $x_j = \Sigma_{i=1}^j a_i$. Define the map τ from $[0, 1]$ into itself by setting $\tau(x_i) = x_{\sigma(i)}$ and making τ linear on the intervals $[x_{i-1}, x_i]$. The slope of τ on $[x_{i-1}, x_i]$ is equal to $|x_{\sigma(i)} - x_{\sigma(i-1)}|/a_i = \pm(\Sigma_{j=1}^n a_{ij}a_j)/a_i = \pm\lambda$ since $A \cdot \boldsymbol{a} = \lambda \boldsymbol{a}$. Thus τ is a map of constant slope (in absolute value) and its topological entropy, $h(\tau)$, is therefore equal to $\log_2(\lambda)$ [7].

The map τ defines a directed graph G with vertices $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$ and arrows $I_i \rightarrow I_j$ and only if $\tau(I_i) \supset I_j$. The adjacency matrix of this graph is precisely the above matrix A_σ. If the topological entropy of G, $h(G)$, is defined to be the spectral radius of its adjacency matrix we have

$$h(G) = \lambda = h(\tau).$$

Continuous endomorphisms. Now let f be any continuous map from a closed interval J into itself. Suppose f has a periodic orbit $p_0 < p_1 < \ldots < p_n$ of minimal period $n + 1$. Of course f defines the cyclic permutation σ by $f(p_i) = p_{\sigma(i)}$ $i = 0, \ldots, n$. The partition of J into subintervals $J_i = [p_{i-1}, p_i]$ gives us a directed graph for f which contains the graph G above as a subgraph. As in [3] the theory developed in [7] can be used to show that $h(f) \geqslant h(G)$. Thus when σ is irreducible we have $h(f) \geqslant h(G) = \lambda = h(\tau)$.

This lower bound is the spectral radius of the matrix A_σ. To calculate it we must be able to write down the characteristic polynomial for $A = A_\sigma$. To this end we have det $(I - zA) = \exp[\mathrm{Tr}\log(I - zA] = \exp[-\Sigma_{k=1}^\infty z^k/k \,\mathrm{Tr}\, A^k]$ which converges for small z. Now $\mathrm{Tr}\, A^k$ is the number of loops in the graph G which have length k which is equal to the number of periodic points of period k for the piecewise-linear map τ. Thus

$\det(I - zA) = \rho^{-1}(\tau, z)$, the reciprocal of the zeta function discussed in [7]. We now need the following:

LEMMA. *If the permutation σ is irreducible, the spectral radius of the matrix A_σ is strictly greater than one.*

PROOF. Since A_σ is primitive the spectral radius is bounded below by the minimum number of 1's in any row, [L], the spectral radius is greater or equal to 1. Suppose it is equal to one. If τ is the piecewise linear map induced by σ then τ has slope ± 1. Suppose $[0, 1] = \bigcup_{i=1}^{n} I_i$ as above. If $J \subseteq [0, 1]$ is a closed interval, then the length of $\tau(J)$, $|\tau(J)| = |\bigcup_{i=1}^{n} \tau(J \cap I_i| \leq |J|$. Thus if $J = I_i$ we cannot have $\tau^k(J) = [0, 1]$ for any $k \geqslant 1$. This is a contradiction since we have $A_\sigma^k > 0$ for some k and this implies that for every $i = 1, \ldots, n$ we have $\tau^k(I_i) = [0, 1]$.

Thus τ is an expanding map and so no two periodic points can be montonely-equivalent. (This would mean that for some k, τ^k is a homeomorphism from the interval between them onto itself). Thus, $\det(I - zA) = \hat{\zeta}^{-1}(\tau, z)$ the inverse of the so-called reduced zeta-function, which is therefore a polynomial of degree n. Suppose this polynomial is $Q(z) = a_0 + a_1 z + \ldots + a_n z^n$. Then $\det(A - xI) = (-x)^n \det(I - (1/x)A) = (-1)^n(a_n + a_{n-1}x + \ldots + a_0 x^n)$. Let $a_n + a_{n-1}x + \ldots + a_0 x^n = P(x)$. The polynomials P and Q are related to the dynamics of the map τ by the following:

THEOREM (Milnor and Thurston [8]). *If each turning point belongs to the same periodic orbit (of period $n + 1$) then the power series $\hat{\zeta}^{-1}(\tau z)$ equals $(1 - z^{n+1})D(z)$ where $D(z)$ is the kneading determinant.*

In any order to avoid the (lengthy) definition of $D(z)$ in general we shall now restrict ourselves to the case of permutations σ which induce unimodal maps τ. Such permutations clearly satisfy the condition that there exists an integer $k_0 \in \{1, 2, \ldots, n - 1\}$ such that either $(a)\sigma(i) < \sigma(j)$ for $i < j \leqslant k_0$ and $\sigma(i) > \sigma(j)$ for $k_0 \leqslant i < j$ or $(b)\sigma(i) > \sigma(j)$ for $i < j \leqslant k_0$ and $\sigma(i) < \sigma(j)$ for $k_0 \leqslant i < j$. In fact if σ satisfies (b) then the permutation $\sigma'(m) = n - \sigma(n - m)$ satisfies (a). The resulting matrices are related by $(A'_\sigma)_{ij} = (A_\sigma)_{n-i+1, n-j+1}$. Thus they have the same characteristic polynomial and so we only consider permutations satisfying (a).

