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COARSE AMENABILITY AND DISCRETENESS

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Abstract

This paper is devoted to dualization of paracompactness to the coarse category via the concept of R-disjointness. Property A of Yu can be seen as a coarse variant of amenability via partitions of unity and leads to a dualization of paracompactness via partitions of unity. On the other hand, finite decomposition complexity of Guentner, Tessera, and Yu and straight finite decomposition complexity of Dranishnikov and Zarichnyi employ R-disjointness as the main concept. We generalize both concepts to that of countable asymptotic dimension and our main result shows that it is a subclass of spaces with Property A. In addition, it gives a necessary and sufficient condition for spaces of countable asymptotic dimension to be of finite asymptotic dimension.

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1. Introduction

Property A of Yu [19] was introduced in the context of the Novikov conjecture. It is a large-scale variant of amenability. See [18] for a survey of results on Property A. Subsequently, it was generalized to the concept of *exact spaces* by Dadarlat and Guentner [4]. In [1], exact spaces were narrowed down to *large-scale paracompact spaces* and [3] (see also [2]) contains an analysis of interrelationships between various concepts.

As explained in [3], all the above concepts can be unified using existence (for each $\epsilon > 0$) of (ϵ, ϵ)-Lipschitz (see Definition 2.8) partitions of unity $f : X \to \Delta(S)$ (see Definition 2.6) that are cobounded (see Definition 2.7). Property A corresponds to f being a barycentric partition of unity (see Definition 2.6), exact spaces correspond to arbitrary partitions of unity, and large-scale paracompact spaces correspond to the case of f having Lebesgue number at least $1/\epsilon$ (see Definition 2.6).

One may summarize that the three concepts (Property A, exact spaces, and large-scale paracompact spaces) deal with dualizing paracompactness via partitions of unity.

In [3], the concept of *Strong Property A* was introduced as a way of dualizing paracompactness via covers.

This paper is devoted to developing large-scale paracompactness from the point of view of discreteness. More precisely, it deals with dualizing the following two classical results of general topology.

THEOREM 1.1 [11]. *A Hausdorff space X is paracompact if and only if it is weakly paracompact and collectionwise normal.*

THEOREM 1.2 (See [11, Theorem 5.1.12, page 303]). *A regular space X is paracompact if and only if every open cover has a σ-discrete open refinement.*

Since topological discreetness naturally dualizes to R-disjointness (see Definition 2.2), one arrives at the following question.

PROBLEM 1.3. Characterize metric spaces X such that for each $R > 0$ there exist $M > 0$ and a finite sequence of R-disjoint families $\mathcal{U}_n$, $i \le k$, such that $X = \bigcup_{i=1}^{k} \mathcal{U}_n$ and diameters of elements of each $\mathcal{U}_n$ are at most M.

It turns out that special cases of Problem 1.3 were considered in the past. The most restrictive property expressed in terms of R-disjointness is the following definition.

DEFINITION 1.4 [5]. A metric space X has *asymptotic property C* if for every sequence $R_1 < R_2 < \cdots$ there exists $n \in \mathbb{N}$ such that X is the union of R_i-disjoint families $\mathcal{U}_i$, $1 \le i \le n$, that are uniformly bounded.

Subsequently, Guentner *et al.* introduced the concept of *finite decomposition complexity* (see [13]), which was weakened as follows.

DEFINITION 1.5 [7]. The metric space X is of *straight finite decomposition complexity* if for any increasing sequence of positive real numbers $R_1 < R_2 < \cdots$ there is a sequence $\mathcal{V}_i$, $i \le n$, of families of subsets of X such that the following conditions are satisfied:

(1) $\mathcal{V}_1 = \{X\}$;
(2) each element $U \in \mathcal{V}_i$, $i < n$, can be expressed as a union of at most two families from $\mathcal{V}_{i+1}$ that are R_i-disjoint;
(3) $\mathcal{V}_n$ is uniformly bounded.

It turns out that straight finite decomposition complexity is a variant of coarse amenability.

THEOREM 1.6 [7]. *Every space of straight finite decomposition complexity has Property A.*

Our view is that straight finite decomposition complexity is a special case of countable asymptotic dimension (see Definition 7.1). Namely, it corresponds to the fact that, in topology, one can define spaces of countable covering dimension as either countable unions of zero-dimensional spaces or as countable unions of spaces of finite

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dimension. Our main result, Theorem 7.6, states that spaces X of countable asymptotic dimension are actually of finite asymptotic dimension provided some finite skeleton of $\Delta(X)$ is a large-scale absolute extensor of X. It generalizes Theorem 1.6 as well.

2. Basic concepts

In this section we recall basic concepts used in the paper.

DEFINITION 2.1. The *cardinality* of a set S is denoted by $\mathrm{card}(S)$.

DEFINITION 2.2. Given $R > 0$, a family $\{U_s\}_{s\in S}$ of subsets of a metric space X is called *R-disjoint* if $d(x, y) > R$ whenever $x \in U_s$, $y \in U_t$, and $s \neq t$.

DEFINITION 2.3. A family $\{U_s\}_{s\in S}$ of subsets of a metric space X is called *uniformly bounded* if there is $M > 0$ such that diameters of all sets of the family are at most M.

