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STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE

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The Liapunov-Razumikhin technique has been employed to study L^p-stability properties of solutions of functional differential equations of delay type where the delay becomes unbounded as $t \to +\infty$. These results have been applied to investigate sufficient conditions for L^2-stability of Volterra integro-differential equations.

1. Introduction

The Liapunov second method involving an energy like function has come to be a powerful tool in the qualitative study of ordinary differential equations. Over the years, this method has also been extended to functional differential equations by various authors. While Liapunov functions are employed in the study of ordinary differential equations, more generally Liapunov functionals are used in studying functional differential equations (*cf.* [4], [10]). As Volterra integro-differential equations can also be treated as functional differential equations, Liapunov functionals have been constructed exclusively for Volterra integro-differential equations by Burton ([1], [2]) in order to study the stability and uniform stability properties of Volterra integro-differential equations. As distinguished from this line, Razumikhin [7] obtained

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stability properties of functional differential equations using Liapunov functions rather than functionals. This technique of Razumikhin involves an estimation of the derivative of the Liapunov function along the solutions of the given equation on a certain set in which the solutions satisfy a specific inequality. In recent years, a good number of papers ([3], [5], [6], [8], [9]) have appeared employing this technique and various results have been reported. In particular, Seifert [8], [9] and Grimmer and Seifert [3] have studied the properties of solutions of Volterra integro-differential equations - such as boundedness, stability, asymptotic stability and uniform stability - by studying the corresponding properties of functional differential equations. In this paper the Liapunov-Razumikhin technique is further explored to obtain L^p-stability of the zero solution of a functional differential equation. This result has been applied to obtain sufficient conditions for L^2-stability properties of solutions of a Volterra integro-differential equation.

2. Preliminaries

In this section the employed notations are explained and then it is indicated how a Volterra integro-differential equation can be thought of as an ordinary differential equation involving an interval of delay which becomes unbounded as $t \to +\infty$. In addition, two results of Seifert [8] are stated which ensure the stability of the point $x = 0$ and boundedness of solutions, for a functional differential equation.

For an element x of R^n, by $|x|$ we mean the usual euclidean norm. We denote by $x_t(\cdot)$ a function continuous on the interval $0 \le s \le t < \infty$ to R^n and by S_t we mean the set $\{x_t(\cdot)\}$ for all such functions. If $x(s)$ is a function defined and continuous on $0 \le s < \infty$ to R^n, then for each t, $0 \le t < \infty$, this function defines a member $x_t(\cdot)$ of S_t given by $x(s)$, $0 \le s < \infty$. We call this function $x_t(\cdot)$ a segment of $x(s)$.

For fixed $t \ge 0$, let $F(t, x_t(\cdot))$ be a function on S_t to R^n. For each function $x(s)$ continuous on $0 \le s < \infty$ to R^n, we assume that $F(t, x_t(\cdot))$ is continuous in t, where $x_t(\cdot)$ is a segment of $x(s)$.

By a solution of the equation (functional differential equation)

$$x'(t) = F\big(t, x_t(\cdot)\big) , \tag{2.1}$$

we mean a continuously differentiable function $x(s)$ on $0 \le s < \infty$ such that (2.1) is satisfied on $0 \le t < \infty$ for $x_t(\cdot)$ a segment of $x(s)$. Let $x(t, x_0)$ be any solution of (2.1) existing for all $t \ge 0$ such that $x(0, x_0) = x_0$.

A special case of (2.1) is the Volterra integro-differential equation

$$x'(t) = H\big(x(t)\big) + \int_0^t g\big(t, s, x(s)\big)ds , \tag{2.2}$$

for $t \ge 0$. Here $H(x)$ is continuous on R^n and $g(t, s, x)$ is continuous on $R^+ \times R^+ \times R^n$.

It can be noted that if $x(t)$ is a solution of (2.2), then it is also a solution of (2.1) where

$$F\big(t, \xi(\cdot)\big) = H\big(\xi(t)\big) + \int_0^t g\big(t, s, \xi(s)\big)ds$$

and $x_t(\cdot)$ is a segment of $x(s)$, $0 \le s < \infty$.

DEFINITION 2.1. The point $x = 0$ for (2.1) is stable if given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $|x_0| < \delta$, then every solution $x(t, x_0)$ of (2.1) is defined for all $t \ge 0$ and satisfies $|x(t, x_0)| < \varepsilon$ for $t \ge 0$.

DEFINITION 2.2. The point $x = 0$ for (2.1) is L^p-stable $(0 < p < \infty)$, if it is stable and if there exists a $\delta_0 > 0$ such that

$$\int_0^\infty |x(t, x_0)|^p dt < \infty$$

whenever $|x_0| < \delta_0$.

We shall now state the following results due to Seifert [8] which are useful in our subsequent discussion.

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LEMMA 2.3 ([8]). *Let there exist functions* $u(s)$, $v(s)$ *and* $f(s)$ *continuous for* $s \geq 0$ *and such that* $u(0) = v(0) = 0$, $u(s)$ *is increasing,* $f(s) > s$ *for* $s > 0$, *and suppose* $V(t, x)$ *is a real-valued continuous function in* (t, x) *for* $t \geq 0$ *and* x *in* D, *an open subset of* R^n *containing the zero vector. Let* V *satisfy*

(i) $u(|x|) \leq V(t, x) \leq v(|x|)$ *for* $t \geq 0$, $x \in D$,

(ii) $$V'(t, x(t)) = \limsup_{h \to 0^+} \frac{[V(t+h, x(t+h)) - V(t, x(t))]}{h} \leq 0 ,$$

for any solution $x(t)$ *of* (2.1) *for which* $x(s)$ *is in* D *and* $f(V(t, x(t))) > V(s, x(s))$ *for* $0 \leq s \leq t$.

