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EXISTENCE OF POSITIVE SOLUTION FOR A QUASI-LINEAR PROBLEM WITH CRITICAL GROWTH IN $\mathbb{R}^N_+$

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Abstract. In this paper we show existence of positive solutions for a class of quasi-linear problems with Neumann boundary conditions defined in a half-space and involving the critical exponent.

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1. Introduction. In the last years, many authors have considered quasi-linear problems of the kind

$$-\Delta_p u + V(x)u^{p-1} = f(u), \quad \text{in} \quad \Omega, \quad \text{and} \quad u > 0, \quad \text{in} \quad \Omega$$

where Δ_p is the p-Laplacian operator given by

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u),$$

Ω is a domain in $\mathbb{R}^N$ with $N > p$, which can be bounded or unbounded, f has subcritical or critical growth, V is a continuous function and, in general, it is assumed Dirichlet or Neumann boundary conditions. This class of problems arises in a lot of applications, such as image processing, non-Newtonian fluids and pseudo-plastic fluids. For more details see [**5**, **11**, **13**].

From the mathematical viewpoint, this class of problems is also very interesting because the p-Laplacian operator is non-linear and many important properties that hold for the $p = 2$ case (Laplacian operator) are no longer valid when we are working with $p \neq 2$; for example, classical regularity and bootstrap arguments.

Different approaches and techniques were explored and developed in papers related to this class of problem, such as the symmetry of the solutions and of the domain, the methods of symmetrization and the Principle of Concentration–Compactness given by Lions [**19**]. One of the main difficulties that appear, when the domain Ω is unbounded or the non-linearity f has a critical growth, is the lack of compactness, which, in connection with the variational method approach, leads the energy functional associated to the problem not to verify the Palais–Smale condition.

In [**8**], Brézis and Nirenberg considered a critical problem of the type

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $N \geq 3$ and Ω is a smooth bounded domain. In that paper, Brézis and Nirenberg developed a new approach to overcome the technical difficulty associated with the presence of the term u^{2^*-1}. This approach consists in showing that the Palais Conditions hold for levels in the interval $(0, 1/NS^{N/2})$, where S is the best Sobolev constant related to the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. After Brézis and Nirenberg [**8**], this technique was widely used by many authors, such as, Struwe [**22**], Benci and Cerami [**6**], Cerami and Passaseo [**9**], Passaseo [**21**], Ben-Naoum, Troestler and Willem [**7**], Tarantello [**24**], Chabrowski and Yang [**10**], Wang [**25**] and see also the references therein.

For the special case $\Omega = \mathbb{R}^N$, in [**6**], Benci and Cerami showed that the problem

$$-\Delta u + V(x)u = u^{2^*-1} \text{ in } \mathbb{R}^N \tag{2}$$

has a positive solution if $V(x)$ is a non-negative function, strictly positive somewhere, having $L^{N/2}(\mathbb{R}^N)$ norm satisfying $|V|_{L^{N/2}(\mathbb{R}^N)} < S(2^{2/N} - 1)$ and belonging to $L^t(\mathbb{R}^N)$, for t in a neighbourhood of $N/2$. In their paper, to establish the existence of a non-trivial critical point, they used the Struwe's Global Compactness to prove a Compactness Theorem for the functional related to (2), together with a variant of a deformation lemma on manifolds.

In [**9**], assuming to $V(x)$ similar hypotheses to those in [**6**], Cerami and Passaseo considered the following class of problem:

$$\begin{cases} -\Delta u + V(x)u = u^{2^*-1}, & \text{in } \mathbb{R}^N_+ \\ u > 0, & \text{in } \mathbb{R}^N_+ \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{3}$$

where

$$\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N, x_N > 0\}$$

and

$$\partial\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N; x_N = 0\}.$$

Quasi-linear problems related to (1), that is, problems involving the p-Laplacian operator, have also been considered in a lot of papers. In many papers where there are the presence of the p-Laplacian and a non-linearity with critical growth, the Concentration–Compactness Principle due to Lions [**19**] is a key tool to prove that the weak limit of the (PS) sequence related to energy functional is a critical point, and thus, a weak solution for the problem. In this direction, we cite the papers of Garcia Azorero and Peral Alonso [**16**, **17**], Alves [**1**–**3**], Alves and El Hamidi [**4**], Drabek and Pohozaev [**14**], Medeiros [**12**], Guedda and Veron [**18**], Hegnell [**15**], Noussair, Wei and Jianfu [**20**] and the references therein.

In [**17**], Garcia Azorero and Peral Alonso showed that the main results proved in [**8**] also hold for a larger class of problems involving the p-Laplacian, more precisely, for problems of the kind

$$\begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{2^*-1}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

In [**2**], Alves, motivated by some results proved in [**6**], has established the existence of positive solution for the following class of problem:

$$-\Delta_p u + V(x)u^{p-1} = u^{p^*-1} \text{ in } \mathbb{R}^N. \tag{5}$$

The main results of that paper completes the study made in [**6**], in the sense that they are valid for $p \geq 2$.

In this paper, motivated by [**9**], we show the existence of positive solution for the following class of quasi-linear problems:

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = u^{p^*-1}, & \text{in } \mathbb{R}^N_+ \\ u > 0, \quad \text{in} \quad \mathbb{R}^N_+ \\ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{P}$$

where $p^* = \frac{Np}{N-p}$, $N > p \geq 2$ and $V : \mathbb{R}^N_+ \to \mathbb{R}$ is a non-negative continuous function. Our main result completes the study initialized in [**2**], once we are considering the same equation in $\mathbb{R}^N_+$, but with Neumann boundary conditions. However, since we are considering the p-Laplacian operator, it is necessary to make a careful analysis of some estimates found in [**9**], because we need to use different functions for the general case $p \geq 2$ and the calculus related to these new estimates are not immediate (see, for example, the proof of Lemma 2.2 in Section 2).