DEFINITION 2.4. The *Lebesgue number* of a family $\{U_s\}_{s\in S}$ of subsets of a metric space X is at least $M > 0$ if the family of M-balls $\{B(x, M)\}_{x\in X}$ refines $\{U_s\}_{s\in S}$.

DEFINITION 2.5. By $\Delta(S)$, we mean the subspace of $l_1(S)$ (S is the set of vertices of the simplicial complex $\Delta(S)$) consisting of nonnegative functions $f : S \to [0, 1]$ of finite support such that $\sum_{v\in S} f(v) = 1$. The *star* $st(v)$ of vertex v consists of all $f \in \Delta(S)$ such that $f(v) > 0$.

By $\Delta(S)^{(n)}$, we mean the *n-skeleton* of $\Delta(S)$.

DEFINITION 2.6. A (point-finite) *partition of unity* on a set X is a function $f : X \to \Delta(S)$ for some S. The function f is a *barycentric partition of unity* if $f(x)(v) = f(x)(w)$ whenever $f(x)(v) > 0$ and $f(x)(w) > 0$.

The *Lebesgue number* of f is synonymous with the Lebesgue number of $\{f^{-1}(st(v))\}_{v\in S}$.

DEFINITION 2.7. Suppose that X is a metric space. A partition of unity $f : X \to \Delta(S)$ is *M-cobounded* if $\mathrm{diam}(f^{-1}(st(v))) \leq M$ for all $v \in S$.

A function f is *cobounded* if it is M-cobounded for some $M > 0$.

DEFINITION 2.8. A function $f : X \to Y$ is (λ, C)-Lipschitz if $d_Y(f(x), f(y)) \leq \lambda \cdot d_X(x, y) + C$ for all $x, y \in X$.

LEMMA 2.9. *Suppose that $f : X \to \Delta(S)$ is a partition of unity and $\epsilon \geq 2/(R + 1)$ for some $R > 0$. If $d(x, y) < R$ implies $d(f(x), f(y)) \leq \epsilon \cdot d(x, y) + \epsilon$, then f is (ϵ, ϵ)-Lipschitz.*

PROOF. If $d(x, y) \geq R$, then $\epsilon \cdot d(x, y) + \epsilon \geq \epsilon \cdot (R + 1) \geq 2 \geq d(f(x), f(y))$. □

For basic facts related to the coarse category, see [17].

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3. Large-scale weak paracompactness

A dualization of weak paracompactness was developed in [3] via coarsening of covers. Using R-disjointness, one is led to a different concept and we do not know if it is equivalent to large-scale weak paracompactness (see Problems 1.3 and 3.5).

DEFINITION 3.1 [2, 3]. A metric space X is *large-scale weakly paracompact* if for each $r, s > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X of Lebesgue number at least s such that every r-ball $B(x, r)$ is contained in only finitely many elements of $\mathcal{U}$.

PROPOSITION 3.2 [3]. *The following conditions are equivalent for each metric space X:*

(a) *for each $r > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X such that every r-ball $B(x, r)$ intersects only finitely many elements of $\mathcal{U}$;*
(b) *X is large-scale weakly paracompact;*
(c) *for every uniformly bounded cover $\mathcal{U}$ of X there exists a uniformly bounded point-finite cover $\mathcal{V}$ such that $\mathcal{U}$ is a refinement of $\mathcal{V}$.*

The following proposition is a partial answer to Problem 1.3.

PROPOSITION 3.3. *If for every $r > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X that can be written as the union $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ of r-disjoint families $\mathcal{U}_i$, then X is large-scale weakly paracompact.*

PROOF. Suppose that $s > 0$. Pick a uniformly bounded cover $\mathcal{U}$ of X that can be written as the union $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ of $2s$-disjoint families $\mathcal{U}_i$.

Given $U \in \mathcal{U}_k$, define $U' = U \setminus \bigcup_{i<k} \{B(V, s) | V \in \mathcal{U}_i\}$ and $U^* = B(U', s)$. Since $\{U'\}_{U \in \mathcal{U}}$ is a uniformly bounded cover of X, $\{U^*\}_{U \in \mathcal{U}}$ is of Lebesgue number at least s and is uniformly bounded. Given $x \in X$, choose $m \geq 1$ so that $x \in U$ for some $U \in \mathcal{U}_m$. Therefore, $B(x, s) \cap V' = \emptyset$ and $x \notin V^*$ for all $V \in \mathcal{U}_i$, $i > m$. If we fix $k \leq m$, then there is at most one $V \in \mathcal{U}_k$ such that $x \in V^*$. Thus, $\{U^*\}_{U \in \mathcal{U}}$ is a point-finite cover of X. By Proposition 3.2(c), X is large-scale weakly paracompact. □

COROLLARY 3.4 [3]. *If X is separable at some scale $r > 0$ (that means that there is a countable subset S of X with $\bigcup_{x \in S} B(x, r) = X$), then X is large-scale weakly paracompact.*

PROOF. The family $\{B(x, r)\}_{x \in S}$ is uniformly bounded and is the union of countably many ∞-disjoint families. □

PROBLEM 3.5. Suppose that X is large-scale weakly paracompact and $r > 0$. Is there a uniformly bounded cover $\mathcal{U}$ of X that can be written as the union $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ of r-disjoint families $\mathcal{U}_i$?

