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ON UNIFORM APPROXIMATION BY RATIONAL FUNCTIONS WITH AN APPLICATION TO CHORDAL CLUSTER SETS*

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For a closed and bounded set E in the complex plane, let $A(E)$ denote the collection of all functions continuous on E and analytic on E°, its interior; let $R(E)$ denote the collection of all functions which are uniform limits on E of rational functions with poles outside E. Then let $\mathscr{A}$ denote the collection of all closed, bounded sets for which $A(E) = R(E)$. The purpose of this paper is to formulate a condition on a set, which is essentially of a geometric nature, in order that the set belong to $\mathscr{A}$. Then using approximation techniques, we shall construct a meromorphic function having a certain boundary behavior on a perfect set; this answers a question raised in [1].

Uniform Approximation

For any subset H of the complex plane, let $C(H)$ denote the set of all functions each of which is continuous on the whole plane, analytic outside some closed subset of H, bounded in modulus by the constant one, and equal to zero at infinity. Let

$$\alpha(H) = \sup_{f \in C(H)} \lim_{z \to \infty} |zf(z)|.$$

Then $\alpha(H)$ is called the analytic C-capacity of H.

The result we obtain does not depend on the rather complicated definition of the analytic C-capacity of a set, but depends instead only on the formal relationship appearing in the following theorem of A.G. Vituškin [6, Theorem 2].

THEOREM A. *Let E be a closed and bounded set. Then $E \in \mathscr{A}$ if and only if for every open set G, the equality $\alpha(G - E) = \alpha(G - E^\circ)$ is satisfied.*

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THEOREM 1. *Let* $E, F \in \mathscr{A}$, *and suppose that* $\overline{E^\circ} \cap \overline{F^\circ} = \phi$. *Then* $E \cup F \in \mathscr{A}$.

Proof. An immediate consequence of E being in $\mathscr{A}$ is that $\overline{E^\circ} \in \mathscr{A}$. To prove this we need only observe that any function in $A(\overline{E^\circ})$ can be continuously extended to E and then approximated by rational functions on this larger set. Similarly $\overline{F^\circ} \in \mathscr{A}$. It then follows that $\overline{E^\circ} \cup \overline{F^\circ} \in \mathscr{A}$ since $\overline{E^\circ}$ and $\overline{F^\circ}$ are disjoint. (This is easily established by approximating any function in $A(\overline{E^\circ} \cup \overline{F^\circ})$ on each of the individual sets, and then using [5, p. 15] to obtain the desired approximation on $\overline{E^\circ} \cup \overline{F^\circ}$.)

Let $H = E \cup F$, and let G be any open set. We then have, using Theorem A several times:

$$\begin{aligned}
\alpha(G - H) &= \alpha(G - (E \cup F)) \\
&= \alpha((G - E) - F) \\
&= \alpha((G - E) - F^\circ), \text{ since } F \in \mathscr{A} \text{ and } G - E \text{ is open}, \\
&= \alpha((G - E) - \overline{F^\circ}), \text{ since } \overline{F^\circ} \in \mathscr{A} \text{ and } G - E \text{ is open}, \\
&= \alpha((G - \overline{F^\circ}) - E) \\
&= \alpha((G - \overline{F^\circ}) - E^\circ), \text{ since } E \in \mathscr{A} \text{ and } G - \overline{F^\circ} \text{ is open}, \\
&= \alpha((G - \overline{F^\circ}) - \overline{E^\circ}), \text{ since } \overline{E^\circ} \in \mathscr{A} \text{ and } G - \overline{F^\circ} \text{ is open}, \\
&= \alpha(G - (\overline{E^\circ} \cup \overline{F^\circ})) \\
&= \alpha(G - (\overline{E^\circ} \cup \overline{F^\circ})^\circ), \text{ since } \overline{E^\circ} \cup \overline{F^\circ} \in \mathscr{A}.
\end{aligned}$$

The proof is completed by noting that since $\overline{E^\circ}$ and $\overline{F^\circ}$ are disjoint, $H^\circ = (\overline{E^\circ} \cup \overline{F^\circ})^\circ$ so that $\alpha(G - (\overline{E^\circ} \cup \overline{F^\circ})^\circ) = \alpha(G - H^\circ)$. Connecting the first and last expressions we have $\alpha(G - H) = \alpha(G - H^\circ)$, and hence by Theorem A, $H = E \cup F \in \mathscr{A}$.