Throughout this work, $D^{1,p}(\mathbb{R}^N_+)$ denotes the closure of $C^\infty(\mathbb{R}^N_+)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N_+} |\nabla u|^p\right)^{\frac{1}{p}},$$

where we state that $\Psi \in C^\infty(\mathbb{R}^N_+)$, if there is $\widehat{\Psi} \in C_0^\infty(\mathbb{R}^N)$ such that $\widehat{\Psi}(x) = \Psi(x), \quad \forall x \in \mathbb{R}^N_+$. We denote by $J : D^{1,p}(\mathbb{R}^N_+) \to \mathbb{R}$ the functional given by

$$J(u) = \int_{\mathbb{R}^N_+} (|\nabla u|^p + V(x)\,|u|^p) \tag{6}$$

and by $\mathcal{M}$ the manifold

$$\mathcal{M} = \left\{ u \in D^{1,p}(\mathbb{R}^N_+); \int_{\mathbb{R}^N_+} |u|^{p^*} = 1 \right\}.$$

It is well known that positive critical points of J constrained on $\mathcal{M}$ are solutions of (P).

To state our main result, we need some previous definitions and notations. In relation to function V, we will assume the following hypotheses:

$$\begin{cases} \text{(i) } V(x) \geq 0, \quad \forall x \in \mathbb{R}^N_+ \\ \text{(ii) } V \in L^{N/p}(\mathbb{R}^N_+), \quad |V|_{L^{N/p}(\mathbb{R}^N_+)} \neq 0. \end{cases} \tag{7}$$

In the sequel, we denote by S and Σ, respectively, the best Sobolev constants of the embeddings

$$D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$$

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and

$$D^{1,p}(\mathbb{R}^N_+) \hookrightarrow L^{p^*}(\mathbb{R}^N_+),$$

which are given by

$$S = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^p \,;\, u \in D^{1,p}(\mathbb{R}^N),\ |u|_{L^{p^*}(\mathbb{R}^N)} = 1\right\} \tag{8}$$

and

$$\Sigma = \inf\left\{\int_{\mathbb{R}^N_+} |\nabla u|^p \,;\, u \in D^{1,p}(\mathbb{R}^N_+),\ |u|_{L^{p^*}(\mathbb{R}^N_+)} = 1\right\}. \tag{9}$$

From Talenti [**23**], the constant S is achieved by the function

$$\psi_{1,0}(x) = \frac{U(x)}{|U|_{L^{p^*}(\mathbb{R}^N)}}, \quad \text{where} \quad U(x) = \frac{1}{\left[1 + |x|^{\frac{p}{p-1}}\right]^{\frac{N-p}{p}}},$$

and all the minimizers for S are of the type

$$\psi_{\sigma,y}(x) = \sigma^{-\frac{(N-p)(p-1)}{p^2}} \psi_{1,0}\left(\frac{x-y}{\sigma^{\frac{p-1}{p}}}\right),\ \sigma > 0,\ y \in \mathbb{R}^N. \tag{10}$$

Moreover, by direct calculus, the above information yields $\Sigma = 2^{-p/N} S$, the constant Σ is achieved by the function

$$\widetilde{\psi}_{1,0}(x) = 2^{1/p^*} \psi_{1,0}(x) \ \ \forall x \in \mathbb{R}^N_+$$

and all the minimizers for Σ are of the type

$$\widetilde{\psi}_{\sigma,y}(x) = \sigma^{-\frac{(N-p)(p-1)}{p^2}} \widetilde{\psi}_{1,0}\left(\frac{x-y}{\sigma^{\frac{p-1}{p}}}\right),\ \sigma > 0,\ y \in \partial\mathbb{R}^N_+. \tag{11}$$

THEOREM 1.1. *Assume (7) and*

$$|V|_{L^{N/p}(\mathbb{R}^N_+)} < S - \Sigma. \tag{12}$$

Then, problem (P) has a positive solution $u \in D^{1,p}(\mathbb{R}^N_+)$.

To conclude this introduction, we would like to emphasize that the restriction $p \geq 2$ is assumed here because we use a Global Compactness Lemma for the p-Laplacian operator given by Alves [**2**], that was proved assuming such restriction.

2. Preliminary remarks. In this section, we will show some properties of the constant Σ and prove the Palais–Smale condition for J constrained to $\mathcal{M}$.

Our first proposition shows that it is impossible to find a solution for (P) by direct minimization of the functional J constrained on $\mathcal{M}$. The proof of this result for $p = 2$ can be found in [**9**]. For the general case $p \geq 2$ the proof is similar, since we can use the estimates proved in [**1**, **16**, **17**]. This way, this proof will be omitted.

PROPOSITION 2.1. *Assume that (7) holds and consider*

$$\Sigma^* = \inf\{J(u) : u \in \mathcal{M}\}. \tag{13}$$

Then, $\Sigma^* = \Sigma$ and the minimization problem (13) has no solution.

The next lemma is very important in our arguments, because it shows a sufficient condition which guarantees that the critical point of J on $\mathcal{M}$ does not change its sign.