DEFINITION 3.6 [2]. A metric space X is *large-scale finitistic* if for every $r > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X whose Lebesgue number is at least r and there is $n(\mathcal{U}) \in \mathbb{N}$ such that each $x \in X$ belongs to at most $n(\mathcal{U})$ elements of $\mathcal{U}$.

PROBLEM 3.7. Suppose that X is large-scale finitistic and $r > 0$. Is there a uniformly bounded cover $\mathcal{U}$ of X that can be written as the union $\bigcup_{i=1}^{m} \mathcal{U}_i$ of finitely many r-disjoint families $\mathcal{U}_i$?

4. Pasting partitions of unity

This section contains the main technical tool of the paper: pasting partitions of unity so that the resulting partition of unity is (ϵ, ϵ)-Lipschitz and K-cobounded. Given a partition of unity $f : A \to \Delta(S)$, by the *carrier* of f we mean the minimal subcomplex of $\Delta(S)$ containing $f(A)$.

LEMMA 4.1. *Suppose that the following are given:*

(a) *A is a subset of a metric space X;*
(b) *$f : A \to \Delta(S)$ is a (δ, δ)-Lipschitz partition of unity on A for some $\delta > 0$;*
(c) *$g : X \to \Delta(S)$ is a (δ, δ)-Lipschitz partition of unity on X;*
(d) *$p : X \to A$ is a retraction such that $d(x, p(x)) < \mathrm{dist}(x, A) + 1$ for all $x \in A$;*
(e) *$\alpha : X \to [0, 1]$ is $1/r$-Lipschitz, $\alpha(A) \subset \{0\}$, and $\alpha(X \setminus B(A, r)) \subset \{1\}$;*
(f) *$h : X \to \Delta(S)$ is defined as $h(x) = \alpha(x) \cdot g(x) + (1 - \alpha(x)) \cdot f(p(x))$.*

In order for h to be (ϵ, ϵ)-Lipschitz, it suffices that $r \geq 4/\epsilon$, $\delta \leq \epsilon/3 - 2/3r$, and $\delta \leq \epsilon/(4r + 7)$.

If, in addition, the carriers of $f(A)$ and $g(X)$ are disjoint and both f and g are M-cobounded, then h is $(M + 2r + 2)$-cobounded.

PROOF. Notice that h is an extension of f.

We need to show that $|h(x) - h(y)| \leq \epsilon \cdot d(x, y) + \epsilon$ for $x, y \in X$. Notice that $h(x) - h(y) = \alpha(x) \cdot g(x) + (1 - \alpha(x)) \cdot f(p(x)) - [\alpha(y) \cdot g(y) + (1 - \alpha(x)) \cdot f(p(y))] = (\alpha(x) - \alpha(y)) \cdot g(x) + \alpha(y) \cdot (g(x) - g(y)) + [f(p(x)) - f(p(y))] - [\alpha(x) \cdot f(p(x)) - \alpha(y) \cdot f(p(y))]$.

The terms $(\alpha(x) - \alpha(y)) \cdot g(x)$ and $\alpha(y) \cdot (g(x) - g(y))$ have universal estimates $|(\alpha(x) - \alpha(y)) \cdot g(x)| \leq |\alpha(x) - \alpha(y)| \leq 1/r \cdot d(x, y)$ and $|\alpha(y) \cdot (g(x) - g(y))| \leq |g(x) - g(y)| \leq \delta \cdot d(x, y) + \delta$, so we need to estimate the remaining terms depending on where x and y belong.

Case 1. $x \notin B(A, r)$ and $y \in B(A, r)$.

Here $\alpha(x) = 1$, so $[f(p(x)) - f(p(y))] - [\alpha(x) \cdot f(p(x)) - \alpha(y) \cdot f(p(y))] = (\alpha(y) - \alpha(x)) \cdot f(p(y))$ and this term is at most $1/r \cdot d(x, y)$. Thus, in that case, we have $|h(x) - h(y)| \leq ((2/r) + \delta) \cdot d(x, y) + \delta \leq \epsilon \cdot d(x, y) + \epsilon$.

Case 2. $x \in B(A, r)$ and $y \in B(A, r)$.

We know that $|f(p(x)) - f(p(y))| \leq \delta \cdot d(p(x), p(y)) + \delta$. Notice that $d(p(x), p(y)) \leq d(p(x), x) + d(x, y) + d(y, p(y)) \leq \mathrm{dist}(x, A) + 1 + d(x, y) + d(y, A) + 1 \leq 2r + 2 + d(x, y)$. Also, $\alpha(x) \cdot f(p(x)) - \alpha(y) \cdot f(p(y)) = \alpha(x) \cdot (f(p(x)) - f(p(y))) + (\alpha(x) - \alpha(y)) \cdot f(p(y))$, resulting in $|\alpha(x) \cdot f(p(x)) - \alpha(y) \cdot f(p(y))| \leq |f(p(x)) - f(p(y))| + |\alpha(x) - \alpha(y)| \leq \delta \cdot (2r + 2 + d(x, y)) + \delta + (1/r) \cdot d(x, y)$.

