Can. J. Math., Vol. XXVI, No. 5, 1974, pp. 1090–1097

ON THE NUMBER OF CLASSES OF A FINITE GROUP INVARIANT FOR CERTAIN SUBSTITUTIONS

A. J. VAN ZANTEN AND E. DE VRIES

1. Introduction. In this paper we consider representations of groups over the field of the complex numbers.

The nth-Kronecker power $\sigma^{\otimes n}$ of an irreducible representation σ of a group can be decomposed into the constituents of definite symmetry with respect to the symmetric group S_n. In the special case of the general linear group $\mathrm{GL}(N)$ in N dimensions the decomposition of the defining representation at once provides irreducible representations of $\mathrm{GL}(N)$ [**9**; **10**; **11**]. For an arbitrary group the above constituents (which are sometimes called plethysms [**9**]) are in general no longer irreducible. However, in any case this decomposition provides a partial reduction of the Kronecker nth-power, which gives us a tool for deriving some properties of the characters of an arbitrary group, as was also done in [**13**]. In particular we derived there some properties concerning the relationship between the group averages $(1/g)\Sigma_R\{\chi(R)\}^n$ and $(1/g)\Sigma_R\chi(R^n)$ where $\chi(R)$ is the character of the element R in the irreducible representation σ of a finite group $\mathscr{G}$ of order g. For the special case $n = 3$ this relationship is of some importance for theoretical physics. It was shown in [**4**] that the so-called 3jm-symbols or Clebsch-Gordan coefficients of a group $\mathscr{G}$ have simple symmetry-properties if and only if for all irreducible representations σ one has

$$\frac{1}{g}\sum_R \chi(R^3) = \frac{1}{g}\sum_R \{\chi(R)\}^3.$$

(Clebsch-Gordan coefficients are used for the explicit reduction of a Kronecker product of two irreducible representations.) The work of [**13**] was in fact inspired by this problem. Groups for which the above relation holds for all irreducible representations are called simple phase groups (S.P. groups). With the results of [**13**] we were able to derive some criteria for non-simple phase groups (see [**12**; **13**]).

Furthermore, we considered in [**13**] the equation $X^n = S$ where X and S are elements of $\mathscr{G}$ (S fixed) and where n is a positive integer. The relations between $(1/g)\Sigma_R\{\chi(R)\}^n$ and $(1/g)\Sigma_R\chi(R^n)$ were used to derive some theorems relating the number of roots of this equation and the existence of irreducible

Received March 2, 1973 and in revised form, May 14, 1973. This research was partially supported by the National Science Foundation and the Army Research Office (Durham). The research of the first named author was also supported by the Netherlands America Commission for Educational Exchange.

representations with the property that their Kronecker nth-power contains the trivial representation $1_{\mathscr{G}}$ of $\mathscr{G}$.

The work in this paper is a continuation of the work begun in [**13**]. The same method is used here to derive relations between the number of classes, which are invariant under the substitution $R \to R^{1-n}$ (where n is again a positive integer) and the existence of irreducible representations with the property that their Kronecker nth-power contains $1_{\mathscr{G}}$.

The theorems in this paper as well as those of [**13**] are examples of theorems which relate properties of classes with properties of irreducible representations.

One of the theorems is used to derive another criterion for non-simple phase groups.

2. Preliminaries. We shall here present a number of formulae from [**13**], which we intend to make use of in the following sections. Let $\chi(R)$ be the character of the element R in the irreducible representation σ of a finite group $\mathscr{G}$, whereas $\chi^{\lambda}(R)$ stands for the character of R in that part of the nth-Kronecker power of σ which is denoted by the partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_\mu)$ of n into μ parts. We then have (see [**13**, Equation (2)])

$$(1) \qquad \{\chi(R)\}^{l_1}\{\chi(R^2)\}^{l_2} \ldots \{\chi(R^m)\}^{l_m} = \sum_{\lambda} \phi_l^{\lambda}\chi^{\lambda}(R).$$