LEMMA 2.1. *Let V be a function verifying (7). If u is a critical point of J on $\mathcal{M}$ satisfying $J(u) \le S$, then u does not change sign.*

Proof. The proof follows using the definition of Σ and the relation $\Sigma = 2^{-p/N} S$. □

The next proposition establishes the existence of an interval where the functional J verifies the Palais-Smale condition on $\mathcal{M}$.

PROPOSITION 2.2. *Assume that V satisfies (7) and let $\{u_n\} \subset \mathcal{M}$ be a sequence verifying*

$$J(u_n) \to c \quad \textit{and} \quad J'|_{\mathcal{M}}(u_n) \to 0,$$

with $c \in (\Sigma,\ S)$. Then $\{u_n\}$ has a subsequence strongly convergent in $D^{1,p}(\mathbb{R}^N_+)$.

Proof. If u_n^* and V^* denote the functions obtained by u_n and V extended to the whole $\mathbb{R}^N$ by reflection, we have that

$$u_n^* \in D^{1,p}(\mathbb{R}^N) \quad \forall n \in \mathbb{N}.$$

Moreover,

$$\left|\frac{u_n^*}{2^{1/p^*}}\right|_{L^{p^*}(\mathbb{R}^N)} = 1, \qquad \frac{1}{2^{p/p^*}} \int_{\mathbb{R}^N} \left(|\nabla u_n^*|^p + V^*\, |u_n^*|^p\right) \to 2^{p/N} c$$

and

$$\int_{\mathbb{R}^N} \left(|\nabla u_n^*|^{p-2}\, \nabla u_n^*\, \nabla v + V^*\, |u_n^*|^{p-2}\, u_n^*\, v\right) + (2^{p/N} + o_n(1)) \int_{\mathbb{R}^N} |u_n^*|^{p-2}\, u_n^*\, v = o_n(1),$$

for all $v \in D^{1,p}(\mathbb{R}^N)$.

Since $2^{p/N} c \in (S,\ 2^{p/N} S)$, from [**2**, Corollary 4], $\{\frac{u_n^*}{2^{1/p^*}}\}$ is relatively compact, and thus $\{u_n\}$ is also relatively compact. □

Hereafter, Π denotes the projection of $\mathbb{R}^N$ on $\partial\mathbb{R}^N_+$, i.e.

$$\Pi(x_1, x_2, \ldots, x_N) = (x_1, x_2, \ldots, x_{N-1}, 0).$$

Using the function Π, we consider the functions $\beta : D^{1,p}(\mathbb{R}^N_+) \to \partial\mathbb{R}^N_+$ and $\gamma : D^{1,p}(\mathbb{R}^N_+) \to \mathbb{R}$ given by

$$\beta(u) = \frac{\displaystyle\int_{\mathbb{R}^N_+} \frac{\Pi(x)}{1 + |\Pi(x)|} |u|^{p^*}}{|u|^{p^*}_{L^{p^*}(\mathbb{R}^N_+)}}$$

and

$$\gamma(u) = \frac{\displaystyle\int_{\mathbb{R}^N_+} \left| \frac{\Pi(x)}{1 + |\Pi(x)|} - \beta(u) \right| |u|^{p^*}}{|u|^{p^*}_{L^{p^*}(\mathbb{R}^N_+)}}.$$

For all $\rho > 0$ and $y \in \mathbb{R}^N$, let us denote by $\Lambda_\rho(y)$ the following set:

$$\Lambda_\rho(y) = \left\{ x \in \mathbb{R}^N_+ : |\Pi(x) - \Pi(y)| < \rho \right\}.$$

The next lemma is a technical result and a key point in this work, because we use it in the proof of other results that will appear later.

LEMMA 2.2. *Let $\{u_n\}$ be a sequence in $D^{1,p}(\mathbb{R}^N_+)$ verifying*

$$\{u_n\} \subset \mathcal{M}, \ \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} |\nabla u_n|^p = \Sigma, \ \beta(u_n) = 0 \quad \textit{and} \quad \gamma(u_n) = \frac{1}{3}.$$

Then, up to subsequences, there are three sequences $\{\sigma_n\} \subset \mathbb{R}_+$, $\{y_n\} \subset \partial\mathbb{R}^N_+$ and $\{w_n\} \subset D^{1,p}(\mathbb{R}^N_+)$ such that

- $u_n = \widetilde{\psi}_{\sigma_n, y_n} + w_n$,
- $\{\sigma_n\}$ *and* $\{y_n\}$ *are bounded,*

and

- $w_n \to 0$ *in* $D^{1,p}(\mathbb{R}^N_+)$.

Proof. From the result by Alves [**2**, Lemma 2], we have that

$$u_n(x) = \widetilde{\psi}_{\sigma_n, y_n}(x) + w_n(x), \quad \forall x \in \mathbb{R}^N_+,$$

where

$$\sigma_n \in \mathbb{R}^+ \backslash \{0\}, \quad y_n \in \partial\mathbb{R}^N_+$$

and w_n is a sequence that goes strongly to zero in $D^{1,p}(\mathbb{R}^N_+)$. Consequently, for all $\rho > 0$

$$\int_{\Lambda_\rho(0)} |u_n|^{p^*}\, dx = \int_{\Lambda_\rho(0)} |\widetilde{\psi}_{\sigma_n, y_n}|^{p^*} + o_n(1). \tag{14}$$

Using the last equality, we have the following claim:

Claim 2.1. If $\{\sigma_n\}$ is unbounded, for some subsequence, still denoted by $\{\sigma_n\}$, we have that

$$\lim_{n\to+\infty} \int_{\Lambda_\rho(0)} |u_n|^{p^*} = \lim_{n\to+\infty} \int_{\Omega_n} \left| \widetilde{\psi}_{1,0}\left(x - \frac{y_n}{\sigma_n^{(p-1)/p}} \right) \right|^{p^*} = 0, \tag{15}$$

where $\Omega_n = \Lambda_{\frac{\rho}{\sigma_n^{(p-1)/p}}}(0)$.