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The final outcome is

$$
\begin{aligned}
|h(x)-h(y)| \le {} & \frac{1}{r}\cdot d(x,y) + \delta\cdot d(x,y) \\
& + \delta + \delta\cdot(2r+2+d(x,y)) + \delta + \delta\cdot(2r+2+d(x,y)) + \delta + \frac{1}{r}\cdot d(x,y) \\
= {} & \left(\frac{2}{r} + 3\delta\right)\cdot d(x,y) + 4r\delta + 7\delta.
\end{aligned}
$$

To achieve $|h(x)-h(y)| \le \epsilon\cdot d(x,y) + \epsilon$, it suffices that $(2/r) + 3\delta \le \epsilon$ and $4r\delta + 7\delta \le \epsilon$. That amounts to $\delta \le (\epsilon/3) - (2/3r)$ and $\delta \le \epsilon/(4r+7)$.

Case 3. $x \notin B(A,r)$ and $y \notin B(A,r)$.

In that case $h(x) = g(x)$ and $h(y) = g(y)$, so $|h(x)-h(y)| \le \delta\cdot d(x,y) + \delta \le \epsilon\cdot d(x,y) + \epsilon$.

Suppose that the carriers of $f(A)$ and $g(X)$ are disjoint and there is $M > 0$ such that $\mathrm{diam}(f^{-1}(st(v))), \mathrm{diam}(g^{-1}(st(v))) \le M$ for all $v \in S$.

If $v \in S$ belongs to the carrier of $g(X)$ and $h(x)(v) > 0$, then x must belong to $g^{-1}(st(v))$. Thus, $\mathrm{diam}(h^{-1}(st(v))) \le M$ in that case. If $v \in S$ belongs to the carrier of $f(A)$ and $h(x)(v) > 0$, then $x \in B(A,r)$ and $p(x) \in f^{-1}(st(v))$. Since $d(x,p(x)) \le r+1$, $\mathrm{dist}(x, f^{-1}(st(v)) \le r+1$ and $\mathrm{diam}(h^{-1}(st(v))) \le M + 2r + 2$. □

5. Coarse normality

In this section we dualize one part of Theorem 1.1.

It is shown in [8, Theorem 9.1(5)] that a topological space X is collectionwise normal if and only if partitions of unity on each closed subset A of X extend over X. In other words, certain spaces are absolute extensors of X. The work of Dydak and Mitra [10] is devoted to dualizing the concept of absolute extensors to the coarse category.

The following result may be seen as stating that every metric space X is large-scale collectionwise normal.

THEOREM 5.1. *For every $\epsilon > 0$ there is $\delta > 0$ such that any (δ,δ)-Lipschitz partition of unity $f : A \to \Delta(S)$, A a subset of a metric space X, extends to an (ϵ,ϵ)-Lipschitz partition of unity $g : X \to \Delta(S)$.*

PROOF. Pick $r = 8/\epsilon$. Once r is fixed, choose δ smaller than both $(\epsilon/3) - (2/3r) = \epsilon/4$ and $\epsilon/(4r+7)$. Suppose that $f : A \to \Delta(S)$ is a (δ,δ)-Lipschitz partition of unity on A. Obviously, there is a retraction $p : X \to A$ such that $d(x,p(x)) < \mathrm{dist}(x,A) + 1$ for all $x \in A$. Consider $\alpha : X \to [0,1]$ defined by $\alpha(x) = \min((d(x,A))/r, 1)$. Notice that it is $1/r$-Lipschitz. Define $g : X \to \Delta(S)$ via $g(x) = \alpha(x)\cdot v + (1-\alpha(x))\cdot f(p(x))$, where v is some fixed point in S. By Lemma 4.1, g extends f and is (ϵ,ϵ)-Lipschitz. □

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6. Unifying asymptotic dimension and large-scale paracompactness

In this section we develop a result that allows a unified approach to both asymptotic dimension and large-scale paracompactness.

Classical dimension theory of topological spaces has the following three threads that are relevant to this paper (the fourth thread is that of inductive definitions of dimension):

- dimension defined using multiplicity of covers (commonly known as the covering dimension);
- Ostrand–Kolmogorov version of covering dimension (see [15] and [14]);
- dimension defined via extending maps to spheres.

Gromov [12] defined asymptotic dimension by interpreting the first thread. It turns out that the definition also generalizes the second thread, as seen in [17, Theorem 9.9, page 131]. The definition of asymptotic dimension in [17, page 129] can be translated using [9] to the language of uniformly bounded covers (as opposed to the language of controlled sets of [17]), as follows.

Definition 6.1. A coarse space X has *asymptotic dimension* at most n (n a given nonnegative integer) if for every uniformly bounded cover $\mathcal{U}$ of X there exist uniformly bounded families $\mathcal{V}_0, \dots, \mathcal{V}_n$ that are $\mathcal{U}$-disjoint (that is, each element of $\mathcal{U}$ intersects at most one element of $\mathcal{V}_i$) and $X = \bigcup_{i=0}^{n} \mathcal{V}_i$.

Definition 6.1 is in the spirit of Ostrand–Kolmogorov and is equivalent to the following result (see [17, Theorem 9.9, page 131]): a coarse space X has *asymptotic dimension* at most n (notation: $\text{asdim}(X) \le n$, n a given nonnegative integer) if for every uniformly bounded cover $\mathcal{U}$ of X there exists a uniformly bounded cover $\mathcal{V}$ of X such that each element of $\mathcal{U}$ intersects at most $n+1$ elements of $\mathcal{V}$.

