A DECOMPOSITION THEOREM FOR SUBMEASURES

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1. Introduction. In recent years versions of the Lebesgue and the Hewitt–Yosida decomposition theorems have been proved for group-valued measures. For example, Traynor [**4**], [**6**] has established Lebesgue decomposition theorems for exhaustive group-valued measures on a ring using (1) algebraic and (2) topological notions of continuity and singularity, and generalizations of the Hewitt–Yosida theorem have been given by Drewnowski [**2**], Traynor [**5**] and Khurana [**3**]. In this paper we consider group-valued submeasures and in particular we have established a decomposition theorem from which analogues of the Lebesgue and Hewitt–Yosida decomposition theorems for submeasures may be derived. Our methods are based on those used by Drewnowski in [**2**] and the main theorem established generalizes Theorem 4.1 of [**2**].

2. Notation and terminology. Let G be a commutative lattice group (abbreviated to l-group). A quasi-norm (resp. norm) q on G is said to be an *l-quasi-norm* (*l-norm*) if $q(x)\leq q(y)$ for all x, y in G with $|x|\leq|y|$. A G-valued function μ defined on a ring $\mathcal{R}$ of subsets of a set X is said to be a submeasure if $\mu(\varnothing)=0$, $\mu(E\cup F)\leq\mu(E)+\mu(F)$ for all E, F in $\mathcal{R}$ with $E\cap F=\varnothing$, and $\mu(E)\leq(F)$ for all E, F in $\mathcal{R}$ with $E\subseteq F$. A G-valued submeasure μ on $\mathcal{R}$ is said to be *exhaustive* if and only if, for any disjoint sequence $\{E_n\}$ in $\mathcal{R}$, $\lim_{n\to\infty}\mu(E_n)=0$ in (G, q). An l-group G is said to be *order complete* if every bounded increasing net in G has a supremum. An l-quasi-norm q on G is said to be *order continuous* if $\varnothing\subset A\uparrow x$ in $G^+=\{x\in G: x\geq 0\}$ implies $q(x)=\sup\{q(y): y\in A\}$ and $B\downarrow x$ in G^+ implies $q(x)=\inf\{q(y): y\in B\}$.

Let $\mathcal{D}$ denote a collection of pairwise disjoint sets in $\mathcal{R}$ and let Δ be the set of all such collections. If $\mathcal{D}_1$, $\mathcal{D}_2\in\Delta$, then we write $\mathcal{D}_1\leq\mathcal{D}_2$ if and only if $\mathcal{D}_2$ is a refinement of $\mathcal{D}_1$. With each $E\in\mathcal{R}$ we associate members of $\mathcal{D}$; the collection of all such pairs $(E,\mathcal{D})$ is denoted by $\mathcal{G}$ and we let

$$\mathcal{G}(E)=\{\mathcal{D}\in\Delta:(E,\mathcal{D})\in\mathcal{G}\}\quad\text{and}\quad\Delta_{\mathcal{G}}=\bigcup_{E\in\mathcal{R}}\mathcal{G}(E).$$

In the sequel we use $\bigcup\mathcal{D}$ to mean the set theoretic union of the members of $\mathcal{D}$. Following Drewnowski's terminology ([**2**], Definition 2.1), the collection $\mathcal{G}$ is said to be an *additivity* on $\mathcal{R}$ if it satisfies the following conditions:

(a) $\Delta_f\subseteq\Delta_{\mathcal{G}}$, where Δ_f consists of those collections $\mathcal{D}$ which have only a finite number of members;

(b) if $E\in\mathcal{R}$ and $\mathcal{D}\in\mathcal{G}(E)$, then $\bigcup\mathcal{D}=E$;

(c) if $E\in\mathcal{R}$, $\mathcal{D}_1$, $\mathcal{D}_2\in\mathcal{G}(E)$, then $\mathcal{D}_1\cap\mathcal{D}_2\in\mathcal{G}(E)$, where $\mathcal{D}_1\cap\mathcal{D}_2=\{D_1\cap D_2: D_i\in\mathcal{D}_i, i=1,2\}$.