The summation in Equation (1) has to be extended over all partitions of n with at most $\chi(1)$ parts (the unit element of the group $\mathscr{G}$ is denoted by 1). In Equation (1), ϕ_l^{λ} is the character of the irreducible representation (λ) and the class (l) of the symmetric group S_n. Here, (l) stands for the cycle structure $[1^{l_1}, 2^{l_2}, \ldots, m^{l_m}]$ of the classes of S_n. Special cases of (1) which play a role in subsequent sections are

$$(2) \qquad \{\chi(R)\}^n = \sum_{\lambda} \phi_{[1^n]}^{\lambda}\chi^{\lambda}(R),$$

$$(3) \qquad \chi(R^n) = \sum_{\lambda} \phi_{[n]}^{\lambda}\chi^{\lambda}(R)$$

and

$$(4) \qquad \chi(R)\chi(R^{n-1}) = \sum_{\lambda} \phi_{[1,n-1]}^{\lambda}\chi^{\lambda}(R).$$

The characters $\phi_{[1^n]}^{\lambda}$ are the degrees of the representations (λ) of the symmetric group S_n, whereas the characters $\phi_{[n]}^{\lambda}$ are ± 1 or 0 (see [**10**, Equation (5.3)]). Special cases which we shall need are

$$(5) \qquad \phi_{[n]}^{(n)} = +1,$$

$$(6) \qquad \phi_{[n]}^{(1^n)} = (-1)^{n-1}.$$

https://doi.org/10.4153/CJM-1974-101-6 Published online by Cambridge University Press

Furthermore one has for $n \geqq 2$

(7) $\phi^{(1^n)}_{[1,n-1]} = (-1)^n.$

3. Number of classes invariant for certain substitutions. First we introduce the notion of invariance of a class for certain substitutions. Consider the substitution

(8) $R \to R^\mu,$

where μ is an integer. If R runs through all the elements of a class $\mathscr{C}_i$ of conjugate elements, then R^μ runs once or more times through all elements of a class $\mathscr{C}_i^{(\mu)}$. A class is said to be invariant for the substitution (8), if $\mathscr{C}_i^{(\mu)} = \mathscr{C}_i$. Sometimes this is also expressed by saying that the class $\mathscr{C}_i$ admits the substitution (8) (see [**6**, § 6]). We shall derive now some properties of the total number of classes $N(\mu)$ of a finite group $\mathscr{G}$, which are invariant for substitutions of the form (8).

THEOREM 1. *Let $N(1-n)$ be the number of classes of a finite group $\mathscr{G}$, which admit the substitution $R \to R^{1-n}$, where n is a positive integer. Let σ be an irreducible representation of $\mathscr{G}$ with character χ. Let $s_n(\chi)$ be the number of times that the trivial representation $1_\mathscr{G}$ is contained in the nth-Kronecker power $\sigma^{\otimes n}$. Then the following inequality holds:*

(9) $N(1-n) \leqslant \sum_\chi s_n(\chi).$

Proof. First of all we shall prove that for the number $N(1-n)$ the following equation holds:

(10) $N(1-n) = \sum_\lambda \phi^\lambda_{[1,n-1]} \frac{1}{g} \sum_R \sum_\chi \chi^\lambda(R).$

In Equation (10), g is the order of $\mathscr{G}$, whereas the summation runs over all partitions (λ) of n, all elements R of $\mathscr{G}$ and all irreducible representations χ of $\mathscr{G}$.

Consider

(11) $\frac{1}{g} \sum_R \sum_\chi \chi^*(R)\chi(R^{1-n}),$

where the asterisk denotes complex conjugation. The sum

(12) $\frac{1}{g} \sum_\chi \chi^*(R)\chi(R^{1-n})$

is according to an orthogonality relation of characters equal to 0 if R and R^{1-n} do not belong to the same class and equal to $1/g_i$ if R and R^{1-n} belong to

https://doi.org/10.4153/CJM-1974-101-6 Published online by Cambridge University Press

the same class $\mathscr{C}_i$ (g_i is the number of elements in $\mathscr{C}_i$). If we sum the expression (12) over the elements of a class $\mathscr{C}_i$, then

$$(13)\quad \frac{1}{g}\sum_{R\in\mathscr{C}_i}\sum_{\chi}\chi^*(R)\chi(R^{1-n})$$

is equal to 1 if $\mathscr{C}_i$ admits the substitution $R \to R^{1-n}$ and is equal to 0 if $\mathscr{C}_i$ does not admit this substitution. From this we see that the expression (11) gives the number of classes of $\mathscr{G}$, which admit the substitution $R \to R^{1-n}$.