For an irreducible permutation, σ, satisfying (a)

$$\text{set } \epsilon_i = \begin{cases} +1 & \text{if} \quad \sigma^i(k_0) < k_0 \\ -1 & \text{if} \quad \sigma^i(k_0) > k_0 \end{cases}$$

and choose $\epsilon_n + 1$ so that $\epsilon_1 \epsilon_2 \ldots \epsilon_n + 1 = 1$. Then $D(z) = Q(z)/1 - z^{n+1}$ where $Q(z) = 1 + \epsilon_1 z + \epsilon_1 \epsilon_2 z^2 + \ldots + \epsilon_1 \ldots \epsilon_n z^n$ [7]. Thus $\det(I - zA) = (1 - z^{n+1}) \times D(z) = Q(z)$ and $\det(A - xI) = (-1)^n P(x)$ where $P(x) = x^n + \epsilon_1 x^{n-1} + \ldots + \epsilon_1 \ldots \epsilon_n$. Thus we have the following:

THEOREM: *Suppose f is a continuous endomorphism of an interval with a periodic orbit of period $n + 1$ which gives rise to an irreducible permutation σ as above. Then the topological entropy of f is greater or equal to the logarithm of the largest root, λ_σ,* *of the polynomial* $P(x) = P_\sigma(x)$, *the characteristic polynomial of the matrix* A_σ. *In the case where* σ *induces a unimodal map* τ *(condition* (a) *or* (b) *above), the polynomial* $P(x)$ *is given explicitly by the above formula.*

NOTE. (1) If f is unimodal so is the induced map τ.

(2) This estimate is the best possible since for an irreducible permutation σ, the piecewise linear map τ has entropy equal to $\log(\lambda_\sigma)$.

(3) Suppose the continuous map f has a finite invariant set $\{p_0 < p_1 < \ldots < p_n\}$ on which f induces the finite map σ and matrix A_σ as above. If A_σ is primitive with spectral radius λ_σ the same reasoning shows that $h(f) \geqslant \log(\lambda_\sigma)$.

EXAMPLE 1. *Suppose f has a periodic point of period 4,* $P_0 < P_1 < P_2 < P_3$, *where* $f(p_i) = p_{i+1}$; $i = 0, 1, 2$ *and* $f(P_3) = P_0$. *Then the permutation*

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

induces the companion matrix

$$A_\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

with characteristic polynomial $x^3 - x^2 - x - 1 = 0$. *(This may also be obtained by noting that* σ *satisfies condition* (a) *and that* $\epsilon_1 = -1$ *and* $\epsilon_2 = \epsilon_3 = +1$.

The maximal root of this polynomial is approximately 1.839287 and so the topological entropy satisfies $h(f) \geqslant 0.879147 \ldots$. Simply knowing that the point has period 4 gives no useful information in itself since there are maps with points of period 4 and entropy zero. Even using the associated directed graph to show that such a map f must have a point of period 3 would only give us $h(f) \geqslant 0.694242 \ldots$ by previous estimates. Thus we certainly get sharper estimates for $h(f)$ in this way.

EXAMPLE 2. *Previous estimates* ([3], [5]) *depend, in our view, on the result of Stêfan* [9] *which states that if a continuous map has a periodic point of period* $2n - 1$ *(and no point of smaller odd period) then it must give rise to the permutation*

$$\sigma = \begin{pmatrix} 1 & 2 & n-1 & n & n+1 & 2n-1 \\ n & 2n-1 & n+2 & n+1 & n-1 & 1 \end{pmatrix}$$

$$\left[\text{or (equivalently) to} \begin{pmatrix} 1 & n-1 & n & n\,1 & 2n-2 & 2n-1 \\ 2n-1 & n\,1 & n\,1 & n-2 & 1 & n \end{pmatrix}\right]$$

It is straightforward to check that these permutations are irreducible and they both induce unimodal maps. In the case of σ we have $\epsilon_1 = \epsilon_3 = \epsilon_4 = \epsilon_{2n-2} = -1$ and $\epsilon_2 = +1$. Thus $P(X) = X^{2n-2} - X^{2n-3} - X^{2n-4} + X^{2n-5} - \ldots - 1 = 0$. Multiplying by $(X + 1)$ we get the polynomial $X^{2n-1} - 2X^{2n-3} - 1 = 0$ which is the polynomial obtained in the papers we have mentioned above.

EXAMPLE 3. *There are many examples of maps whose associated permutations are not irreducible. Block's "simple" periodic points* ([1], [2]) *fall into this category. An elementary example is given by a piecewise-linear realization of*

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix}.$$

The sets $\{0, 1\}$ *and* $\{2, 3\}$ *are maximal. The associated matrix is reducible.*

REFERENCES

1. L. Block, *Simple periodic orbits of mappings of the interval*. Trans. Amer. Math. Soc. **254** (1979), pp. 391–398.

2. L. Block and D. Hart, *Stratification of the space of unimodal interval maps*, Ergod. Th. and Dynam. Syst. **3** (1983), pp. 533–539.

3. L. Block, J. Guckenheimer, M. Misiurewicz, and L.-S. Young, *Periodic points and topological entropy of one dimensional maps*, Lect. Notes Math. **819** (Springer 1980), pp. 18–34.

4. R. Bowen and J. Franks, *The periodic points of maps of the disk and the interval*, Topology **15** (1976), pp. 337–342.

5. L. Jonber and D. Rand, *The periodic orbit and entropy of certain maps of the unit interval*, J. London Math. Soc. (2) **22** (1980), pp. 175–181.

6. P. Lancaster, *Theory of Matrices*, Academic Press, New York (1969).

7. M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), pp. 45–63.

8. J. Milnor and W. Thurston, *On iterated maps of the interval I and II*, Princeton (1977).

9. P. Stefan, *A Theorem of Sarkovskii on the coexistence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys. **54** (1977), pp. 237–248.

10. Y. Takahashi, *A formula for topological entropy of one dimensional dynamics*, Sci. Papers College Gen. Ed. Univ. Tokyo **30** (1980), pp. 11–22.

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