The first attempt to generalize the third thread of dimension theory was initiated by Dranishnikov [5]. The work of Dydak and Mitra [10] contains a different take on that issue and it centers on the concept of a large-scale absolute extensor. Recall that, in case K is a bounded metric space, K is a *large-scale absolute extensor* of X if for all $\epsilon > 0$ there is $\delta > 0$ such that for any subset A of X any (δ, δ)-Lipschitz function $f : A \to K$ extends to an (ϵ, ϵ)-Lipschitz function $g : X \to K$ (see [10]).

It turns out (see [10]) that S^n being a large-scale absolute extensor of X is related to the dimension of the Higson corona of X being at most n (in case X is a proper metric space) and, if X is of finite asymptotic dimension, then it is equivalent to $\text{asdim}(X) \le n$. It remains an open problem if $\text{asdim}(X) \le n$ provided S^n is a large-scale absolute extensor of X. In this section we propose another version of generalizing the third thread of dimension theory, as follows.

Definition 6.2. Let X be a metric space, $n \le \infty$, α be a function on a subset D_α of $(0, \infty)$ to $(0, \infty)$, and $M : D_\alpha \times (0, \infty) \to (0, \infty)$ be a function. We say that the *large-scale extension dimension of X with respect to α* and M is at most n (notation $\text{LsExtDim}(X, \alpha, M) \le n$) if for any set S of cardinality bigger than $\text{card}(X \times \mathbb{N})$,

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any $K > 0$ any $(\alpha(\delta), \alpha(\delta))$-Lipschitz map $f : A \subset X \to \Delta(S)^{(n)}$ $(\delta \in D_\alpha)$ that is K-cobounded extends to a (δ, δ)-Lipschitz map $g : X \to \Delta(S)^{(n)}$ that is $M(\delta, K)$-cobounded.

Remark 6.3. Notice that if Definition 6.2 holds for one set S, then it holds for any set of cardinality bigger than $\mathrm{card}(X \times \mathbb{N})$. Indeed, given a partition of unity $f : A \subset X \to \Delta(S)$, the carrier of f has vertices forming a set of cardinality at most $\mathrm{card}(X \times \mathbb{N})$. That can be easily established by noticing that, for each $k \geq 0$, vertices generated by $x \in A$ such that $f(x)$ is in the geometric interior of a k-simplex form a set of cardinality at most $\mathrm{card}(X \times \mathbb{N})$.

Theorem 6.4. *Let X be a metric space, $n \leq \infty$, and S a set of cardinality bigger than* $\mathrm{card}(X \times \mathbb{N})$. *The following conditions are equivalent.*

(1) *For each $\epsilon > 0$ there is an (ϵ, ϵ)-Lipschitz partition of unity $f : X \to \Delta(S)^{(n)}$ such that the family $\{f^{-1}(st(v))\}_{v \in S}$ is uniformly bounded.*
(2) *There are functions $\alpha : (0, \infty) \to (0, \infty)$, $M : (0, \infty) \times (0, \infty) \to (0, \infty)$ such that* $\mathrm{LsExtDim}(X, \alpha, M) \leq n$.

Proof. (2) $\Longrightarrow$ (1). Let A be a point in X and let $f : A \to \Delta(S)^{(n)}$ be a constant map to a vertex. For each $\epsilon > 0$, f is $(\alpha(\epsilon), \alpha(\epsilon))$-Lipschitz and 1-cobounded, so it extends to an (ϵ, ϵ)-Lipschitz $g : X \to \Delta(S)^{(n)}$ that is $M(\epsilon, 1)$-cobounded.

(1) $\Longrightarrow$ (2). Suppose that $\epsilon > 0$ and $K > 0$. Pick $\mu > 0$ with the property that for any (μ, μ)-Lipschitz partition of unity $g : X \to \Delta(S)$ there is an (ϵ, ϵ)-Lipschitz $h : X \to \Delta(S)^{(n)}$ so that $g(x) \in \Delta(S)^{(n)}$ implies $h(x) = g(x)$ and $h(x)(v) > 0$ implies $g(x)(v) > 0$ for all $x \in X$ and $v \in S$. For $n < \infty$, existence of μ is established in [1]; for $n = \infty$, we put $\mu = \epsilon$ (as $h = g$ works).

Pick $r = 8/\mu$. Once r is fixed, choose δ smaller than both $(\mu/3) - (2/3r) = \mu/4$ and $\mu/(4r + 7)$. Put $\alpha(\epsilon) = \delta$. Suppose that $f : A \to \Delta(S)$ is a (δ, δ)-Lipschitz partition of unity on A that is K-cobounded. Obviously, there is a retraction $p : X \to A$ such that $d(x, p(x)) < \mathrm{dist}(x, A) + 1$ for all $x \in A$. Consider $\gamma : X \to [0, 1]$ defined by $\gamma(x) = \min(d(x, A)/r, 1)$. Notice that it is $1/r$-Lipschitz. Define $g : X \to \Delta(S)$ via $g(x) = \gamma(x) \cdot u(x) + (1 - \alpha(x)) \cdot f(p(x))$, where u is some (δ, δ)-Lipschitz partition of unity $u : X \to \Delta(S)^{(n)}$ that is Q-cobounded for some $Q > 0$. By Lemma 4.1, g extends f, is (μ, μ)-Lipschitz, and is $(\max(K, Q) + 2r + 2)$-cobounded. Now, modify g to obtain an (ϵ, ϵ)-Lipschitz $h : X \to \Delta(S)^{(n)}$ so that $g(x) \in \Delta(S)^{(n)}$ implies $h(x) = g(x)$ and $h(x)(v) > 0$ implies $g(x)(v) > 0$ for all $x \in X$ and $v \in S$. Notice that h is $\max(K, Q) + 2r + 2$-cobounded. That means that putting $M(\epsilon, K) = \max(K, Q) + 2r + 2$ works and the proof is completed. □

Remark 6.5. Notice that Theorem 6.4 provides a very good unification of Property A and asymptotic dimension. For n finite, Condition 1 in Theorem 6.4 amounts to $\mathrm{asdim}(X) \leq n$. For $n = \infty$, that condition is equivalent to X being large-scale paracompact, which, in case of X being of bounded geometry, is equivalent to X having Property A (see [3]).