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(d) if $E_1, E_2 \in \mathscr{R}$, $E_1 \cap E_2 = \varnothing$ and $\mathscr{D}_i \in \mathscr{G}(E_i)$ $(i = 1, 2)$, then $\mathscr{D}_1 \cup \mathscr{D}_2 \in \mathscr{G}(E_1 \cup E_2)$, where $\mathscr{D}_1 \cup \mathscr{D}_2 = \{D_1 \cup D_2 : D_i \in \mathscr{D}_i,\ i = 1, 2\}$;

(e) if $E, F \in \mathscr{R}$, $E \subseteq F$ and $\mathscr{D} \in \mathscr{G}(F)$, then $\mathscr{D} \cap E \in \mathscr{G}(E)$.

Examples of additivities are

1. $\mathscr{G}_f = \{(E, \mathscr{D}) : E \in \mathscr{R}, \mathscr{D} \in \Delta_f, \bigcup \mathscr{D} = E\}$
2. $\mathscr{G}_c = \{(E, \mathscr{D}) : E \in \mathscr{R}, \mathscr{D} \in \Delta_c, \bigcup \mathscr{D} = E\}$, where Δ_c is the collection of all $\mathscr{D}$ which contain a countable number of disjoint sets in $\mathscr{R}$.

A topology τ on $\mathscr{R}$ is said to be a ring topology if the mappings $(A, B) \to A \,\Delta\, B$ and $(A, B) \to A \cap B$ of $\mathscr{R} \times \mathscr{R} \to \mathscr{R}$ are continuous, continuity being with respect to the product topology on $\mathscr{R} \times \mathscr{R}$. A ring topology τ is said to be an FN-*topology* (Fréchet–Nikodym) if and only if, for each τ-neighbourhood U of $\varnothing$ in $\mathscr{R}$, there exists a τ-neighbourhood V of $\varnothing$ in $\mathscr{R}$ such that $B \in U$ for all $B \subseteq A \in V$, $B \in \mathscr{R}$. The notion of an FN-topology was introduced and studied by Drewnowski in ([**1**], pp. 271–5). In particular, a family $\mathscr{F} = \{\eta_i : i \in I\}$ of $\mathbb{R}_+^*$-valued submeasures on a ring defines an FN-topology $\Gamma(\eta_i : i \in I)$; a base of $\Gamma(\eta_i : i \in I)$-neighbourhoods of $\varnothing$ in $\mathscr{R}$ being given by finite intersections of sets of the form $U_{\varepsilon,i} = \{A \in \mathscr{R} : \eta_i(A) < \varepsilon\}(\varepsilon > 0, \eta_i \in \mathscr{F})$. Conversely, for each FN-topology Γ on $\mathscr{R}$, there is a family $\{\xi_j : j \in J\}$ *of* $\mathbb{R}_+^*$-submeasures on $\mathscr{R}$ such that $\Gamma = \Gamma(\xi_j : J \in J)$.

Let $f(\mathscr{D})$ denote finite collections of members of $\mathscr{D}$. If Γ is an FN-topology on $\mathscr{R}$ and $E \in \mathscr{R}$, we say that $E = \Gamma\text{-}\lim f(\mathscr{D})$ if and only if, for each Γ-neighbourhood U *of* $\varnothing$ in $\mathscr{R}$, there exists a $\mathscr{D}_U \in f(\mathscr{D})$ such that $E \,\Delta \bigcup \mathscr{D}' \in U$ for all $\mathscr{D}_U \subseteq \mathscr{D}' \in f(\mathscr{D})$. We shall also use the following example of an additivity.

