4

Lecture notes on variational models for incompressible Euler equations

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Abstract

These notes briefly summarise the lectures for the Summer School "Optimal Transportation: Theory and Applications" held by the second author in Grenoble during the week of 22–26 June 2009. Their goal is to describe some recent results on Brenier's variational models for incompressible Euler equations [Ambrosio and Figalli, *Arch. Ration. Math. Anal.*, **194** (2009) 421–462; Ambrosio and Figalli, Calc. Var. Partial Dif. Equations, **31** (2008) 497–509; Bernot *et al.*, *J. Math. Pures Appl.*, **91** (2008) 137–155].

4.1 Euler incompressible equations and Arnold geodesics

Let D denote either a bounded domain of $\mathbb{R}^d$ or the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. We consider an incompressible fluid moving inside D with velocity $\boldsymbol{u}$. The Euler equations for $\boldsymbol{u}$ describe the evolution in time of the velocity field, and are given by

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p & \text{in } [0,T]\times D, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } [0,T]\times D, \end{cases}$$

coupled with the boundary condition

$$\boldsymbol{u}\cdot\nu = 0 \qquad \text{on } [0,T]\times\partial D$$

when $D \neq \mathbb{T}^d$. Here, p is the pressure field, and arises as a Lagrange multiplier for the divergence-free constraint on the velocity $\boldsymbol{u}$.

If $\boldsymbol{u}$ is smooth we can write the above equations in Lagrangian coordinates: let g denote the flow map of $\boldsymbol{u}$; that is,

$$\begin{cases} \dot g(t,a) = \boldsymbol{u}(t, g(t,a)), \\ g(0,a) = a. \end{cases}$$

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By the incompressibility condition, and the classical differential identity

$$\frac{d}{dt}\det\nabla_a g(t,a) = \operatorname{div} \boldsymbol{u}(t, g(t,a)) \det\nabla_a g(t,a),$$

(here and in the following div denotes the spatial divergence of a possibly time-dependent vector field) we get $\det\nabla_a g(t,a) \equiv 1$. This means that $g(t,\cdot) : D \to D$ is a measure-preserving diffeomorphism of D:

$$g(t,\cdot)_\# \mu_D = \mu_D \qquad \big(\text{i.e. } \mu_D(g(t,\cdot)^{-1}(E)) = \mu_D(E)\ \ \forall E\big).$$

Here and in the following $f_\#\mu$ is the push-forward of a Borel measure μ through a map $f : X \to Y$ (i.e. $\int_Y \phi \, d f_\# \mu = \int_X \phi \circ f \, d\mu$ for all Borel bounded functions $\phi : Y \to \mathbb{R}$), and μ_D is the volume measure of D, renormalised by a constant so that $\mu_D(D) = 1$.

Writing Euler's equations in terms of g we obtain an ordinary differential equation (ODE) for $t \mapsto g(t)$ in the space $\mathrm{SDiff}(D)$ of measure-preserving smooth diffeomorphisms of D:

$$\begin{cases} \ddot{g}(t,a) = -\nabla p\,(t, g(t,a)) & (t,a) \in [0,T] \times D, \\ g(0,a) = a & a \in D, \\ g(t,\cdot) \in \mathrm{SDiff}(D) & t \in [0,T]. \end{cases} \tag{4.1}$$

4.1.1 Weak solutions to Euler's equations

In the case $d = 2$, existence of distributional solutions can be proved through the *vorticity* formulation; setting $\omega_t(\cdot) = \operatorname{curl} \boldsymbol{u}(t,\cdot)$, so that $\boldsymbol{u}(t,\cdot) = \nabla^\perp \Delta^{-1} \omega_t$, the Euler equations can be read as follows:

$$\frac{d}{dt}\omega_t(x) + \operatorname{div}\big(\omega_t(x)\boldsymbol{u}(t,x)\big) = 0.$$

Formally, this equation preserves all L^p norms of solutions, and indeed existence is not hard to obtain if $\omega_0 \in L^p$ for $1 < p \leq \infty$. Delort improved the existence theory up to L^1 or measure initial conditions ω_0 whose positive (or negative) part is absolutely continuous, and the problem of getting a solution for all measure initial data is still open. As shown by Yudovitch [15,16], uniqueness holds for $p = \infty$, while it is still open in all the other cases.

In the case $d > 2$ much less is known: *no* general global existence results of distributional solutions are presently available.