We can now write

$$\begin{aligned} N(1-n) &= \frac{1}{g}\sum_{R}\sum_{\chi}\chi^*(R)\chi(R^{1-n}) \\ &= \frac{1}{g}\sum_{R}\sum_{\chi}\chi^*(R^{-1})\chi(R^{n-1}) \\ &= \frac{1}{g}\sum_{R}\sum_{\chi}\chi(R)\chi(R^{n-1}) \\ &= \sum_{\lambda}\phi^{\lambda}_{[1,n-1]}\frac{1}{g}\sum_{R}\sum_{\chi}\chi^{\lambda}(R), \end{aligned}$$

where we used Equation (1) for $(l) = [1, n-1]$. This proves (10).

Now we know that the numbers

$$\frac{1}{g}\sum_{R}\sum_{\chi}\chi^{\lambda}(R)$$

are non-negative integers (cf. [**13**]) and furthermore

$$\phi^{\lambda}_{[1,n-1]} \leqq \phi^{\lambda}_{[1^n]}.$$

Hence

$$\begin{aligned} N(1-n) &= \sum_{\lambda}\phi^{\lambda}_{[1,n-1]}\frac{1}{g}\sum_{R}\sum_{\chi}\chi^{\lambda}(R) \\ &\leqslant \sum_{\lambda}\phi^{\lambda}_{[1^n]}\frac{1}{g}\sum_{R}\sum_{\chi}\chi^{\lambda}(R) \\ &= \sum_{\chi}s_n(\chi), \end{aligned}$$

where we have used (2) and the equation

$$(14)\quad s_n(\chi) = \frac{1}{g}\sum_{R}\{\chi(R)\}^n.$$

COROLLARY. *If in a finite group $\mathscr{G}$ the number of classes which admit the substitution $R \to R^{1-n}$, where n is a positive integer, is larger than one, then there*

https://doi.org/10.4153/CJM-1974-101-6 Published online by Cambridge University Press

exists at least one irreducible representation σ other than the trivial one, such that the nth-Kronecker power $\sigma^{\otimes n}$ contains the trivial representation at least once.

THEOREM 2. *Let n be a positive integer. Let $N(1-n)$ be the number of classes of a finite group $\mathcal{G}$, which admit the substitution $R \to R^{1-n}$. Let σ be an irreducible representation of $\mathcal{G}$ with character χ. Let $s_n(\chi)$ be the number of times that the trivial representation $1_{\mathcal{G}}$ is contained in the nth-Kronecker power $\sigma^{\otimes n}$. Then the following relations hold:*

(i) *if $n = p^a + 1$, where p is a prime and a is a non-negative integer, then*

$$N(1-n) \equiv \sum_{\chi} s_n(\chi) \pmod p; \tag{15}$$

(ii) *if n is an even positive integer and if $s_n(\chi) \leqq 1$ if $n = 4$ and $s_n(\chi) \leqq n-2$ if $n \geqq 6$ for all irreducible representations σ of $\mathcal{G}$, then*

$$N(1-n) = \sum_{\chi} s_n(\chi); \tag{16}$$

(iii) *if n is an odd integer not less than 3 and if $s_n(\chi) \leqq n-2$ for all irreducible representations σ of $\mathcal{G}$, then*

$$N(1-n) \equiv \sum_{\chi} s_n(\chi) \pmod 2. \tag{17}$$

In order to prove this theorem we first shall state two lemmas, the proofs of which can be found in the appendix of [**13**].

LEMMA A. *Let p be a prime number and let a be a non-negative integer. If two classes (l) and (l') of the symmetric group S_n have the same cycles save that p^a cycles of order 1 in (l) are replaced by a cycle of order p^a in (l'), then the characters of the two classes are congruent to modulus p for every representation.*

LEMMA B. *The minimum of the degrees of the non-linear characters of the symmetric group S_n equals $n-1$ if $n \geqq 3$, $n \neq 4$. (For $n = 4$ this minimum equals 2.)*

Proof of Theorem 2. (i) From Lemma A we know that

$$\phi^{\lambda}_{[1,n-1]} \equiv \phi^{\lambda}_{[1^n]} \pmod p \tag{18}$$

for $n = p^a + 1$. Hence,

$$\begin{aligned} N(1-n) &= \sum_{\lambda} \phi^{\lambda}_{[1,n-1]} \frac{1}{g} \sum_{R} \sum_{\chi} \chi^{\lambda}(R) \\ &\equiv \sum_{\lambda} \phi^{\lambda}_{[1^n]} \frac{1}{g} \sum_{R} \sum_{\chi} \chi^{\lambda}(R) \pmod p, \end{aligned} \tag{19}$$

from which (15) follows, by using (2) and (14).