3. $\mathscr{G}_c(\Gamma) = \{(E, \mathscr{D}) : E \in \mathscr{R}, \mathscr{D} \in \Delta_c, \bigcup \mathscr{D} = E, E = \Gamma\text{-}\lim f(\mathscr{D})\}$. The above additivity is called the additivity generated by Γ. In particular, if η is an $\mathbb{R}_+^*$-valued submeasure on $\mathscr{R}$ we abbreviate $\mathscr{G}_c(\Gamma(\eta))$ to $\mathscr{G}(\eta)$; in this case we note that, if $E \in \mathscr{R}$ and $\mathscr{D} = \{D_n : n = 1, 2, \ldots\} \in \Delta_c$, then $E = \eta\text{-}\lim f(\mathscr{D})$ if and only if

$$\eta\left(E \setminus \bigcup_{k=1}^{n} D_k\right) \to 0 \quad \text{as} \quad n \to \infty.$$

In proving our decomposition theorem we require the notions of $\mathscr{G}$-continuity and $\mathscr{G}$-singularity as given by Drewnowski in [**2**], Definitions 2.4 and 2.17 respectively. For the sake of completeness we include these definitions as follows.

Definition 1. Let $\mathscr{G}$ be an additivity on $\mathscr{R}$. An FN-topology Γ on $\mathscr{R}$ is said to be $\mathscr{G}$*-continuous* if and only if, for each $E \in \mathscr{R}$ and $\mathscr{D} \in \mathscr{G}(E)$, $\Gamma\text{-}\lim E \setminus \bigcup_{\mathscr{D}' \in f(\mathscr{D})} \mathscr{D}' = \varnothing$.

Definition 2. An FN-topology Γ is said to be $\mathscr{G}$*-singular* if and only if the only $\mathscr{G}$-continuous FN-topology weaker than Γ is the trivial one.

If (G, q) is an l-quasi-normed group and η is a G-valued submeasure on $\mathscr{R}$, then clearly $\Gamma(q \circ \eta)$ is $\mathscr{G}(\eta)$-continuous. We also see that, if $\mathscr{G}$ is an additivity on $\mathscr{R}$, then $\Gamma(q \circ \eta)$ is $\mathscr{G}$-continuous if and only if, for each $E \in \mathscr{R}$ and $\mathscr{D} \in \mathscr{G}(E)$, $\lim_{\mathscr{D}' \in f(\mathscr{D})} q(\eta(E \setminus \bigcup \mathscr{D}')) = 0$; in this case we simply say that η is $\mathscr{G}$-continuous. It is also straightforward to show that an FN-topology Γ is $\mathcal{G}_c$-continuous if and only if it is order continuous; that is, if $\{E_n : n = 1, 2, \ldots\}$ is a sequence in $\mathcal{R}$, $E_n \downarrow \emptyset$, then $\Gamma\text{-}\lim_n E_n = \emptyset$. In a similar way we say that η is $\mathcal{G}$-singular if and only if $\Gamma(q \circ \eta)$ is $\mathcal{G}$-singular. It is not difficult to prove that η is $\mathcal{G}$-singular if and only if any $\mathcal{G}$-continuous G-valued submeasure λ on $\mathcal{R}$ such that $\lambda \ll \eta$ is identically zero.

3. The decomposition theorem. In this section we assume that G is an order complete l-group and that q is an order continuous l-quasi-norm on G. Let μ be an exhaustive G-valued submeasure on $\mathcal{R}$ and suppose that $\mathcal{G}$ is an additivity on $\mathcal{R}$. For each $E \in \mathcal{R}$, define

$$S_\mu(E) = \bigwedge_{\mathcal{D} \in \mathcal{G}(E)} \bigvee_{\mathcal{D}' \in f(\mathcal{D})} \mu\left(\bigcup \mathcal{D}'\right)$$

and

$$S'_\mu(E) = \bigvee_{\mathcal{D} \in \mathcal{G}(E)} \bigwedge_{\mathcal{D}' \in f(\mathcal{D})} \mu\left(E \setminus \bigcup \mathcal{D}'\right).$$