4.1.2 Arnold's geodesic interpretation

At least formally, one can view the space $\mathrm{SDiff}(D)$ of measure-preserving diffeomorphisms of D as an infinite-dimensional manifold with the metric

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inherited from the embedding in $L^2(D;\mathbb{R}^d)$, and with tangent space made by the divergence-free vector fields. Using this viewpoint, Arnold interpreted the ODE (4.1), and therefore Euler's equations, as a *geodesic* equation on $\mathrm{SDiff}(D)$ [3]. Therefore, one can look for solutions of Euler's equations on $[0,1]\times D$ by minimizing the action functional

$$\mathscr{A}(g) := \int_0^1 \int_D \frac{1}{2}|\dot{g}(t,x)|^2\, d\mu_D(x)\, dt$$

among all paths $g(t,\cdot) : [0,1] \to \mathrm{SDiff}(D)$ with $g(0,\cdot) = f$ and $g(1,\cdot) = h$ prescribed (typically, by right invariance, f is taken as the identity map $\boldsymbol{i}$). Ebin and Marsden proved in [10] that this problem has indeed a unique solution when $h \circ f^{-1}$ is sufficiently close, in a strong Sobolev norm, to $\boldsymbol{i}$. We shall denote by $\delta(f,h)$ the Arnold distance in $\mathrm{SDiff}(D)$ induced by this minimisation problem.

Of course, this variational problem differs from Euler's problem, because the initial and final diffeomorphisms, and not the initial velocity, are prescribed. Nevertheless, the investigation of this problem leads to difficult and still not completely understood questions (typical of calculus of variations); namely:

(a) necessary and sufficient optimality conditions;
(b) regularity of the pressure field;
(c) regularity of (relaxed) curves with minimal length.

Before describing some of the main contributions in this field, let us recall some 'negative' results that motivate somehow the necessity of relaxed formulations of this minimisation problem.

4.1.3 Non-attainment and non-existence results

Shnirelman [12, 13] found the example of a map $\bar{g} \in \mathrm{SDiff}([0,1]^2)$ which cannot be connected to $\boldsymbol{i}$ by a path with finite action, i.e. $\delta(\boldsymbol{i},\bar{g}) = +\infty$. Furthermore, he proved that for $h \in \mathrm{SDiff}([0,1]^3)$ of the form

$$h(x_1,x_2,x_3) = (\bar{g}_1(x_1,x_2), \bar{g}_2(x_1,x_2), x_3), \quad \text{with} \quad (\bar{g}_1,\bar{g}_2) = \bar{g} \text{ as above},$$

$\delta(\boldsymbol{i},h)$ is not attained, i.e. no minimizing path between $\boldsymbol{i}$ and h exists (although there exist paths with a finite action). This fact can be easily explained as follows (see also [8, Paragraph 1.3]): since there is no two-dimensional path with finite action connecting $\boldsymbol{i}$ to $\bar{g}$ while in three dimensions it is known that the minimal action is finite [12], if a minimising path $t \mapsto g(t)$ exists then it has a non-trivial third component, i.e. $g_3(t,x) \not\equiv x_3$. Set $\eta(x_3) := \min\{2x_3, 2-2x_3\}$, and let $\boldsymbol{u}$ denote the velocity field associated with g, i.e. $\boldsymbol{u} = \dot{g}\circ g^{-1}$. Then it is easily

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seen that the velocity field

$$\tilde{\boldsymbol{u}}(x_1, x_2, x_3) := \begin{cases} \boldsymbol{u}_1(x_1, x_2, \eta(x_3)) \\ \boldsymbol{u}_2(x_1, x_2, \eta(x_3)) \\ \frac{1}{2}\boldsymbol{u}_3(x_1, x_2, \eta(x_3)) \end{cases}$$

induces a path $\tilde{g}$ which still joins $\boldsymbol{i}$ to h, but with strictly less action (since $\boldsymbol{u}_3$ is not identically zero). This contradicts the minimality of g, and proves that there is no minimizing path between $\boldsymbol{i}$ and h. (See also [8, Paragraph 1.3].)

Let us point out that the above argument shows that minimising sequences exhibit oscillations on small scales, and strongly suggest the analysis of weak solutions.