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7. Countable asymptotic dimension

This section is devoted to generalizing Definition 6.1 to the case of infinite asymptotic dimension. Using the Ostrand–Kolmogorov approach as a blueprint (and in analogy to the concept of countable covering dimension), we propose the following definition.

Definition 7.1. A metric space X is of *countable asymptotic dimension* if there is a sequence of integers $n_i \geq 1$, $i \geq 1$, such that for any sequence of positive real numbers R_i, $i \geq 1$, there is a sequence $\mathcal{V}_i$ of families of subsets of X such that the following conditions are satisfied:

(1) $\mathcal{V}_1 = \{X\}$;
(2) each element $U \in \mathcal{V}_i$ can be expressed as a union of at most n_i families from $\mathcal{V}_{i+1}$ that are R_i-disjoint;
(3) at least one of the families $\mathcal{V}_i$ is uniformly bounded.

Proposition 7.2. *If a metric space X is of straight finite decomposition complexity, then X is of countable asymptotic dimension.*

Proof. Recall that X is of *straight finite decomposition complexity* [7] if for any increasing sequence of positive real numbers $R_1 < R_2 < \cdots$ there is a sequence $\mathcal{V}_i$, $i \leq n$, of families of subsets of X such that the following conditions are satisfied:

(1) $\mathcal{V}_1 = \{X\}$;
(2) each element $U \in \mathcal{V}_i$, $i < n$, can be expressed as a union of at most two families from $\mathcal{V}_{i+1}$ that are R_i-disjoint;
(3) $\mathcal{V}_n$ is uniformly bounded.

That means that $n_i = 2$ for $i \geq 1$ works. □

Our next concept generalizes Definition 6.2.

Definition 7.3. Suppose that X is a subset of a metric space Y, $n \leq \infty$, α is a function on a subset D_α of $(0, \infty)$ to $(0, \infty)$, and $M : D_\alpha \times (0, \infty) \to (0, \infty)$ is a function. We say that the *large-scale extension dimension of X with respect to Y*, α, and M is at most n (notation $\text{LsExtDim}(X, Y, \alpha, M) \leq n$) if for any set S of cardinality bigger than $\text{card}(Y \times \mathbb{N})$ and any $K > 0$, any $(\alpha(\delta), \alpha(\delta))$-Lipschitz map $f : A \subset Y \to \Delta(S)^{(n)}$ ($\delta \in D_\alpha$) that is K-cobounded extends to a (δ, δ)-Lipschitz map $g : A \cup X \to \Delta(S)^{(n)}$ that is $M(\epsilon, K)$-cobounded.

Lemma 7.4. *Suppose that $\alpha : [a, \infty) \to [b, \infty)$ and $\beta : [b, \infty) \to (0, \infty)$ are functions. Let $\{W_t\}_{t \in T}$ be an R-disjoint family of subsets of X such that $\text{LsExtDim}(W_t, X, \alpha, M) \leq n$ for each $t \in T$. If $\text{LsExtDim}(B, X, \beta, M_B) \leq n$ for some $B \subset X$, then*

$$\text{LsExtDim}\Big(B \cup \bigcup_{t \in T} W_t, X, \beta \circ \alpha, M_1\Big) \leq n$$

provided $a \geq 2/(R + 1)$ and $M_1(u, K) = 2 \cdot M(u, M_B(\alpha(u), K)) + M_B(\alpha(u), K)$.

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Proof. Suppose that $A \subset X$ and $f : A \to \Delta(S)^n$ is $(\beta \circ \alpha(u), \beta \circ \alpha(u))$-Lipschitz and K-cobounded for some $u \geq a$. Extend it to $g : A \cup B \to \Delta(S)^n$, which is $(\alpha(u), \alpha(u)$-Lipschitz and $M_B(\alpha(u), K)$-cobounded. Now, for any $t \in T$, g extends over W_t to a g_t function that is (u, u)-Lipschitz and $M(u, M_B(\alpha(u), K))$-cobounded. We may arrange so that for $t_1 \neq t_2$, new vertices introduced during extension are different. Since $u \geq 2/(R + 1)$, $h = f \cup \bigcup_{t \in T} g_t$ is (u, u)-Lipschitz by Lemma 2.9. The function h is $(2 \cdot M(u, M_B(\alpha(u), K)) + M_B(\alpha(u), K))$-cobounded. Indeed, new vertices have point inverses of their stars arising from a single map g_t, so they are bounded by $M(u, M_B(\alpha(u), K))$. Old vertices v have their main part $g^{-1}(st(v)) \neq \emptyset$ (of diameter at most $M_B(\alpha(u), K)$) enlarged by adding $g_t^{-1}(st(v))$ for each $t \in T$. Each union $g^{-1}(st(v)) \cup g_t^{-1}(st(v))$ is of diameter at most $M(u, M_B(\alpha(u), K))$, resulting in h being $M_1(u, K)$-cobounded. □