Then we have the following

Lemma 1. *S_μ and S'_μ are G-valued exhaustive submeasures on $\mathcal{R}$.*

Proof. Let $E \in \mathcal{R}$ and $\mathcal{D} \in \mathcal{G}(E)$. By property (b) of an additivity $\bigcup \mathcal{D} = E$ and so $0 \leq \mu(\bigcup \mathcal{D}') \leq \mu(E)$ for all $\mathcal{D}' \in f(\mathcal{D})$; the net $\{\mu(\bigcup \mathcal{D}') : \mathcal{D}' \in f(\mathcal{D})\}$ is $\uparrow$ and bounded and so by the order completeness of G $\bigvee_{\mathcal{D}' \in f(\mathcal{D})} \mu(\bigcup \mathcal{D}')$ exists. Similarly, by property (c) of an additivity the net $\left\{\bigvee_{\mathcal{D}' \in f(\mathcal{D})} \mu(\bigcup \mathcal{D}') : \mathcal{D} \in \mathcal{G}(E)\right\}$ is $\downarrow$ and bounded and so by the order completeness of G $\bigwedge_{\mathcal{D} \in \mathcal{G}(E)} \bigvee_{\mathcal{D}' \in f(\mathcal{D})} \mu(\bigcup \mathcal{D}')$ exists in G^+ for each $E \in \mathcal{R}$. By a similar argument we can prove that $S'_\mu(E)$ exists in G^+ for each $E \in \mathcal{R}$.

The subadditivity of S_μ (resp. S'_μ) follows from the subadditivity of μ and property (d) (property (e)) of an additivity. Similarly the monotonicity of S_μ (resp. S'_μ) follows from the monotonicity of μ and property (e) (resp. (d)) of an additivity.

For any $E \in \mathcal{R}$, $S_\mu(E) \leq \mu(E)$ and $S'_\mu(E) \leq \mu(E)$, and so, since q is an l-quasi-norm, $q(S_\mu(E)) \leq q(\mu(E))$ and $q(S'_\mu(E)) \leq q(\mu(E))$; this implies that both S_μ and S'_μ are exhaustive and μ-continuous.

Lemma 2. (i) *S_μ is $\mathcal{G}$-continuous.*
(ii) *S'_μ is $\mathcal{G}$-singular.*

Proof. (i) Suppose that S_μ is not $\mathcal{G}$-continuous. Then there exist a positive number ε, $E \in \mathcal{R}$ and $\mathcal{D} \in \mathcal{G}(E)$ such that $q(S_\mu(E \setminus \bigcup \mathcal{D}')) > \varepsilon$ for all $\mathcal{D}' \in f(\mathcal{D})$. Since S_μ is a submeasure and q has the l-property we have

$$q(S_\mu(E)) \geq q\left(S_\mu\left(E \setminus \bigcup \mathcal{D}'\right)\right) > \varepsilon \tag{1}$$

for all $\mathcal{D}' \in f(\mathcal{D})$; also, $S_\mu(E) \leq \bigvee_{\mathcal{D}' \in f(\mathcal{D})} \mu(\bigcup \mathcal{D}')$ and since q is order continuous

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$\sup_{\mathcal{D}' \in f(\mathcal{D})} q(\mu(\bigcup \mathcal{D}')) \geq q(S_\mu(E)) > \varepsilon$. Thus there exists a $\mathcal{D}_1 \in f(\mathcal{D})$ such that $q(\mu(\bigcup \mathcal{D}_1)) > \varepsilon$. By property (e) of an additivity $\mathcal{D} \setminus \mathcal{D}_1 \in \mathcal{G}(E \setminus \bigcup \mathcal{D}_1)$, where $\mathcal{D} \setminus \mathcal{D}_1 = \{D \in \mathcal{D} : D \notin \mathcal{D}_1\}$, and from (1) $q(S_\mu(E \setminus \bigcup \mathcal{D}_1)) > \varepsilon$. It follows from the order continuity of q that $\sup_{\mathcal{D}' \in f(\mathcal{D} \setminus \mathcal{D}_1)} q(\mu(\bigcup \mathcal{D}')) \geq q(S_\mu(E \setminus \bigcup \mathcal{D}_1)) > \varepsilon$ and so there exists a $\mathcal{D}_2 \in f(\mathcal{D} \setminus \mathcal{D}_1)$ such that $q(\mu(\bigcup \mathcal{D}_2)) > \varepsilon$.