4.1.4 Time discretisation, minimal projection and optimal transport

Before describing the concept of relaxed solutions to the Euler equations introduced by Brenier, let us first see what happens when one tries to attack the above variational problem by time-discretisation: assume $D \subset \mathbb{R}^d$, and fix $g_0, g_1 \in \mathrm{SDiff}(D)$. We want to find the 'midpoint' $g_{1/2}$ between g_0 and g_1; that is, we consider

$$\min_{g \in \mathrm{SDiff}(D)} \left\{ \frac{1}{2}\|g - g_0\|^2_{L^2(D;\mathbb{R}^d)} + \frac{1}{2}\|g_1 - g\|^2_{L^2(D;\mathbb{R}^d)} \right\}.$$

Up to rearranging the terms and removing all the quantities independent on g, the above problem is equivalent to minimising

$$\min_{g \in \mathrm{SDiff}(D)} \left\| g - \frac{g_0 + g_1}{2} \right\|^2_{L^2(D;\mathbb{R}^d)},$$

i.e. we have to find the L^2-projection on $\mathrm{SDiff}(D)$ of the function $\frac{g_0+g_1}{2} \in L^2(D;\mathbb{R}^d)$. Since the set $\mathrm{SDiff}(D)$ is neither closed nor convex, no classical theory is available to ensure the existence of such projection.

In order to make the problem more treatable, let us close $\mathrm{SDiff}(D)$: as shown for instance in [9], if $D = [0, 1]^d$ or $D = \mathbb{T}^d$ then the L^2-closure of $\mathrm{SDiff}(D)$ in $L^2(D;\mathbb{R}^d)$ coincides with the space $S(D)$ of measure-preserving maps:

$$S(D) := \left\{ g : D \to D : \ \mu_D(g^{-1}(A)) = \mu_D(A) \ \forall A \in \mathcal{B}(D) \right\}.$$

Then the general problem we want to study becomes the following: given $h \in L^2(D;\mathbb{R}^d)$, solve

$$\min_{s \in S(D)} \int_D |h - s|^2 \, d\mu_D. \tag{4.2}$$

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As in the classical optimal transport problem, one can consider the following Kantorovich relaxation: denoting by $\Pi(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ_D and second marginal $\nu := h_\# \mu_D$, we minimise

$$\min_{\gamma \in \Pi(\mathbb{R}^d)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\gamma(x, y). \tag{4.3}$$

Assume the *non-degeneracy condition* $\nu \ll dx$. Then we can apply the classical theory of optimal transport with quadratic cost for the problem of sending ν onto μ_D [6]: there exists a unique optimal transport map $\nabla\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $(\nabla\phi)_\# \nu = \mu_D$. Moreover, the unique optimal measure $\bar{\gamma}$ which solves (4.3) is given by

$$\bar{\gamma} = (\nabla\phi \times \mathrm{Id})_\# \nu.$$

Then it is easily seen that the map

$$\bar{s} := \nabla\phi \circ h$$

belongs to $S(D)$ and uniquely solves (4.2) (see [6] or [14, Chapter 3] for more details).

4.2 Relaxed solutions

In Section 4.1 we have seen how the attempt of attacking Arnold's geodesics problem by time discretisation leads to study the existence of the L^2-projection onto $\mathrm{SDiff}(D)$, and that the projection of a function h onto its closure $S(D)$ exists and is unique whenever h satisfies a non-degeneracy condition. Instead of going on with this strategy, we now want to change the point of view, attacking the problem by a relaxation in 'space'.

Two levels of relaxation can be imagined: the first one is to relax the smoothness and injectivity constraints, and this leads to the definition of the space $S(D)$ of *measure-preserving maps*. However, we will see that a second level is necessary, giving up the idea that $g(t, \cdot)$ is a map, but allowing it to be a *measure-preserving plan* (roughly speaking, a multivalued map). This leads to the space

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \ \eta(A \times D) = \mu_D(A) = \eta(D \times A) \ \ \forall A \in \mathcal{B}(D)\}.$$

The space $S(D)$ 'embeds' into $\Gamma(D)$ considering

$$S(D) \ni g \mapsto (\boldsymbol{i} \times g)_\# \mu_D \in \Gamma(D).$$

Conversely, any $\eta \in \Gamma(D)$ concentrated on a graph is induced by a map $g \in S(D)$.