In fact, note firstly that

$$\int_{\Lambda_\rho(0)} |u_n|^{p^*} = \int_{\Omega_n} \left| \widetilde{\psi}_{1,0}\left(x - \frac{y_n}{\sigma_n^{(p-1)/p}} \right) \right|^{p^*} + o_n(1).$$

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Assuming that for some subsequence

$$\lim_{n\to+\infty} \sigma_n = +\infty,$$

and studying the cases

(i) $\left\{ \dfrac{y_n}{\sigma_n^{(p-1)/p}} \right\}$ is unbounded

(ii) $\dfrac{y_n}{\sigma_n^{(p-1)/p}} \to \overline{y}$ for some subsequences

it is easy to check that (15) holds and the proof of the claim is complete.

From Claim 2.1, we are able to prove that $\{\sigma_n\}$ is bounded. Arguing by contradiction, if $\{\sigma_n\}$ is unbounded, there is a subsequence, still denoted by $\{\sigma_n\}$, such that

$$\lim_{n\to+\infty} \sigma_n = +\infty.$$

This limit combined with Claim 2.1 yields

$$\lim_{n\to+\infty} \int_{\Lambda_\rho(0)} |u_n|^{p^*} = 0. \tag{16}$$

Since $\beta(u_n) = 0$, for all $\rho > 0$ we derive that

$$\liminf_{n\to+\infty} \gamma(u_n) \geq \frac{\rho}{1+\rho}, \quad \forall \rho > 0$$

and thus

$$\liminf_{n\to+\infty} \gamma(u_n) \geq 1. \tag{17}$$

On the other hand,

$$0 \leq \gamma(u_n) \leq \int_{\mathbb{R}^N_+} |u_n|^{p^*} = 1,$$

then

$$\limsup_{n\to+\infty} \gamma(u_n) \leq 1. \tag{18}$$

From (17) and (18)

$$\lim_{n\to+\infty} \gamma(u_n) = 1$$

obtaining, therefore, an absurd. Thus $\{\sigma_n\}$ is bounded and we can assume that

$$\lim_{n\to+\infty} \sigma_n = \overline{\sigma} \quad \text{with} \quad \overline{\sigma} \geq 0.$$

We claim that $\overline{\sigma}$ is positive. In fact, if $\overline{\sigma} = 0$, for all $\rho > 0$ we have that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N_+ \setminus \Lambda_\rho(y_n)} |u_n|^{p^*} = 0. \tag{19}$$

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From (19), there is $M > 0$ verifying

$$\frac{|y_n|}{1+|y_n|} \leq M\,\rho + o_n(1). \tag{20}$$

Hence,

$$\limsup_{n\to+\infty} \frac{|y_n|}{1+|y_n|} \leq M\rho, \ \ \forall \rho > 0$$

from where it follows

$$\lim_{n\to+\infty} |y_n| = 0.$$

On the other hand,

$$\lim_{n\to+\infty} \gamma(u_n) = \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} \left| \frac{\Pi(x)}{1+|\Pi(x)|} - \beta(u_n) \right| |u_n|^{p^*} = \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} \left| \frac{\Pi(x)}{1+|\Pi(x)|} \right| |u_n|^{p^*}$$

leading to the limit

$$\lim_{n\to+\infty} \gamma(u_n) = \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} \left| \frac{\Pi(x)}{1+|\Pi(x)|} - \frac{y_n}{1+|y_n|} \right| |u_n|^{p^*} = 0 \tag{21}$$

obtaining again an absurd.

Now, we are able to prove that $\{y_m\}$ is bounded. We again argue by contradiction, supposing that there is a subsequence, still denoted by $\{y_m\}$, verifying

$$\lim_{m\to+\infty} |y_m| = +\infty.$$

Then, fixed $\epsilon > 0$, there is $R > 0$ and $m_0 \in \mathbb{N}$ such that

$$|\Pi(x) - y_m| < R \Rightarrow \left| \frac{\Pi(x)}{1+|\Pi(x)|} - \frac{y_m}{1+|y_m|} \right| < \epsilon \quad \forall m \geq m_0 \tag{22}$$

and

$$\int_{\mathbb{R}^N_+ \setminus \Lambda_R(y_m)} |\widetilde{\psi}_{\overline{\sigma},\,y_m}|^{p^*} = \int_{\mathbb{R}^N_+ \setminus \Lambda_R(0)} |\widetilde{\psi}_{\overline{\sigma},\,0}|^{p^*} < \epsilon. \tag{23}$$

From (22) and (23),

$$\left| \beta(u_m) - \frac{y_m}{1+|y_m|} \right| \leq \epsilon + 2\epsilon + o_m(1) = 3\epsilon + o_m(1)$$

leading to the limit

$$|\beta(u_m)| \to 1, \ \text{as} \quad m \to +\infty,$$

which contradicts the fact that $\beta(u_m) = 0$. Therefore, $\{y_m\}$ is bounded. □

The next two propositions establish important properties involving the functions β, γ and the constant Σ.