In this way we construct by induction a disjoint sequence $\{\mathcal{D}_n : n = 1, 2, \ldots\}$ such that $q(\mu(\bigcup \mathcal{D}_n)) > \varepsilon$. This contradicts the exhaustive property of μ, and so μ is $\mathcal{G}$-continuous, as required.

(ii) Suppose that S'_μ is not $\mathcal{G}$-singular. Then there exists a $\mathcal{G}$-continuous G-valued submeasure λ such that $\lambda \ll S'_\mu$ and λ is not identically zero. This implies that there is a set $E \in \mathcal{R}$ and a positive number η such that $q(\lambda(E)) > \eta > 0$. Since $\lambda \ll S'_\mu$ there is a positive number δ such that

$$q(S'_\mu(F)) < \delta \Rightarrow q(\lambda(F)) < \eta/2 \qquad (F \in \mathcal{R}). \tag{2}$$

Thus $q(S'_\mu(E)) \geq \delta$; since q is order continuous there exists a $\mathcal{D} \in \mathcal{G}(E)$ such that $q(\mu(E \setminus \bigcup \mathcal{D}')) \geq \delta$ for all $\mathcal{D}' \in f(\mathcal{D})$. Now λ is $\mathcal{G}$-continuous and so there exists a $\mathcal{D}'_0 \in f(\mathcal{D})$ such that $q(\lambda(E \setminus \bigcup \mathcal{D}'_0)) < \eta/2^2$. Let $E_1 = \bigcup \mathcal{D}'_0$ and $A_1 = E \setminus E_1$. Then $q(\mu(A_1)) \geq \delta$, $q(\lambda(A_1)) < \eta/2^2$ and $q(\lambda(E_1)) > \eta/2 + \eta/2^2$. Thus from (2) $q(S'_\mu(E_1)) \geq \delta$ and so there exists a $\mathcal{D} \in \mathcal{G}(E_1)$ such that $q(\mu(E_1 \setminus \bigcup \mathcal{D}')) \geq \delta$ for all $\mathcal{D}' \in f(\mathcal{D})$. Again since λ is $\mathcal{G}$-continuous there exists a $\mathcal{D}'_1 \in f(\mathcal{D})$ such that $q(\lambda(E_1 \setminus \bigcup \mathcal{D}'_1)) < \eta/2^3$. Let $E_2 = \bigcup \mathcal{D}'_1$ and $A_2 = E_1 \setminus E_2$. Then $q(\mu(A_2)) \geq \delta$, $q(\lambda(A_2)) < \eta/2^3$ and $q(\lambda(E_2)) > \eta/2 + \eta/2^3$. In this way we construct by induction a disjoint sequence $\{A_n : n = 1, 2, \ldots\}$ in $\mathcal{R}$ such that $q(\mu(A_n)) \geq \delta$ for $n = 1, 2, \ldots$. This contradicts the property that μ is exhaustive.

LEMMA 3. (i) *If λ is a $\mathcal{G}$-continuous G-valued submeasure on $\mathcal{R}$ such that $\lambda \ll \mu$, then $\lambda \ll S_\mu$.*

(ii) *If ν is a $\mathcal{G}$-singular G-valued submeasure on $\mathcal{R}$ such that $\nu \ll \mu$, then $\nu \ll S'_\mu$.*

Proof. (i) Since $\lambda \ll \mu$, given any $\varepsilon > 0$, there exists a positive δ such that

$$q(\mu(E)) < \delta \Rightarrow q(\lambda(E) \leq \varepsilon \qquad (E \in R). \tag{3}$$