Even from the Lagrangian viewpoint, it is natural to follow the path of each particle, and to relax the smoothness and injectivity constraints, allowing fluid paths to split, forward or backward in time. These remarks led Brenier in 1989 to the following model [5]: let

$$\Omega(D) := C([0,1]; D), \qquad e_t(\omega) := \omega(t),\ \ t \in [0,1].$$

Then, denoting by $\mathscr{P}(\Omega(D))$ the family of probability measures in $\Omega(D)$, we minimise the action functional

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega), \qquad \boldsymbol{\eta} \in \mathscr{P}(\Omega(D))$$

with the endpoint and incompressibility constraints

$$(e_0, e_1)_\# \boldsymbol{\eta} = (\boldsymbol{i} \times h)_\# \mu_D, \qquad (e_t)_\# \boldsymbol{\eta} = \mu_D \ \ \forall t \in [0, T].$$

In Brenier's model, a flow is modelled by a random path with some constraints on the expectations of this path. As we will see below, this problem can be recast in the optimal transportation framework, dealing properly with the incompressibility constraint.

Classical flows $g(t, a)$ induce generalised ones, with the same kinetic action, via the relation $\boldsymbol{\eta} = (\Phi_g)_\# \mu_D$, with

$$\Phi_g : D \to \Omega(D), \qquad \Phi_g(a) := g(\cdot, a).$$

In this relaxed model, some obstructions of the original one disappear; for instance, in the case $D = [0,1]^d$ or $D = \mathbb{T}^d$ it is always possible to connect any couple of measure-preserving diffeomorphism by a path with action less than $\sqrt{d}$. Actually, this allows are to prove that finite-action paths exist in many situation: as shown in [1, Theorem 3.3], given a domain D for which there exists a bi-Lipschitz measure-preserving diffeomorphism $\Phi : D \to [0,1]^d$, by considering composition of generalised flows with Φ one can easily constructs a generalised flow with finite action between any $h_0, h_1 \in \mathrm{SDiff}(D)$. Moreover, standard compactness/lower semicontinuity arguments in the space $\mathscr{P}(\Omega(D))$ provide existence of generalised flows with minimal action.

4.2.1 Eulerian–Lagrangian model

Coming back to the relaxed model described above, we observe that the endpoint constraint $(e_0, e_1)_\# \boldsymbol{\eta} = (\boldsymbol{i} \times h)_\# \mu_D$ cannot be modified to deal with the more general problem of connecting $f \in S(D)$ to $h \in S(D)$; indeed, by right invariance, this is clear only if f is invertible (in this case, one looks for the optimal connection between $\boldsymbol{i}$ and $h \circ f^{-1}$). These remarks led to a more general model, which allows are to connect $\eta = \eta_a \otimes \mu_D$ to $\gamma = \gamma_a \otimes \mu_D$ [2]. (Here, we are disintegrating both the initial and final plans with respect to the first variable.) The idea, which appears first in Brenier's *Eulerian–Lagrangian* model [8] is to 'double' the state space, adding to the Eulerian state space D a Lagrangian state space A. Even though A could be thought of as an identical copy of D, it is convenient to denote it by a different symbol.

Let

$$\Omega^*(D) := \Omega(D) \times A.$$

Then, consider probability measures $\boldsymbol{\eta} = \boldsymbol{\eta}_a \otimes \mu_D$ in $\Omega^*(D)$; this means that $\boldsymbol{\eta}$ has μ_D as second marginal, and that

$$\int \phi(\omega, a)\, d\boldsymbol{\eta}(\omega, a) = \int_A \left(\int_{\Omega(D)} \phi(\omega, a)\, d\boldsymbol{\eta}_a \right) d\mu_D(a)$$

for all bounded Borel functions ϕ on $\Omega^*(D)$.

Again, one minimises the action

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega^*(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2\, dt\, d\boldsymbol{\eta}(\omega, a)$$

with the incompressibility constraint $(e_t)_\# \boldsymbol{\eta} = \mu_D$ for all t (here, $e_t(\omega, a) = \omega(t)$) and the family of endpoint constraints

$$(e_0)_\# \boldsymbol{\eta}_a = \gamma_a, \quad (e_1)_\# \boldsymbol{\eta}_a = \eta_a \qquad \text{for } \mu_D\text{-a.e. } a \in D.$$

As in Section 4.2, we are using $\eta_a \otimes \mu_D$ and $\gamma_a \otimes \mu_D$ to denote the disintegrations of η and γ respectively.