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PROPOSITION 2.3. *Let $V \in L^{N/p}(\mathbb{R}^N_+)$ be a non-negative function with $|V|_{L^{N/p}(\mathbb{R}^N_+)} \neq 0$. Then,*

$$\Sigma < \inf\left\{\int_{\mathbb{R}^N_+} (|\nabla u|^p + V\,|u|^p)\colon u \in D^{1,p}(\mathbb{R}^N_+),\ |u|_{L^{p^*}(\mathbb{R}^N_+)} = 1,\ \beta(u) = 0,\ \gamma(u) = \frac{1}{3}\right\}. \tag{24}$$

Proof. From the definition of Σ,

$$\inf\left\{\int_{\mathbb{R}^N_+} (|\nabla u|^p + V\,|u|^p)\colon u \in D^{1,p}(\mathbb{R}^N_+),\ |u|_{L^{p^*}(\mathbb{R}^N_+)} = 1,\ \beta(u) = 0,\ \gamma(u) = \frac{1}{3}\right\} \geq \Sigma.$$

To prove (24), we argue by contradiction by supposing that the equality holds in the above relation. Thus, there is a sequence $\{u_n\} \subset D^{1,p}(\mathbb{R}^N_+)$ verifying

$$\begin{cases} \text{(a)} \quad |u_n|_{L^{p^*}(\mathbb{R}^N_+)} = 1,\ \beta(u_n) = 0,\ \gamma(u_n) = \frac{1}{3} \\ \text{(b)} \quad \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} (|\nabla u_n|^p + V\,|u_n|^p) = \Sigma. \end{cases} \tag{25}$$

Since $V(x) \geq 0 \quad \forall x \in \mathbb{R}^N_+$,

$$\Sigma = \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} (|\nabla u_n|^p + V\,|u_n|^p) \geq \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} |\nabla u_n|^p \geq \Sigma$$

from where it follows

$$\Sigma = \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} |\nabla u_n|^p.$$

Using the uniqueness of the family of functions $\widetilde{\psi}_{\sigma,\,y}$ and a result by Alves [**2**, Lemma 2], we deduce that

$$u_n(x) = \widetilde{\psi}_{\sigma_n,\,y_n}(x) + w_n(x), \quad \forall x \in \mathbb{R}^N_+$$

where

$$\sigma_n \in \mathbb{R}^+\backslash\{0\}, \quad y_n \in \partial\mathbb{R}^N_+$$

and w_n is a sequence that goes strongly to zero in $D^{1,p}(\mathbb{R}^N_+)$.

From Lemma 2.2, without loss of generality, we can assume that

$$\lim_{n\to+\infty} \sigma_n = \overline{\sigma} > 0, \quad \lim_{n\to+\infty} y_n = \overline{y} \in \partial\mathbb{R}^N_+$$

and

$$\widetilde{\psi}_{\sigma_n,\,y_n} \to \widetilde{\psi}_{\overline{\sigma},\,\overline{y}} \quad \text{in} \quad D^{1,p}(\mathbb{R}^N_+) \text{ and } \quad L^{p^*}(\mathbb{R}^N_+). \tag{26}$$

From (25b) and (26), it follows that

$$\int_{\mathbb{R}^N_+} V\,|\widetilde{\psi}_{\overline{\sigma},\,\overline{y}}|^p = 0,$$

which is an absurd, because $\widetilde{\psi}_{\overline{\sigma},\overline{y}}$ is positive. Thus, inequality (24) is proved. $\square$

3. Technical results. Hereafter, we assume that V verifies (7) and (12). Moreover, let us denote by C_V the following real number:

$$C_V = \inf\left\{\int_{\mathbb{R}^N_+}(|\nabla u|^p + V|u|^p);\ u \in \mathcal{M},\ \beta(u) = 0,\ \gamma(u) = \frac{1}{3}\right\}.$$

From (7) and Proposition 2.3, we derive that

$$C_V > \Sigma.$$

Using the numbers C_V and Σ, we consider a new number $\overline{C}$ given by

$$\overline{C} = \frac{C_V + \Sigma}{2}$$

and remark that the following inequality holds:

$$\Sigma < \overline{C} < C_V.$$

In the sequel, we denote by φ a function that belongs to $W_0^{1,p}(B_1(0))$ and has the following properties:

$$\begin{cases} \text{(i) } \varphi \in C_0^\infty(B_1(0)),\ \varphi(x) > 0\ \ \forall\, x \in B_1(0) \\ \text{(ii) } \varphi \text{ is radially symmetric and } |x_1| < |x_2| \Rightarrow \varphi(x_1) > \varphi(x_2) \\ \text{(iii) } |\varphi|_{L^{p^*}(\mathbb{R}^N_+\cap B_1(0))} = 1 \\ \text{(iv) } \Sigma < \int_{\mathbb{R}^N_+\cap B_1(0)} |\nabla\varphi|^p dx \equiv \overline{\Sigma} < \min\left\{\overline{C},\ S - |V|_{L^{N/p}(\mathbb{R}^N_+)}\right\}. \end{cases} \tag{27}$$