(ii) We substitute (2) in (14), which provides us with

$$s_n(\chi) = \sum_{\lambda} \phi^{\lambda}_{[1^n]} \sum_{R} \sum_{\chi} \chi^{\lambda}(R). \tag{20}$$

The non-negative integer $(1/g)\Sigma_R\chi^\lambda(R)$, which occurs in (20) gives the number of times that the trivial representation is contained in the representation corresponding to the character $\chi^\lambda(R)$. Because of the conditions of this part of the theorem and because of Lemma B, it follows from (20), that all integers $(1/g)\Sigma_R\chi^\lambda(R)$ have to be zero, except possibly when $(\lambda) = (n)$ or $(\lambda) = (1^n)$. Hence,

$$(21)\quad \sum_R \{\chi(R)\}^n = \sum_R \chi^{(n)}(R) + \sum_R \chi^{(1^n)}(R)$$

and

$$(22)\quad \sum_R \chi(R)\chi(R^{n-1}) = \sum_R \chi^{(n)}(R) + \sum_R \chi^{(1^n)}(R),$$

where we used (7). Summing over all irreducible representations σ and using (10) provides us immediately with (16).

(iii) Exactly as in the proof of (ii) also here (21) holds, whereas now

$$(23)\quad \sum_R \chi(R)\chi(R^{n-1}) = \sum_R \chi^{(n)}(R) - \sum_R \chi^{(1^n)}(R).$$

Again summing over all irreducible representations gives immediately (17).

We remark that for $n = 2$, part (ii) of Theorem 2 amounts to the well-known result that the number of ambivalent (or real) classes equals the number of real characters (see [**5**, Theorem 12.4]). Note that $s_2(\chi) = 1$ for a real character and $s_2(\chi) = 0$ for a complex character.

4. Application to simple phase groups. Theorem 1 of the previous section can be applied to a problem from theoretical physics. It is shown in [**4**] that so-called 3jm-symbols (or Clebsch-Gordan coefficients) of a group $\mathscr{G}$ can be symmetrized if and only if for all irreducible representations σ with character χ one has

$$(24)\quad \frac{1}{g}\sum_R \chi(R^3) = \frac{1}{g}\sum_R \{\chi(R)\}^3.$$

Groups for which (24) holds for all irreducible representations are called simple phase groups or S.P. groups (cf. [**1**; **2**; **7**; **12**; **13**]).

First we shall present a necessary and sufficient condition for a finite group $\mathscr{G}$ to be non-S.P.

PROPOSITION. *Let $\mathscr{G}$ be a finite group. Let $\zeta^{(3)}(1)$ be the number of solutions of the equation $X^3 = 1$ $(X \in \mathscr{G})$ and let $s_3(\chi)$ be the number of times that the trivial representation $1_\mathscr{G}$ is contained in the Kronecker 3rd-power of the irreducible representation σ with character χ. The group $\mathscr{G}$ is a non-S.P. group if and only if the following inequality holds*:

$$(25)\quad \zeta^{(3)}(1) < \sum_\lambda \chi(1)s_3(\chi).$$

https://doi.org/10.4153/CJM-1974-101-6 Published online by Cambridge University Press

Proof. From Equations (2) and (3) we have

$$(26)\quad \frac{1}{g}\sum_R \{\chi(R)\}^3 = \frac{1}{g}\sum_R \chi^{(3)}(R) + \frac{2}{g}\sum_R \chi^{(2,1)}(R) + \frac{1}{g}\sum_R \chi^{(1^3)}(R)$$

and

$$(27)\quad \frac{1}{g}\sum_R \chi(R^3) = \frac{1}{g}\sum_R \chi^{(3)}(R) - \frac{1}{g}\sum_R \chi^{(2,1)}(R) + \frac{1}{g}\sum_R \chi^{(1^3)}(R),$$

which gives with (14),

$$(28)\quad \frac{1}{g}\sum_R \chi(R^3) = s_3(\chi) - \frac{3}{g}\sum_R \chi^{(2,1)}(R).$$

Hence,

$$(29)\quad \frac{1}{g}\sum_R \sum_\chi \chi(1)\chi(R^3) = \sum_\chi \chi(1)s_3(\chi) - \frac{3}{g}\sum_R \sum_\chi \chi(1)\chi^{(2,1)}(R).$$

Now the left hand side of (29) is equal to the number of solutions $\zeta^{(3)}(1)$ of the equation $X^3 = 1$, $(X \in \mathscr{G})$ [**13**]. The inequality (25) follows immediately.