Denoting by $\overline{\delta}(\eta, \gamma)^2$ the minimal action, it turns out that one can define natural operations of *reparameterisation*, *restriction* and *concatenation* in this class of flows. These imply that $(\overline{\delta}, \Gamma(D))$ is a metric space.

Indeed, it is proved in [1] that it is *complete* and a *length* space, whose convergence is stronger than weak convergence in $\mathscr{P}(D \times D)$.

4.2.2 Motivation for the extension to $\Gamma(D)$

Even for deterministic initial and final data, there exist examples of minimising geodesics $\boldsymbol{\eta}$ that are *not deterministic* in between; this means that $(e_0, e_t)_\# \boldsymbol{\eta} \in \Gamma(D) \setminus S(D)$, $t \in (0, 1)$.

To show this phenomenon, consider the problem of connecting up to additive constants in $D = B_1(0) \subset \mathbb{R}^2$ the identity map $\boldsymbol{i}$ to $-\boldsymbol{i}$. For convenience, up to

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a reparameterisation, we can choose the time interval as $[0,\pi]$. Two classical solutions are

$$[0,\pi] \ni t \mapsto (x_1 \cos \pm t + x_2 \sin \pm t, x_1 \sin \pm t + x_2 \cos \pm t),$$

corresponding to a clockwise and an anti-clockwise rotation.

On the other hand, one can consider the family of maps $\omega_{x,\theta}$ connecting x to $-x$:

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1-|x|^2}(\cos\theta, \sin\theta) \sin t \qquad \theta \in (0,\pi) \tag{4.4}$$

and define $\boldsymbol{\eta} := (\omega_{x,\theta})_\sharp(\frac{1}{2\pi^2}\mathscr{L}^2 \lfloor D \times \mathscr{L}^1 \lfloor (0, 2\pi))$.

It turns out that $\boldsymbol{\eta}$ is optimal as well, and non-deterministic in between. Moreover, as shown in [4], it is possible to construct infinitely many other solutions to the above minimisation problem which are not induced by maps. For instance, one can split the measure $\boldsymbol{\eta}$ above as $\frac{1}{2}(\boldsymbol{\eta}_+ + \boldsymbol{\eta}_-)$, where $\boldsymbol{\eta}_+$ consists of the curves such that $(\cos\theta, \sin\theta) \cdot x^\perp \geq 0$, and $\boldsymbol{\eta}_-$ consists of the curves such that $(\cos\theta, \sin\theta) \cdot x^\perp \leq 0$, where $x^\perp = (x_2, -x_1)$, and the two flows $\boldsymbol{\eta}_+$ and $\boldsymbol{\eta}_-$ can be shown to be still incompressible (see [4, Paragraph 4.1]). We will say more about these important examples later on, as more results on the theory will be available.

4.3 The pressure field

Brenier proved in [7] a surprising result: even though geodesics are not unique in general, given the initial and final conditions, there is a *unique*, up to an additive time-dependent constant, pressure field. The pressure field arises if one relaxes the incompressibility constraint, considering *almost incompressible* flows $\boldsymbol{\nu}$. Denoting by $\rho^{\boldsymbol{\nu}}$ the density produced by the flow, defined by

$$(e_t)_\# \boldsymbol{\nu} = \rho^{\boldsymbol{\nu}}(t,\cdot)\mu_D \quad \left(\text{i.e.} \int \phi(\omega(t))\, d\boldsymbol{\eta}(\omega) = \int_D \phi\rho^{\boldsymbol{\nu}}(t,\cdot)\, d\mu_D \text{ for all } \phi\right),$$

we say that $\boldsymbol{\nu}$ is almost incompressible if $\|\rho^{\boldsymbol{\nu}} - 1\|_{C^1} \leq 1/2$.

Theorem 4.1 (Pressure as a Lagrange multiplier, [1, 7]). *Let $\boldsymbol{\eta}$ be optimal between η and γ. There exists a distribution $p \in (C^1)^*$ such that*

$$\mathscr{A}(\boldsymbol{\nu}) + \langle p, \rho^{\boldsymbol{\nu}} - 1\rangle \geq \mathscr{A}(\boldsymbol{\eta}) \tag{4.5}$$

for all almost incompressible flows $\boldsymbol{\nu}$ between η and γ with $\rho^{\boldsymbol{\nu}}(t,\cdot) = 1$ for t sufficiently close to 0 *and to* 1.