Lemma 7.5. *Suppose that $\alpha : (0, \infty) \to (0, \infty)$ is a nondecreasing function such that the q-fold composition α^q satisfies $\alpha^q(a) \geq 2/(R + 1)$ for some $R > 0$, $q \geq 1$, and all $a > 0$. Let $M : [\alpha^q(a), \infty) \times (0, \infty) \to (0, \infty)$ be a function and consider the family $\mathcal{V}$ of all subsets W of X satisfying* $\mathrm{LsExtDim}(W, X, \alpha|[\alpha^q(a), \infty), M) \leq n$. *There is a function $M_1 : [a, \infty) \times (0, \infty) \to (0, \infty)$ such that if $U \subset X$ is the union of q families in $\mathcal{V}$ that are R-disjoint, then* $\mathrm{LsExtDim}(U, X, \alpha^q|[a, \infty), M_1) \leq n$.

Proof. For $q = 1$, it follows from Lemma 7.4. Use induction on q and apply Lemma 7.4 again as follows. Suppose that $B \subset X$ is the union of $q - 1$ families in $\mathcal{V}$ that are R-disjoint. By the inductive assumption (we use $\alpha(a)$ instead of a), $\mathrm{LsExtDim}(B, X, \alpha^{q-1}|[\alpha(a), \infty), M_2) \leq n$ for some function $M_2 : [\alpha(a), \infty) \times (0, \infty) \to (0, \infty)$. If W is the union of a family in $\mathcal{V}$ that is R-disjoint, then put $\beta = \alpha^{q-1}|[\alpha(a), \infty)$. Notice that $\beta \circ \alpha = \alpha^q|[a, \infty)$. Using Lemma 7.4,

$$\mathrm{LsExtDim}(B \cup W, \alpha^q|[a, \infty), M_1) \leq n$$

for $M_1 : [a, \infty) \times (0, \infty) \to (0, \infty)$ defined by $M_1(u, K) = 2 \cdot M(u, M_2(\alpha(u), K)) + M_2(\alpha(u), K)$. □

Theorem 7.6. *Let X be a metric space and $n \leq \infty$ be such that $\Delta(X)^{(n)}$ is a large-scale absolute extensor of X. If X is of countable asymptotic dimension, then* $\mathrm{LsExtDim}(X) \leq n$.

Proof. Pick a function $E : (0, \infty) \to (0, \infty)$ such that $E(x) < x$ for all x and any $(E(x), E(x))$-Lipschitz function $f : A \subset X \to \Delta(X)^{(n)}$ extends to an (x, x)-Lipschitz function $g : X \to \Delta(X)^{(n)}$. We may assume that E is nondecreasing (replace $E(x)$ by $\sup\{E(t)/2 | t < x\}$ if necessary). Suppose that S is a set of cardinality bigger than $\mathrm{card}(X \times \mathbb{N})$. We point out that any $(E(x), E(x))$-Lipschitz function $f : A \subset X \to \Delta(S)^{(n)}$ extends to an (x, x)-Lipschitz function $g : X \to \Delta(S)^{(n)}$. Given $k > 0$, by E^k we mean the composition $E \circ \cdots \circ E$ of k copies of E. Also, $E^0 = \mathrm{id}$.

There is a sequence of integers $n_i \geq 1$, $i \geq 1$, such that for any sequence of positive real numbers R_i, $i \geq 1$, there is a sequence $\mathcal{V}_i$ of families of subsets of X such that the following conditions are satisfied:

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(1) $\mathcal{V}_1 = \{X\}$;
(2) each element $U \in \mathcal{V}_i$ can be expressed as a union of at most n_i families from $\mathcal{V}_{i+1}$ that are R_i-disjoint;
(3) at least one of the families $\mathcal{V}_i$ is uniformly bounded.

Let $N(1) = 0$ and let $N(i) = \prod_{j=1}^{i-1} n_j$ for $i \geq 1$.

Given $2 > \epsilon > 0$, define $R_i > 0$ as satisfying $2/(R_i + 1) = E^{N(i)}(\epsilon)$; then pick a sequence $\mathcal{V}_i$ of families of subsets of X satisfying the above conditions. Choose $m \geq 1$ such that $\mathcal{V}_m$ is uniformly bounded by K.

Claim 1. $\mathrm{LsExtDim}(U, X, E|[\epsilon, \infty), M_m) \leq n$ for all $U \in \mathcal{V}_m$, where $M_m : [\epsilon, \infty) \times (0, \infty) \to (0, \infty)$ is defined by $M_m(x, y) = y + K + R_m$.

Proof of Claim 1. Suppose that $u \geq \epsilon$ and $f : A \subset X \to \Delta(S)^{(n)}$ is $(E(u), E(u))$-Lipschitz and R-cobounded. If $A \cap B(U, R_m) = \emptyset$, then extending f to $g : A \cup U \to \Delta(S)^{(n)}$ by sending U to a vertex v_U not belonging to the carrier of $f(A)$ produces a (u, u)-Lipschitz function by Lemma 2.9 that is $(R + K)$-cobounded. Indeed, $g^{-1}(st(v_U)) = U$ is of diameter at most K and $g^{-1}(st(v)) = f^{-1}(st(v))$ for $v \neq v_U$ is of diameter at most R.