We seek to show that $q(S_\mu(E)) < \delta \Rightarrow q(\lambda(E)) \leq \varepsilon$. Suppose that this assertion is not true. Then there exists an E_0 in $\mathcal{R}$ such that $q(S_\mu(E_0)) < \delta$ and $q(\lambda(E_0)) > \varepsilon + \gamma$ for some positive number γ. Since q is order continuous there exists $\mathcal{D} \in \mathcal{G}(E_0)$ such that $q(\mu(\bigcup \mathcal{D}')) < \delta$ for all $\mathcal{D}' \in f(\mathcal{D})$. Since λ is $\mathcal{G}$-continuous there exists a $\mathcal{D}'_0 \in f(\mathcal{D})$ such that $q(\lambda(E_0 \setminus \bigcup \mathcal{D}'_0)) < \gamma/2$. It follows that $q(\lambda(\bigcup \mathcal{D}'_0)) > \varepsilon + \gamma/2$. Thus $q(\mu(\bigcup \mathcal{D}'_0)) < \delta$ and $q(\lambda(\bigcup \mathcal{D}'_0)) > \varepsilon + \gamma/2$. This contradicts (3), and so $\lambda \ll S_\mu$.

(ii) Since $\nu \ll \mu$, given any $\varepsilon > 0$, there exists a positive number δ such that

$$q(\mu(E)) < \delta \Rightarrow q(\nu(E)) \leq \varepsilon \qquad (E \in R). \tag{4}$$

We seek to prove that $q(S'_\mu(E)) < \delta \Rightarrow q(\nu(E)) \leq \varepsilon$. Suppose that the implication is not true. Then there exists a set E_0 in $\mathcal{R}$ such that $q(S'_\mu(E_0)) < \delta \Rightarrow q(\nu(E_0)) > \varepsilon + \gamma$ for some

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$\gamma > 0$. This implies that for all $\mathcal{D} \in \mathcal{G}(E_0)$, $q\left(\bigwedge_{\mathcal{D}' \in f(\mathcal{D})} \mu(E_0 \setminus \bigcup \mathcal{D}')\right) < \delta$. Since $\nu \ll \mu$ and μ is exhaustive it follows that ν is exhaustive and so, by Lemma 2(i), S_ν is $\mathcal{G}$-continuous. Moreover, $S_\nu \ll \nu$ and so, since ν is $\mathcal{G}$-singular, it follows that $S_\nu = 0$. Thus there exists a $\mathcal{D}_0 \in \mathcal{G}(E_0)$ such that $q(\nu(\bigcup \mathcal{D}')) < \gamma/2$ for all $\mathcal{D}' \in f(\mathcal{D}_0)$. Choose $\mathcal{D}_0' \in f(\mathcal{D}_0)$ so that $q(\mu(E_0 \setminus \bigcup \mathcal{D}_0')) < \delta$ and let $F_0 = \bigcup \mathcal{D}_0'$. Then $q(\nu(E_0 \setminus F_0)) > \varepsilon + \gamma - \gamma/2 = \varepsilon + \gamma/2$. This contradicts (4) and so $\nu \ll S_\mu'$.

Definition 3. Two G-valued submeasures μ, ν defined on a ring $\mathcal{R}$ are said to be *equivalent*, written $\mu \sim \nu$, if and only if $\mu \ll \nu$ and $\nu \ll \mu$.

We now prove our decomposition theorem.

Theorem 1. *Let (G, q) be an l-group and q an order continuous l-norm on G. Let μ be an exhaustive G-valued submeasure on $\mathcal{R}$ and $\mathcal{G}$ an additivity on $\mathcal{R}$. Then $\mu \sim S_\mu + S_\mu'$ $(\sim S_\mu \vee S_\mu')$. If λ, ν are $\mathcal{G}$-continuous and $\mathcal{G}$-singular G-valued submeasures on $\mathcal{R}$ respectively such that $\mu \sim \lambda + \nu$, then $\lambda \sim S_\mu$ and $\nu \sim S_\mu'$.*