Using this result one can make first variations as follows: given a smooth field $\boldsymbol{w}(t,x)$, vanishing for t close to 0 and 1, one can consider the family $(\boldsymbol{X}^t)$

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of flow maps

$$\frac{d}{d\varepsilon}\boldsymbol{X}^t(\varepsilon, x) = \boldsymbol{w}(t, \boldsymbol{X}^t(\varepsilon, x)), \qquad \boldsymbol{X}^t(0, x) = x$$

and perturb (smoothly) the paths ω by $\omega(t) \mapsto \boldsymbol{X}^t(\varepsilon, \omega(t)) \sim \omega(t) + \varepsilon\, \boldsymbol{w}(t, \omega(t))$. Denoting by

$$\Phi_\varepsilon : \Omega^*(D) \to \Omega^*(D), \qquad \Phi_\varepsilon(\omega, a)(t) := \big(\boldsymbol{X}^t(\varepsilon, \omega(t)), a\big),$$

the induced perturbations in $\Omega^*(D)$, these in turn induce perturbations $\boldsymbol{\eta}_\varepsilon := (\Phi_\varepsilon)_\# \boldsymbol{\eta}$ of $\boldsymbol{\eta}$ which are almost incompressible. Then, the first variation gives

$$\int_{\Omega^*(D)} \int_0^1 \dot{\omega}(t) \cdot \frac{d}{dt}\boldsymbol{w}(t, \omega(t))\, dt\, d\boldsymbol{\eta}(\omega, a) + \langle p, \operatorname{div} \boldsymbol{w}\rangle = 0.$$

This equation uniquely determines ∇p as a distribution, independently of the chosen minimizer $\boldsymbol{\eta}$; indeed, $\boldsymbol{\eta}$ enters in (4.5) only through $\mathscr{A}(\boldsymbol{\eta})$, which obviously is independent of the chosen minimiser, and so the above equation holds true for *every* minimiser $\boldsymbol{\eta}$. Since $\boldsymbol{w}$ is arbitrary, the first variation also leads to a weak formulation of Euler's equations

$$\partial_t \overline{\boldsymbol{v}}_t(x) + \operatorname{div}\big(\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t(x)\big) + \nabla_x p(t, x) = 0,$$

where $\overline{\boldsymbol{v}}_t$ and $\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t$ are implicitly defined by

$$\overline{\boldsymbol{v}}_t \mu_D = (e_t)_\#(\dot{\omega}(t)\boldsymbol{\eta}), \qquad \overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t \mu_D = (e_t)_\#(\dot{\omega}(t) \otimes \dot{\omega}(t)\boldsymbol{\eta}).$$

Observe that in general $\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t \neq \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$. Indeed, since these models allow the passage of many fluid paths at the same point at the same time (i.e. branching and multiple velocities are possible), the product $\overline{\boldsymbol{v}}_t(x) \otimes \overline{\boldsymbol{v}}_t(x)$ of the mean velocity $\overline{\boldsymbol{v}}_t(x)$ with itself might be quite different from the mean value $\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t(x)$ of the product. This gap precisely marks the difference between genuine distributional solutions to Euler's equation and 'generalised' ones (see also [4, Section 2 and Paragraph 4.4] for more comments on this fact).

4.4 Necessary and sufficient optimality conditions

In this section we study necessary and sufficient optimality conditions for Brenier's variational problem and its extensions.

The basic remark is that *any* Borel integrable function $q : [0, 1] \times D \to \mathbb{R}$ with $\int_D q(t, \cdot)\, d\mu_D = 0$ for every $t \in [0, 1]$ induces a null-Lagrangian for the minimisation problem, with the incompressibility constraint; indeed,

$$\int_{\Omega^*(D)} \int_0^1 q(t, \omega(t))\, dt\, d\boldsymbol{\eta}(\omega, a) = \int_0^1 \int_D q(t, x)\, d\mu_D(x)\, dt = 0$$

for any generalized incompressible flow $\boldsymbol{\eta}$. If we denote by

$$c_q^{0,1}(x, y) := \inf \left\{ \int_0^1 \frac{1}{2}|\dot{\omega}(t)|^2 - q(t, \omega(t))\, dt : \ \omega(0) = x, \ \omega(1) = y \right\}$$

the value function for the Lagrangian $\mathcal{L}_q(\omega) := \int \frac{1}{2}|\dot{\omega}(t)|^2 - q(t, \omega(t))\, dt$, we also have

$$\int_{\Omega^*(D)} \int_0^1 \frac{1}{2}|\dot{\omega}(t)|^2 - q(t, \omega(t))\, dt\, d\boldsymbol{\eta}(\omega, a) \geq \int_D c_q^{0,1}(a, h(a))\, d\mu_D(a)$$

for any incompressible flow $\boldsymbol{\eta}$ between $\boldsymbol{i}$ and h. Moreover, equality holds if and only if $\boldsymbol{\eta}$-almost every (ω, a) is a $c_q^{0,1}$-minimising path.