For every $\sigma > 0$ and $y \in \mathbb{R}^N$, we set

$$\varphi_{\sigma,y}(x) = \begin{cases} \sigma^{-(N-p)/p^2}\varphi\left(\frac{x-y}{\eta}\right), & x \in B_\eta(y) \\ 0, & x \notin B_\eta(y) \end{cases}$$

where $\eta = \sqrt[p]{\sigma}$. From the definition of $\varphi_{\sigma,y}$, it follows that

$$|\varphi_{\sigma,y}|_{L^{p^*}(\mathbb{R}^N_+)} = |\varphi_{\sigma,y}|_{L^{p^*}(B_\eta(y))} = |\varphi|_{L^{p^*}(B_1(0))}$$

and

$$|\nabla\varphi_{\sigma,y}|_{L^p(\mathbb{R}^N_+)} = |\nabla\varphi_{\sigma,y}|_{L^p(B_\eta(y))} = |\nabla\varphi|_{L^p(B_1(0))}.$$

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LEMMA 3.1. *Let $V \in L^{N/p}(\mathbb{R}^N_+)$ be a non-negative function. Then,*

$$\begin{cases} (a)\ \lim\limits_{\sigma\to 0} \sup\left\{\displaystyle\int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} :\ y \in \partial\mathbb{R}^N_+\right\} = 0 \\ (b)\ \lim\limits_{\sigma\to +\infty} \sup\left\{\displaystyle\int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} :\ y \in \partial\mathbb{R}^N_+\right\} = 0 \\ (c)\ \lim\limits_{r\to +\infty} \sup\left\{\displaystyle\int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} :\ |y| = r,\ \sigma > 0,\ y \in \partial\mathbb{R}^N_+\right\} = 0 \end{cases} \tag{28}$$

Proof. For $y \in \partial\mathbb{R}^N_+$ and $\sigma > 0$,

$$\int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} = \int_{\mathbb{R}^N_+\cap B_\eta(y)} V\varphi^p_{\sigma,y} \le |V|_{L^{N/p}(\mathbb{R}^N_+\cap B_\eta(y))}|\varphi_{\sigma,y}|_{L^{p^*}(\mathbb{R}^N_+\cap B_\eta(y))},$$

and thus

$$\int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} \le |V|_{L^{N/p}(\mathbb{R}^N_+\cap B_\eta(y))}\,.$$

The last inequality implies

$$\sup\left\{\int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} : y \in \partial\mathbb{R}^N_+\right\} \le \sup\left\{|V|_{L^{N/p}(\mathbb{R}^N_+\cap B_\eta(y))} : y \in \partial\mathbb{R}^N_+\right\}.$$

Since, for each $\epsilon > 0$, there is $\sigma_0 > 0$ such that

$$|V|_{L^{N/p}(\mathbb{R}^N_+\cap B_\eta(y))} < \epsilon \ \ \forall\, \sigma \in (0, \sigma_0)\ \text{ e }\ \forall\, y \in \partial\mathbb{R}^N_+$$

we can conclude that (28a) holds.

To prove (28b), note that for all $\rho, \sigma > 0$ and $y \in \partial\mathbb{R}^N_+$ we get

$$\int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} = \int_{\mathbb{R}^N_+\cap B_\rho(0)} V\varphi^p_{\sigma,y} + \int_{\mathbb{R}^N_+\setminus B_\rho(0)} V\varphi^p_{\sigma,y}$$

and consequently

$$\int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} \le C_1|\varphi_{\sigma,y}|^p_{L^{p^*}(B_\rho(0))} + |V|_{L^{N/p}(\mathbb{R}^N_+\setminus B_\rho(0))}.$$

Now, for each $\epsilon > 0$, there is σ_0, $\rho_0 > 0$ verifying

$$|\varphi_{\sigma,y}|^p_{L^{p^*}(B_\rho(0))} < \frac{\epsilon}{2C_1} \quad \forall\, \sigma \in (\sigma_0, +\infty),\ \ \forall\, y \in \partial\mathbb{R}^N_+$$

and

$$|V|_{L^{N/p}(\mathbb{R}^N_+\setminus B_\rho(0))} < \frac{\epsilon}{2} \quad \forall\, \rho \ge \rho_0.$$

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Therefore,

$$\int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} < \epsilon \quad \forall\, \sigma \in (\sigma_0, +\infty), \quad \forall\, y \in \partial\mathbb{R}^N_+$$

which implies

$$\sup_{y\in\partial\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} V\varphi^p_{\sigma,y} < \epsilon \quad \forall\, \sigma \in (\sigma_0, +\infty)$$

and the proof of (28b) is complete.

To prove (28c), we will assume by contradiction that there are sequences $\{y_n\} \subset \partial\mathbb{R}^N_+$ and $\{\sigma_n\} \subset (0, +\infty)$ verifying

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N_+} V\varphi^p_{\sigma_n,y_n} = L > 0 \quad \text{and} \quad |y_n| \to +\infty. \tag{29}$$

From (28a, b), we can suppose

$$\lim_{n\to+\infty} \sigma_n = \overline{\sigma} > 0.$$

Using the hypotheses

$$|y_n| \to +\infty \;\text{ and }\; V \in L^{N/p}(\mathbb{R}^N_+)$$

we get

$$\lim_{n\to+\infty} |V|_{L^{N/p}(\mathbb{R}^N_+\cap B_{\eta_n}(y_n))} = 0, \qquad \eta_n = \sqrt[p]{\sigma_n}.$$