Extend f to $g : A \cup U \to \Delta(S)^{(n)}$ that is (u, u)-Lipschitz. This may give rise to points $x \in U$ and $a \in A$ that are far away but both $g(a)$ and $g(x)$ belong to the same star. To avoid that difficulty, consider the vertices S_1 of the carrier of $f(A \cap B(U, R_m))$ and the vertices $S_2 \supset S_1$ of the carrier of $g(B(U, R_m))$. Let $r : S_2 \to S_1$ be a retraction. Change g to h by changing it on $B(U, R_m)$ to the composition of g and the induced retraction $\Delta(S_2) \to \Delta(S_1)$. The function h is (u, u)-Lipschitz (see Lemma 2.9), it extends f, and to check it is $(R + K + R_m)$-cobounded all one has to do is look at $h^{-1}(st(v))$ for $v \in S_1$. This set contains $a \in A \cap B(U, R_m)$, its intersection with A is of diameter at most R, and the remainder is contained in U. Therefore, any two points of $h^{-1}(st(v))$ are at the distance at most $R + R_m + K$. This completes the proof of Claim 1.

Define $P(m) = 1$ and $P(i) = P(i + 1) \cdot n_i$ for $i < m$.

Claim 2. For each $1 \leq i \leq m$ there is a function $M_i : [E^{N(i)}(\epsilon), \infty) \times (0, \infty) \to (0, \infty)$ such that $\mathrm{LsExtDim}(U, X, E^{P(i)}|[E^{N(i)}(\epsilon), \infty), M_i) \leq n$ for all $U \in \mathcal{V}_i$.

Proof of Claim 2. $i = m$ is taken care of by Claim 1. Suppose that $i < m$ and M_{i+1} exists. Put $q = n(i)$ and $\alpha = E^{P(i+1)} : [E^{N(i+1)}(\epsilon), \infty) \to (0, \infty)$. Applying Lemma 7.5, one gets the existence of a function $M_i : [E^{N(i)}(\epsilon), \infty) \times (0, \infty) \to (0, \infty)$ such that $\mathrm{LsExtDim}(U, X, E^{P(i)}|[E^{N(i)}(\epsilon), \infty), M_i) \leq n$ for all $U \in \mathcal{V}_i$.

Applying Claim 2 to $i = 1$, we get $\mathrm{LsExtDim}(X, X, E^{P(1)}|[\epsilon, \infty), M_1) \leq n$. That implies the existence of an (ϵ, ϵ)-Lipschitz function $g : X \to \Delta(S)^{(n)}$ that is K-cobounded for some $K > 0$. Thus, $\mathrm{LsExtDim}(X) \leq n$. □

Now we can derive a more general result than Theorem 1.6.

Corollary 7.7. *Any space X of countable asymptotic dimension has Property A.*

Proof. We are applying Theorem 7.6 when $n = \infty$, in which case the assumption that $\Delta(X)^{(n)}$ is a large-scale absolute extensor of X is vacuous (in view of Theorem 5.1).

Notice that X is large-scale finitistic (see Definition 3.6) and hence it is large-scale weakly paracompact. In view of Theorem 7.6, for each $\epsilon > 0$ there is an (ϵ, ϵ)-Lipschitz partition of unity on X that is cobounded. As shown in [3] (use Theorem 4.9 there, which says that if X is large-scale weakly paracompact and for each $\epsilon > 0$ there is an (ϵ, ϵ)-Lipschitz partition of unity on X that is cobounded, then X is large-scale paracompact), a large-scale finitistic metric space X has Property A if and only if it is large-scale paracompact. Consequently, X has Property A. □

REMARK 7.8. Theorem 7.6 is related to the problem of Dranishnikov about the equality of the asymptotic dimension asdim(X) of proper metric spaces X to the covering dimension of their Higson corona $\nu(X)$ (see [5]). As is shown in [5] and [6], the two numbers are equal in case of asdim(X) being finite. Theorem 7.6 improves that result for spaces of countable asymptotic dimension. Note (see [10]) that $\dim(\nu(X)) \le n$ is equivalent to the n-sphere S^n being a large-scale absolute extensor of X.

REMARK 7.9. In a recent paper [16], Ramras and Ramsey introduced independently the concept of a metric family $\mathcal{X}$ to have *weak straight finite decomposition complexity* with respect to the sequence $(k_1, k_2, \ldots)$ $(k_i \in \mathbb{N})$ if for every sequence $R_1 < R_2 < \cdots$ of positive numbers, there exist an $n \in \mathbb{N}$ and metric families $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_0$, the family $\mathcal{X}_i$ is (k_{i+1}, R_{i+1})-decomposable over $\mathcal{X}_{i+1}$, and the family $\mathcal{X}_n$ is uniformly bounded. The family $\mathcal{X}$ has weak straight finite decomposition complexity (wsFDC) if it has wsFDC with respect to some sequence $(k_1, k_2, \ldots)$.

Notice that, in case of $\mathcal{X}$ consisting of a single space X, the above definition amounts to saying that X has countable asymptotic dimension. Therefore, our Corollary 7.7 answers positively [16, Question 4.7].

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