The following result, proved in [8, Section 3.6], shows that this lower bound is sharp with $q = p$, if p is sufficiently smooth.

Theorem 4.2. *Let $\boldsymbol{u}$ be a C^1 solution to the Euler equations in $[0, T] \times D$, whose pressure field p satisfies*

$$(*) \qquad T^2 \sup_{t \in [0,T]} \sup_{x \in D,\, |\xi| \leq 1} \langle \nabla_x^2 p(t, x)\xi, \xi \rangle \leq \pi^2.$$

Then the measure $\boldsymbol{\eta}$ induced by $\boldsymbol{u}$ via the flow map is optimal on $[0, T]$.

This follows by the fact that the integral paths of $\boldsymbol{u}$ satisfy $\ddot{\omega}(t) = -\nabla p(t, \omega(t))$, and $(*)$ implies that stationary paths for the action are also minimal for $\mathcal{L}_p$. (This is a consequence of the one-dimensional Poincaré inequality $\int_0^T |\dot{u}(t)|^2\, dt \geq \frac{\pi^2}{T^2} \int_0^T |u(t)|^2 dt$ for all $u : [0, T] \to \mathbb{R}$ such that $\int_0^T u\, dt = 0$; see [5, Section 5] or [8, Proposition 3.2] for more details.)

The question investigated in [1] is: How far are these conditions from being *necessary*? C^1 regularity or even one-sided bounds on $\nabla^2 p$ are not realistic, so one has to look for necessary (and sufficient) conditions under much weaker regularity assumptions on p.

From now on, we restrict for simplicity to the case $D = \mathbb{T}^d$. The following regularity result for the pressure field has been obtained in [2], improving the regularity $\nabla p \in \mathcal{M}_{\text{loc}}((0, 1) \times \mathbb{T}^d)$ obtained in [8].

Theorem 4.3. *For any γ, $\eta \in \Gamma(\mathbb{T}^d)$ the unique pressure field given by Theorem 4.1 belongs to $L^2_{\text{loc}}((0, 1); BV(\mathbb{T}^d))$.*

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The above result says in particular that p is a function, and not just a distribution. This allows one to define the value of p pointwise, which as we will see below will play a key role.

In order to guess the right optimality conditions, we recall that the two main degrees of freedom in optimal transport problems are:

- in moving mass from x to y, the path, or the family of paths, that should be followed;
- the amount of mass that should be moved, on each such path, from x to y.

The second degree of freedom is even more important in situations when more than one optimal path between x and y is available. As we will see, both things will depend on $\mathcal{L}_p$. But, since p is defined only up to negligible sets, the value of the Lagrangian $\mathcal{L}_p$ on a path ω is *not* invariant in the Lebesgue equivalence class; furthermore, no local pointwise bounds on p are available (remember that $p(t,\cdot)$ is only a BV function, with BV norm in $L^2_{\mathrm{loc}}(0,1)$). Therefore, as done in [1], one has to:

- Define a *precise* representative $\bar{p}$ in the Lebesgue equivalence class of p; it turns out that the 'correct' definition is

$$\bar{p}(t,x) := \liminf_{\varepsilon\downarrow 0} p(t,\cdot) * \phi_\varepsilon(x),$$

  where $p(t,\cdot) * \phi_\varepsilon$ are suitable mollifications of $p(t,\cdot)$. Of course, this definition depends on the choice of the mollifiers, but we prove that a suitable choice of them provides a well-behaved (in the sense stated in Theorem 4.4) function $\bar{p}$.
- Consider, in the minimisation problem, only paths ω satisfying

$$Mp(t,\omega(t)) \in L^1_{\mathrm{loc}}(0,1),$$

  where $Mp(t,\cdot)$ is a suitable maximal function of $p(t,\cdot)$ (see [1] for a more precise definition of the maximal operator).

