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Proof of a Theorem in Conics.

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I.

In text books of Plane Coordinate Geometry, two methods are usually given for investigating the condition that the general equation of the second degree:

$$\phi \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

may represent a pair of real or imaginary straight lines.

The first is by identifying ϕ with the product of two linear factors, say $\lambda\lambda' \equiv (lx + my + nz)(l'x + m'y + n'z)$.
Equating coefficients, and eliminating l, m, n, l', m', n', we get

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, \text{ or, Discriminant} = 0$$

as the condition required.

The second method consists in solving $\phi = 0$ as a quadratic equation in x, and deducing the condition that the expression in y and z under the radical sign, should be a perfect square.

This as before, gives the condition : Discriminant $= 0$.

We may note by the way that of these two methods, the former, strictly speaking, proves only the *necessity*, and the latter, only the *sufficiency* of the condition ; so that the propositions proved are converse, one of the other.

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The object of this Note is to point out a short way of performing the elimination required in the former method, by forming the determinant which is the product of the two zero determinants

$$\begin{vmatrix} l, & l', & o \\ m, & m', & o \\ n, & n', & o \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} l', & l, & o \\ m', & m, & o \\ n', & n, & o \end{vmatrix}$$

The product is the symmetrical determinant

$$\begin{vmatrix} ll' + l'l, & lm' + l'm, & ln' + l'n \\ ml' + m'l, & mm' + m'm, & mn' + m'n \\ nl' + n'l, & nm' + n'm, & nn' + n'n \end{vmatrix}$$

which is of course identically equal to zero.

But if ϕ is identical with $\lambda\lambda'$ the determinant is obviously the same as

$$8 \times \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Thus the discriminant of ϕ is zero if ϕ represents a pair of straight lines.

Of course $\lambda\lambda' = 0$ is the standard form when we have a pair of *real* straight lines; and can only represent an *imaginary* pair when some of the coefficients are imaginary. The standard form for a pair of imaginary lines (or point-ellipse) would be $\lambda^2 + \lambda'^2 = 0$, where $\lambda \equiv lx + my + nz$, etc.

In this case the identification with ϕ gives

$$a = l^2 + l'^2, \; f = mn + m'n', \text{ etc., etc.}$$

And the elimination of l, m, n, l', m', n' can here be performed by squaring the zero determinant

$$\begin{vmatrix} l & l' & o \\ m & m' & o \\ n & n' & o \end{vmatrix}$$

and substituting a for $l^2 + l'^2$, f for $mn + m'n'$, etc., in the result.

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II.

It occurred to me recently that this method of getting the condition *discriminant* $=0$ by multiplying two determinants, might be capable of application to discuss the discriminant in the general case. I have only had leisure to make a beginning in this direction, and none to look up the literature of the subject; but the following results seem interesting, and are new to me.

Suppose the general expression ϕ put into the form

$$p\lambda^2+p'\lambda'^2+p''\lambda''^2,$$

where $pp'p''$ are constants and $\lambda \equiv lx+my+nz$, etc.; thus we have

$$a=pl^2+p'l'^2+p''l''^2,\ f=pmn+p'm'n'+p''m''n'', \text{ etc., etc.}$$

and the discriminant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is obviously

the product $\begin{vmatrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{vmatrix} \times \begin{vmatrix} pl, & p'l', & p''l'' \\ pm, & p'm', & p''m'' \\ pn, & p'n', & p''n'' \end{vmatrix}$

which may be written

$$pp'p'' \times \begin{vmatrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{vmatrix}^2$$

and this is $=p\,.\,p'.\,p''.\ \mathrm{NN'N''} \times$ twice area of triangle formed by the lines $\lambda=0$, $\lambda'=0$, $\lambda''=0$; where N, N′, N″, are the minors of n, n', n''.

This of course vanishes when the lines are concurrent, in which case ϕ is expressible as the sum of two squared linear terms; and also when $pp'p''=0$, *i.e.* when one at least of the squared terms is awanting.

The lines $\lambda=0$, $\lambda'=0$, $\lambda''=0$ form a self-conjugate triangle for the conic; and such triangles are triply infinite in number for a given conic. We get the same result as to the possible number of ways of expressing ϕ in the form $p\lambda^2+p'\lambda'^2+p''\lambda''^2$ by noting that

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there are 8 independent ratios between the coefficients of the latter expression, and only 5 in ϕ.

Again, it appears that the discriminant may vanish in virtue of p'' being zero, in which case the value of λ'' might be anything whatever; in fact, it seems that in such a case, while two sides of a self-conjugate triangle must pass through the centre of the conic, the position of the third is quite indeterminate, a result which is obvious also from the geometrical point of view.