The last limit leads to

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N_+} V\varphi^p_{\sigma_n,y_n} \le \lim_{n\to+\infty} |V|_{L^{N/p}(\mathbb{R}^N_+\cap B_{\sigma_n^{1/p}}(y_n))} = 0,$$

which contradicts (29). □

LEMMA 3.2. *The following relations hold:*

$$\begin{cases} (a) \;\; \lim_{\sigma\to 0} \sup\left\{\gamma(\varphi_{\sigma,y}) \colon y \in \partial\mathbb{R}^N_+\right\} = 0 \\ (b) \;\; \lim_{\sigma\to+\infty} \inf\left\{\gamma(\varphi_{\sigma,y})\colon \; y \in \partial\mathbb{R}^N_+, \; |y| \le r\right\} = 1, \;\forall\, r > 0 \\ (c) \;\; \left(\beta(\varphi_{\sigma,y}) \mid y\right)_{\mathbb{R}^N} > 0\colon \;\forall\, y \in \partial\mathbb{R}^N_+\backslash\{0\}, \;\forall\, \sigma > 0 \end{cases} \tag{30}$$

Proof. Let $y \in \partial\mathbb{R}^N_+$ be chosen arbitrarily. Repeating the same arguments as explored by Cerami and Passaseo [**9**], there is $M > 0$ such that

$$0 \le \gamma(\varphi_{\sigma,y}) \le 2M\sqrt[p]{\sigma} \;\; \forall\, y \in \partial\mathbb{R}^N_+ \;\text{ and }\; \forall\, \sigma > 0,$$

and thus

$$0 \le \sup\left\{\gamma(\varphi_{\sigma,y})\colon y \in \partial\mathbb{R}^N_+\right\} \le 2M\sqrt[p]{\sigma} \quad \forall\, \sigma > 0$$

proving (30a).

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To prove (30b), we begin showing that for each $y \in \partial\mathbb{R}^N_+$

$$\lim_{\sigma\to+\infty} |\beta(\varphi_{\sigma,y})| = 0. \tag{31}$$

Since $\beta(\varphi_{\sigma,0}) = 0$,

$$|\beta(\varphi_{\sigma,y})| = |\beta(\varphi_{\sigma,y}) - \beta(\varphi_{\sigma,0})| \leq \int_{\mathbb{R}^N_+} \frac{|\Pi(x)|}{1+|\Pi(x)|} |\varphi^{p^*}_{\sigma,y} - \varphi^{p^*}_{\sigma,0}|$$

then

$$|\beta(\varphi_{\sigma,y})| \leq \int_{\mathbb{R}^N_+} |\varphi^{p^*}_{1,\frac{y}{\sigma}} - \varphi^{p^*}_{1,0}| \overset{\sigma\to\infty}{\longrightarrow} 0.$$

Now, fix $r > 0$ arbitrarily and let $y \in \partial\mathbb{R}^N_+$ with $|y| \leq r$. For any $\sigma > 0$, we get the inequality

$$\gamma(\varphi_{\sigma,y}) \leq 1 + |\beta(\varphi_{\sigma,y})|,$$

which together with (31) implies

$$\limsup_{\sigma\to+\infty} \left[\inf\left\{\gamma(\varphi_{\sigma,y}) \colon\ y \in \partial\mathbb{R}^N_+,\ |y| \leq r\right\}\right] \leq 1. \tag{32}$$

If

$$\limsup_{\sigma\to+\infty} \left[\inf\left\{\gamma(\varphi_{\sigma,y}) \colon\ y \in \partial\mathbb{R}^N_+,\ |y| \leq r\right\}\right] < 1,$$

there is $\{\sigma_n\} \subset (0, +\infty)$ and $\{y_n\} \subset \partial\mathbb{R}^N_+$ satisfying

$$\sigma_n \to +\infty\ , \quad y_n \to y \in \overline{B}_r(0)$$

and

$$\lim_{n\to+\infty} \gamma(\varphi_{\sigma_n,y_n}) = A < 1. \tag{33}$$

From (31),

$$\lim_{n\to+\infty} \gamma(\varphi_{\sigma_n,y_n}) \geq \frac{\rho}{1+\rho} \quad \forall\, \rho > 0.$$

From this, since $\rho > 0$ is arbitrary, we have that

$$\lim_{n\to+\infty} \gamma(\varphi_{\sigma_n,y_n}) \geq 1$$

obtaining a contradiction with (33). Thus, the equality in (32) holds and the proof of (30b) is finished.

Now, we will prove (30c). If $0 \notin B_{\sqrt[p]{\sigma}}(y)$, we have

$$(\Pi(x) \mid y) > 0$$

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and thus

$$(\beta(\varphi_{\sigma,y}) \mid y) > 0.$$

if $0 \in B_{\sqrt[p]{\sigma}}(y)$, for each $x \in B_\sigma(y) \cap \mathbb{R}^N_+$ verifying

$$(\Pi(x) \mid y) < 0,$$

the point $\overline{x}$, symmetrical to $-x$ with respect to $\partial\mathbb{R}^N_+$, belongs to $B_\sigma(y) \cap \mathbb{R}^N_+$ and

$$(\Pi(\overline{x}) \mid y) > 0$$

which leads to

$$\big(\beta(\varphi_{\sigma,y}) \mid y\big) = \int_{\mathbb{R}^N_+ \cap B_{\sqrt[p]{\sigma}}(y)} \frac{(\Pi(x)|y)}{1 + |\Pi(x)|} |\varphi_{\sigma,y}|^{p^*} > 0.$$

□

COROLLARY 3.1. *Let V satisfy (7), (12) and $\epsilon > 0$ verify*

$$\overline{\Sigma} + \epsilon < \min\left\{\overline{C},\ S - |V|_{L^{N/p}(\mathbb{R}^N_+)}\right\}.$$