With these constraints one can talk of *locally minimising path* ω for the Lagrangian $\mathcal{L}_{\bar{p}}$ and, correspondingly, define a family of value functions

$$c^{s,t}_{\bar{p}} : \mathbb{T}^d \times \mathbb{T}^d \to [-\infty,+\infty], \qquad [s,t] \subset (0,1),$$

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representing the cost of the minimal connection between x and y in the time interval (s,t):

$$c_{\bar p}^{s,t}(x,y) := \inf\left\{ \int_s^t \frac{1}{2}|\dot\omega(\tau)|^2 - \bar p(\tau,\omega)\,d\tau : \omega(s) = x, \right.$$
$$\left. \omega(t) = y,\, Mp(\tau,\omega(\tau)) \in L^1(s,t) \right\}.$$

With this notation, the following result proved in [1] provides necessary and sufficient optimality conditions.

Theorem 4.4. *Let $\boldsymbol\eta = \boldsymbol\eta_a \otimes \mu_{\mathbb T}$ be an optimal incompressible flow between $\eta = \eta_a \otimes \mu_{\mathbb T}$ and $\gamma = \gamma_a \otimes \mu_{\mathbb T}$. Then*

(i) $\boldsymbol\eta$ is concentrated on locally minimising paths for $\mathcal L_{\bar p}$;
(ii) for all intervals $[s,t] \subset (0,T)$, for $\mu_{\mathbb T}$-a.e. a, the plan $(e_s,e_t)_\#\boldsymbol\eta_a$ is $c_{\bar p}^{s,t}$-optimal, i.e.

$$\int_{\mathbb T^d\times\mathbb T^d} c_{\bar p}^{s,t}(x,y)\,d(e_s,e_t)_\#\boldsymbol\eta_a \le \int_{\mathbb T^d\times\mathbb T^d} c_{\bar p}^{s,t}(x,y)\,d\lambda$$

for any $\lambda \in \mathscr P(\mathbb T^d \times \mathbb T^d)$ having the same marginals of $(e_s,e_t)_\#\boldsymbol\eta_a$.

Conversely, if (i), (ii) hold with $\bar p$ replaced by some function q satisfying $Mq \in L^1_{\mathrm{loc}}\big((0,1); L^1(\mathbb T^d)\big)$, then $\boldsymbol\eta$ is optimal, and q is the pressure field.

Notice that an optimal transport problem is trivial if either the initial or the final measure is a Dirac mass; therefore, the second condition becomes meaningful when either $(e_s)_\#\boldsymbol\eta_a$ or $(e_t)_\#\boldsymbol\eta_a$ is not a Dirac mass. This corresponds to the case when $(e_s,\pi_a)_\#\boldsymbol\eta$ is *not* induced by a map, a phenomenon that cannot be ruled out, as we discussed in Section 4.2.2. In the example presented in Section 4.2.2 the pressure field $p(x) = |x|^2/2$ is smooth and time independent, but the initial and final conditions are chosen in such a way that a continuum of action-minimising paths (4.4) between x and $-x$ exists. As shown in [4], there are infinitely many incompressible flows connecting the identity map $\boldsymbol i$ to $-\boldsymbol i$, which moreover induce infinitely many distributional solutions to the Euler equations [4, Paragraph 4.4].

The results in [1] show a connection with the theory of action-minimising measures, though in this case the Lagrangian $\int_0^1 \frac{1}{2}|\dot\omega(t)|^2 - \bar p(t,\omega(t))\,dt$ is possibly non-smooth and not given a priori, but *generated* by the variational problem itself.

Here, we see a nice variation on a classical theme of calculus of variations: a field of (smooth, non-intersecting) *extremals* gives rise both to *minimisers*

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and to an incompressible flow *in phase space*. Here, instead, we have a field of (possibly non-smooth, or intersecting) *minimisers* which has to produce an incompressible flow *in the state space*. This structure seems to be rigid, and it might lead to new regularity results for the pressure field.

Let us also recall that, as recently shown in [11], under a $W^{1,p}$-regularity of the pressure p one can show that $\boldsymbol{\eta}$-a.e. ω solves the Euler–Lagrange equations and belongs to $W^{2,p}([0,1]) \subset C^1([0,1])$. This result is a first step towards the BV case, where one can still expect that the minimality of $\boldsymbol{\eta}$ may allow one to prove higher regularity on the minimising curves ω (like $\dot\omega \in BV$).

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