We shall now apply Theorem 1 to the above proposition which gives rise to the following remarkable theorem.

THEOREM 3. *Let $\mathscr{G}$ be a finite group the order of which is not divisible by* 3. *If there is a class other than the class consisting of the unit element, which admits the substitution $R \rightarrow R^{-2}$ then the group $\mathscr{G}$ is a non-S.P. group.*

Proof. Because the order of the group is not divisible by 3 there are no elements of order 3. This means that the equation $X^3 = 1$ has only one solution $(X = 1)$ or $\zeta^{(3)}(1) = 1$. From $N(-2) > 1$ it follows from the corollary of Theorem 1 that there exists at least one irreducible representation $\sigma_1 \neq 1_{\mathscr{G}}$, with character χ_1, such that $s_3(\chi_1) > 0$. For the right hand side of inequality (25) we can now write

$$\sum_\chi \chi(1)s_3(\chi) = 1 + \chi_1(1)s_3(\chi_1) + \ldots > 1.$$

Hence the inequality (25) is satisfied and so $\mathscr{G}$ is a non-S.P. group.

The criterion of Theorem 3 can be applied for example to the K-metacyclic group of order 20, defined by

$$(30)\quad S^5 = T^4 = 1, \quad T^{-1}ST = S^3,$$

(cf. [**3**]). From the defining relations it follows that $S^{-2} = S^3$ lies in the same class as S. So this group is not S.P. (cf. [**13**]).

To apply Theorem 3 it is not always necessary to know the complete group structure. The following example illustrates this. The relations

$$(31)\quad d^5 = 1, \quad y^{-1}dy = d^2,$$

which are part of the defining relations of the simple Suzuki group Su(8) of order 29120 [**8**], already show that this group is a non-S.P. group: $d^{-2} = d^8 = y^{-3}\, d\, y^3$.

This last example also shows that the criterion of Theorem 3 in those cases where it can be applied is a much faster and easier method than checking the equations (24) or the inequality (25).

References

1. P. H. Butler, *Wigner coefficients and $n - j$ symbols for chains of groups* (to appear).

2. P. H. Butler and R. C. King, *Symmetrized Kronecker products of group representations*, Can. J. Math. *26* (1974), 328–339.

3. H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, 2nd ed. (Springer Verlag, Berlin-Göttingen-Heidelberg-New York, 1965).

4. J.-R. Derome, *Symmetry properties of the 3j-symbols for an arbitrary group*, J. Mathematical Phys. *7* (1966), 612–615.

5. W. Feit, *Characters of finite groups* (W. A. Benjamin, New York-Amsterdam, 1967).

6. G. Frobenius and I. Schur, *Über die reellen Darstellungen der endlichen Gruppen*, Sitzungsberichte der kön. preuss. Ak. der Wissenschaften, Jahrgang 1906, S. 186 (Berlin).

7. R. C. King, *Branching rules for* $GL(N) \supset \Sigma_m$ *and the evaluation of inner plethysms* (to appear in J. Mathematical Phys.).

8. J. Leech (ed.), *Computational problems in abstract algebra* (Pergamon Press, Oxford, 1970).

9. D. E. Littlewood, *The theory of group characters and matrix representations of groups*, 2nd ed. (Clarendon Press, Oxford, 1950).

10. F. D. Murnaghan, *The theory of group representations* (Dover Publ., New York, 1963).

11. G. de B. Robinson, *Representation theory of the symmetric group* (University of Toronto Press, Toronto, 1961).

12. A. J. van Zanten, *Some applications of the representation theory of finite groups: a partial reduction method*, Ph.D. thesis, Groningen, 1972.

13. A. J. van Zanten and E. de Vries, *On the number of roots of the equation $X^n = 1$ in finite groups and related properties*, J. Algebra (1973), 475–486

Duke University,
Durham, North Carolina