Then, there are $r, \sigma_1, \sigma_2 > 0$ with

$$0 < \sigma_1 < \frac{1}{3} < \sigma_2$$

such that

$$\gamma(\varphi_{\sigma_1,y}) < \frac{1}{3}\,, \quad \gamma(\varphi_{\sigma_2,y}) > \frac{1}{3} \ \ \forall\, y \in \partial\mathbb{R}^N_+ \tag{34}$$

and

$$\sup\left\{\int_{\mathbb{R}^N_+} \big[|\nabla\varphi_{\sigma,y}|^p + V|\varphi_{\sigma,y}|^p\big]; (y,\sigma) \in \partial K\right\} < \overline{\Sigma} + \epsilon/2 \tag{35}$$

where

$$K = \left\{(y,\sigma) \in \partial\mathbb{R}^N_+ \times \mathbb{R}_+ :\ |y| \le r, \sigma \in [\sigma_1, \sigma_2]\right\}. \tag{36}$$

Proof. The proof follows by using the same type of arguments as found in **[9]**. □

COROLLARY 3.2. *Assume that V satisfies (7)–(12) and let ϵ, σ_1, σ_2 and r be the numbers given in Corollary 3.1 and K defined in (36). Then,*

$$\sup\left\{\int_{\mathbb{R}^N_+} \big[|\nabla\varphi_{\sigma,y}|^p + V|\varphi_{\sigma,y}|^p\big]; (y,\sigma) \in \partial K\right\} < S. \tag{37}$$

Proof. For all $y \in \partial\mathbb{R}^N_+$ and $\sigma > 0$

$$\int_{\mathbb{R}^N_+} V|\varphi_{\sigma,y}|^p \leq |V|_{L^{N/p}(\mathbb{R}^N_+)}|\varphi_{\sigma,y}|^p_{L^{N/p}(\mathbb{R}^N_+)} \leq |V|_{L^{N/p}(\mathbb{R}^N)}.$$

Using the last inequality and (27), it follows that

$$\int_{\mathbb{R}^N_+} \left[|\nabla\varphi_{\sigma,y}|^p + V|\varphi_{\sigma,y}|^p\right] < \overline{\Sigma} + |V|_{L^{N/p}(\mathbb{R}^N)} < S \quad \forall (y,\sigma) \in \partial K,$$

from where it follows the lemma. □

LEMMA 3.3. *Let K be the set defined in (36) with σ_1, σ_2 and r chosen as in Corollary 3.1. Then, there is $(\widehat{y}, \widehat{\sigma}) \in \overset{\circ}{K}$ satisfying*

$$\beta(\varphi_{\widehat{\sigma},\widehat{y}}) = 0 \quad \textit{and} \quad \gamma(\varphi_{\widehat{\sigma},\widehat{y}}) = \frac{1}{3}.$$

Proof. The proof of this proposition follows by adapting arguments found in [**9**]. □

Proof of Theorem 1.1. Hereafter, let us denote by b the following real number

$$b = \sup\{J(\varphi_{\sigma,y})\colon\ (y,\sigma) \in K\}$$

and fix $\epsilon > 0$ verifying

$$\overline{\Sigma} + \epsilon < \min\left\{\overline{C},\ S - |V|_{L^{N/p}(\mathbb{R}^N_+)}\right\}.$$

Using (37) and Lemma 3.3, we get

$$\Sigma < \overline{C} < C_V \leq J(\varphi_{\widehat{\sigma},\widehat{y}}) \leq b < S.$$

We will prove that functional J constrained to $\mathcal{M}$ has a critical level in the interval $(\overline{C}, S)$. In order to achieve this goal, we fix $\delta > 0$ satisfying

$$\overline{C} < C_V - \delta < b + \delta < S \tag{38}$$

and suppose that

$$\left\{u \in \mathcal{M}\colon\ C_V - \delta \leq J(u) \leq b + \delta;\ J'\big|_V(u) = 0\right\} = \emptyset.$$

From (38) and Proposition 2.2, the pair $(J, \mathcal{M})$ satisfies the Palais–Smale condition in $(C_V - \delta, b + \delta)$. Therefore, it is possible to find a continuous map $\eta : [0, 1] \times \mathcal{M} \to \mathcal{M}$ and a positive number $\epsilon_1 < \delta$ verifying

- $\eta(0, u) = u,\ \forall\, u \in \mathcal{M}$
- $\eta(t, u) = u,\ \forall\, u \in J^{C_V-\epsilon_1} \cup (\mathcal{M} \setminus J^{b+\epsilon_1}),\ \forall\, t \in [0, 1]$
- $(J \circ \eta)(t, u) \leq J(u) \quad \forall t \in [0, 1]$
- $\eta(1, J^{b+\epsilon_1}) \subset J^{C_V-\epsilon_1}$.

From the above information,

$$(y,\sigma) \in K \Rightarrow J(\varphi_{\sigma,y}) < b \Rightarrow J(\eta(1, \varphi_{\sigma,y})) < C_V - \epsilon_1. \tag{39}$$

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Repeating the same arguments as found in [**9**], we will find $(\sigma^*, y^*) \in K$ verifying

$$J(\eta(1, \varphi_{\sigma^*,y^*})) \geq C_V > C_V - \epsilon_1$$

which contradicts (39). Therefore, the functional J constrained on $\mathcal{M}$ has at least one critical point $u \in \mathcal{M}$ with $\overline{C} < J(u) < S$. □

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