Proof. Let $E \in \mathcal{R}$, $\mathcal{D} \in \mathcal{G}(E)$ and $\mathcal{D}' \in f(\mathcal{D})$. Now

$$E = (E \setminus \bigcup \mathcal{D}') \cup (\bigcup \mathcal{D}')$$

and so

$$\mu(E) \leq \mu(E \setminus \bigcup \mathcal{D}') + \mu(\bigcup \mathcal{D}') \leq \mu(E \setminus \bigcup \mathcal{D}') + \bigvee_{\mathcal{D}' \in f(\mathcal{D})} \mu(\bigcup \mathcal{D}');$$

it follows that

$$\mu(E) \leq \bigwedge_{\mathcal{D}' \in f(\mathcal{D})} \mu(E \setminus \bigcup \mathcal{D}') + \bigvee_{\mathcal{D}' \in f(\mathcal{D})} \mu(\bigcup \mathcal{D}')$$

and subsequently we have

$$\mu(E) \leq \bigvee_{\mathcal{D} \in \mathcal{G}(E)} \bigwedge_{\mathcal{D}' \in f(\mathcal{D})} \mu(E \setminus \bigcup \mathcal{D}') + \bigwedge_{\mathcal{D} \in \mathcal{G}(E)} \bigvee_{\mathcal{D}' \in f(\mathcal{D})} \mu(\bigcup \mathcal{D}').$$

Thus, for $E \in \mathcal{R}$,

$$\mu(E) \leq S_\mu(E) + S_\mu'(E).$$

Moreover, $S_\mu(E) \leq \mu(E)$ and $S_\mu'(E) \leq \mu(E)$, and so it is clear that $\mu \sim S_\mu + S_\mu'$.

The second part of the theorem deals with the 'uniqueness' of the decomposition.

If $\lambda + \nu \sim \mu$, then λ, $\nu \ll \mu$. Thus, by Lemma 3, $\lambda \ll S_\mu$ and $\nu \ll S_\mu'$. Also $\lambda + \nu \sim S_\mu + S_\mu'$, so that, in particular, $S_\mu \ll \lambda + \nu$ and $S_\mu' \ll \lambda + \nu$. The G-valued submeasure $\lambda + \nu$ is exhaustive and so by Lemma 3

$$S_\mu \ll S_{\lambda+\nu} = S_\lambda + S_\nu \quad \text{and} \quad S_\mu' \ll S_{\lambda+\nu}' = S_\lambda' + S_\nu'.$$

Now S_ν is $\mathcal{G}$-continuous and $S_\nu \ll \nu$ so that, since ν is $\mathcal{G}$-singular, $S_\nu = 0$. Also S_λ' is $\mathcal{G}$-singular by Lemma 2(ii) and since $S_\lambda' \leq \lambda$ and λ is $\mathcal{G}$-continuous it follows that S_λ' is $\mathcal{G}$-continuous; thus $S_\lambda' = 0$.

Therefore $S_\mu \ll S_\lambda \ll \lambda$ and $S_\mu' \ll S_\nu' \ll \nu$.

Thus $S_\mu \sim \lambda$ and $S_\mu' \sim \nu$, as required.

COROLLARY 1. *If $\mathscr{G}=\mathscr{G}_c$, then we have a Hewitt–Yosida type decomposition theorem for exhaustive l-group-valued submeasures. In this case S_μ is order continuous and so is a σ-sub-additive submeasure on $\mathscr{R}$ and S'_μ is 'purely finitely sub-additive' in the sense that, if λ is an order-continuous G-valued submeasure on $\mathscr{R}$ such that $\lambda \ll S'_\mu$, then $\lambda = 0$.*

COROLLARY 2. *Let (E, p) be an l-quasi-normed group and let η be an E-valued submeasure on $\mathscr{R}$. Suppose that the additivity on $\mathscr{R}$ is $\mathscr{G}=\mathscr{G}_c(\Gamma(p\circ\eta))$. In this case we have a Lebesgue-type decomposition theorem for an exhaustive G-valued submeasure μ; the submeasure S_μ is η-continuous and S'_μ is η-singular.*

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