We note that by Mergelyan's theorem [5, p. 367] closed and bounded sets which do not divide the plane are elements of $\mathscr{A}$. Using this we may readily construct by means of Theorem 1 many sets in $\mathscr{A}$ which divide the plane into infinitely many components.

An Application

Let $f(z)$ be a function defined in a domain D, and let ζ be a boundary point of D. By a segment at ζ we mean a half open rectilinear segment contained in D with its open end point at ζ. We say that $f(z)$ has the three-segment property at ζ if there are three segments, say $\Gamma_j(\zeta)$ ($j = 0,1,2$), at ζ for which the intersection $C(f,\Gamma_0(\zeta)) \cap C(f,\Gamma_1(\zeta)) \cap C(f,\Gamma_2(\zeta)) = \phi$; here $C(f,\Gamma(\zeta))$ denotes the cluster set of f at ζ along $\Gamma(\zeta)$. The reader is referred to [3] or [4] for the basic concepts of cluster sets. In answer to a question that appears in [1, p. 32, Question 3], we offer the following:

THEOREM 2. *There exists a meromorphic function in the open unit disk D which has the three-segment property at every point of a perfect subset of the boundary of D.*

Proof. We shall actually construct this function on the right open half plane H instead of D. From the line segment $[0,i]$ we delete the open "middle half" $(i/4, 3i/4)$; from the remaining closed intervals we delete the intervals $(i/16, 3i/16)$ and $(13i/16, 15i/16)$. By continuing this process inductively we arrive at a Cantor set A. Through each $\zeta \in A$ construct the three segments $\Gamma_j(\zeta)$ ($j = 0,1,2$) at ζ, having slopes -1, 0, and $+1$, respectively, and with their free end points on the vertical line through $z = 1$. It was shown in [1, p. 30] that there exists a continuous function $g(z)$ in H having the three-segment property at every point $\zeta \in A$, with $\Gamma_j(\zeta)$ ($j=0,1,2$) as the corresponding segments. We shall use this function to construct our meromorphic function.

We begin by defining a sequence A_n of sets on which we will make our approximations. For n and j fixed ($n \geq 2$), let

$$F_{n,j} = (\bigcup_{\zeta \in A} \Gamma_j(\zeta)) \cap \{z = x + iy:\ 1/(n+1) \leq x \leq 1/n\} \quad (j = 0,1,2).$$

Then $F_{n,j}$ is a closed set which does not divide the plane, so that by Mergelyan's theorem on uniform approximation by polynomials, $F_{n,j} \in \mathscr{A}$. Since $F_{n,j}$ has no interior points,

$$I_n = F_{n,0} \cup F_{n,1} \cup F_{n,2}$$

is in $\mathscr{A}$ by Theorem 1. Let

$$H_n = \{z = x + iy:\ 1/n \leq x \leq n,\ -n \leq y \leq n\} \quad (n = 2,3,4,\cdots).$$

Finally set

$$A_n = H_n \cup I_n \cup I_{n+1}.$$

By Theorem 1 we have $A_n \in \mathscr{A}$.

It follows from [5, p. 15] (by making a second approximation) that we may assume in the sequel that the poles of our approximating functions $r_n(z)$ are always outside the set $I = \cup_{n=2}^{\infty} I_n$. Using a modification of a method devised by F. Bagemihl and W. Seidel, we now define a sequence of continuous functions $\varphi_n(z)$ on A_n and a sequence of rational functions $r_n(z)$ as follows:

$$\varphi_2(z) = \begin{cases} g(z) & \text{for} \quad z \in I_3 \\ 3(1-2x)g(z) & \text{for} \quad z \in I_2 \ (z = x + iy) \\ 0 & \text{for} \quad z \in H_2. \end{cases}$$

The function $\varphi_2(z)$ is continuous on A_2 and analytic at all interior points, so there exists a rational function $r_2(z)$ such that

$$|r_2(z) - \varphi_2(z)| < 1/2^2 \text{ for } z \in A_2.$$

Suppose that we have defined the functions $r_2(z)$, $r_3(z), \cdots r_{n-1}(z)$ in such a way that $\sum_{j=2}^{n-1} r_j(z)$ has no poles on I. Define

$$\varphi_n(z) = \begin{cases} g(z) - \sum_{j=2}^{n-1} r_j(z) & \text{for } z \in I_{n+1} \\ (n+1)(1-nx)[g(z) - \sum_{j=2}^{n-1} r_j(z)] & \text{for } z \in I_n \ (z = x + iy) \\ 0 & \text{for } z \in H_n. \end{cases}$$

Again $\varphi_n(z)$ is continuous on A_n and analytic at all interior points, so there exists a rational function $r_n(z)$ such that

$$|r_n(z) - \varphi_n(z)| < 1/2^n \quad \text{for } z \in A_n.$$

Let

$$f(z) = \sum_{j=2}^{\infty} r_j(z), \ (z \in H).$$

We assert that $f(z)$ is meromorphic in H. To this end, choose $z_0 \in H$, and pick n large enough so that z_0 lies in the interior of H_n. Let G be an open disk about z_0 contained in H_n; then for any $z \in G$ and for all $k \geq n$ we have

$$|r_k(z)| = |r_k(z) - \varphi_k(z)| < 1/2^k.$$

From this it easily follows that $f(z)$ is meromorphic at z_0.

To establish the three-segment property of $f(z)$ it suffices to show that for every $\zeta \in A$

$$|f(z) - g(z)| \longrightarrow 0 \text{ as } z \longrightarrow \zeta \text{ along } \Gamma_j(\zeta), \quad (j = 0, 1, 2).$$

Thus, for fixed $\zeta \in A$ and $\Gamma_j(\zeta)$, let $\varepsilon > 0$ be given. Choose N so large that $1/2^{N-2} < \varepsilon$. Let z be any point on $\Gamma_j(\zeta)$ with $Re(z) < 1/(N+1)$. Then there exists a natural number $n \geq N$ such that

$$z \in I_{n+1} \text{ and } z \in H_{j+2} \text{ for all } j \geq n. \tag{1}$$

We write

$$|f(z) - g(z)| \leq |\textstyle\sum_{j=2}^{n} r_j(z) - g(z)| + |r_{n+1}(z)| + |\sum_{j=n+2}^{\infty} r_j(z)|. \tag{2}$$

Now by (1) for $j \geq n+2$

$$|r_j(z)| = |r_j(z) - \varphi_j(z)| < 1/2^j,$$

so that

$$|\textstyle\sum_{j=n+2}^{\infty} r_j(z)| \leq \sum_{j=n+2}^{\infty} 1/2^j = 1/2^{n+1}. \tag{3}$$

Again by (1) we have $|r_{n+1}(z) - \varphi_{n+1}(z)| < 1/2^{n+1}$ so that

$$\begin{aligned} |r_{n+1}(z)| &< 1/2^{n+1} + |\varphi_{n+1}(z)| \\ &= 1/2^{n+1} + (n+2)(1-(n+1)x)|g(z) - \textstyle\sum_{j=2}^{n} r_j(z)| \quad (z = x + iy) \end{aligned}$$

which, since $(n+2)(1-(n+1)x) \leq 1$ for $z \in I_n$, implies

$$|r_{n+1}(z)| < 1/2^{n+1} + |g(z) - \textstyle\sum_{j=2}^{n} r_j(z)|. \tag{4}$$

Combining (2), (3), and (4) we have

$$|f(z) - g(z)| < 2|\textstyle\sum_{j=2}^{n} r_j(z) - g(z)| + 1/2^n. \tag{5}$$

Using (1) once more, we have $|r_n(z) - \varphi_n(z)| < 1/2^n$, so that

$$\begin{aligned} |\textstyle\sum_{j=2}^{n} r_j(z) - g(z)| &= |r_n(z) - (g(z) - \textstyle\sum_{j=2}^{n-1} r_j(z))| \\ &= |r_n(z) - \varphi_n(z)| < 1/2^n, \end{aligned}$$

or

$$|\textstyle\sum_{j=2}^{n} r_j(z) - g(z)| < 1/2^n. \tag{6}$$

Thus by (5) and (6),

$$|f(z) - g(z)| < 2/2^n + 1/2^n < 1/2^{n-2} \leq 1/2^{N-2} < \varepsilon.$$

It follows from [2, Theorem 4] that a normal meromorphic function cannot have the three-segment property on a set of positive measure. Furthermore, it follows from the Fatou-Nevanlinna theorem that a meromorphic function of bounded characteristic cannot have the three segment property on a set of positive measure. However, it remains an open question whether a continuous or meromorphic function can be so constructed (cf. [1, p. 32, Question 1]).

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