Then the point $x = 0$ *is stable for* (2.1).

LEMMA 2.4 ([8]). *Let there exist a function* V *satisfying the hypotheses of Lemma* 2.3 *except now that* $D = R^n$ *and* $u(r) \to \infty$ *as* $r \to \infty$. *Then the solutions of* (2.1) *are bounded.*

3. Main results

In this section we shall first obtain a general result yielding L^p-stability of the point $x = 0$ for the functional differential equation (2.1). Then this result is applied to investigate sufficient conditions for L^2-stability properties of solutions of an integro-differential equation.

THEOREM 3.1. *Suppose* $V : R^+ \times R^n \to R^+$ *is a continuous function satisfying the following conditions:*

(i) $u(|x|) \leq V(t, x) \leq v(|x|)$ *for all* $(t, x) \in R^+ \times R^n$ *where* $u, v : R^+ \to R^+$ *are continuous function such that* $u(0) = v(0) = 0$ *and* $u(s)$, $v(s)$ *are increasing to* $+\infty$ *as* s *increases to* $+\infty$;

(ii) $V'(t, x(t)) \leq -C|x(t)|^p$, $p > 0$, $C > 0$, *whenever* $f(V(t, x(t))) > V(s, x(s))$ *for* s *in any neighbourhood* $(t-\eta, t)$, $\eta > 0$, *and* $f(s)$ *is any continuous function for* $s \geq 0$ *such that* $f(s) > s$ *for* $s > 0$.

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Then the point $x = 0$ *is* L^p*-stable for* (2.1).

Proof. Define

$$r(t) = V(t, x(t)) + C \int_0^t |x(s, x_0)|^p ds .$$

Clearly $r(0) = V(0, x_0)$ and $r(t)$ is positive. We claim that $r(t) \le r(0)$ for all $t \ge 0$. Suppose not. Then there exists a $\tilde{t}_1 > 0$ such that $r(\tilde{t}_1) > r(0)$. Since $r(t)$ is a continuous function of t, there exist a $t_1 \ge 0$ and a $t_2 > t_1$ such that $r(t) \le r(0)$ on $[0, t_1]$, $r(t_1) = r(0)$ and $r(t) > r(0)$ on (t_1, t_2). Therefore for some $t^* \in (t_1, t_2)$ there exists an $\eta > 0$ such that

(3.1) $$r'(s) > 0 \quad \text{for} \quad s \in (t^*-\eta, t^*] .$$

Further for $t^*-\eta < s < t^*$, we have

(3.2) $$r(s) < r(t^*) .$$

From the hypotheses and Lemma 2.4 it follows that

$$\sup_{0 \le t < \infty} |x(t, x_0)| \le M_1(x_0)$$

where $M_1(x_0)$ is a positive constant and $x(0, x_0) = x_0$. From the continuous dependence of solutions on the initial values, we get $M_1(x_0) \le M$ for $|x_0| < 1$ where M is a positive constant independent of x_0. Hence the monotonocity of v yields that

$$v(|x(t, x_0)|) \le v(M) \quad \text{for} \quad t \ge 0 , \quad |x_0| < 1 .$$

Let $\varepsilon > 0$ be given. We may assume that ε so small that $u(\varepsilon) < v(M)$. Then there is a positive number $a = a(\varepsilon)$ such that

(3.3) $$f(s) > s + a \quad \text{for} \quad s \in [u(\varepsilon), v(M)] .$$

From the hypotheses and Lemma 2.3 it is clear that the point $x = 0$ is stable for (2.1). Therefore, for a given $L = (a/2\eta C)^{1/p}$, there exists a $\tilde{\delta}_0 > 0$ such that

$$|x_0| < \tilde{\delta}_0 \text{ implies } |x(t, x_0)| < L \text{ for } t \geq 0 .$$

In particular for $s \in (t^*-\eta, t^*)$ and $|x_0| < \delta_0 = \min[\tilde{\delta}_0, 1]$ we have

$$|x(s, x_0)|^p < \frac{a}{2\eta C} .$$

This implies that

$$C \int_s^{t^*} |x(\tau, x_0)|^p d\tau < \frac{Ca}{2\eta C} (t^*-s) < a/2 < a . \tag{3.4}$$

From (3.2) and the definition of $r(t)$ we get

$$V(t^*, x(t^*)) + C \int_s^{t^*} |x(\tau, x_0)|^p d\tau > V(s, x(s)) . \tag{3.5}$$

For a given $\varepsilon > 0$ choose a $\delta_1 = \delta_1(\varepsilon) > 0$ such that $v(\delta_1) < u(\varepsilon)$. Since $x = 0$ of (2.1) is stable, it is clear that $u(\delta_1) \leq V(t, x(t)) \leq v(M)$. Hence, from (3.3), (3.4) and (3.5), we have

$$\begin{aligned} f(V(t^*, x(t^*))) &> V(t^*, x(t^*)) + a \\ &> V(t^*, x(t^*)) + C \int_s^{t^*} |x(\tau, x_0)|^p d\tau \\ &> V(s, x(s)) . \end{aligned}$$

Thus condition *(ii)* of the hypotheses gives

$$V'(t^*, x(t^*)) < -C|x(t^*)|^p$$

which in turn implies $r'(t^*) < 0$ contradicting (3.1). Therefore there exists no $\tilde{t}_1 > 0$ such that $r(\tilde{t}_1) > r(0)$. Hence $r(t) \leq r(0)$ for all $t \geq 0$. Thus we have

$$V(t, x(t)) + C \int_0^t |x(\tau, x_0)|^p d\tau < V(0, x_0)$$

and therefore

$$\int_0^t |x(\tau, x_0)|^p d\tau < \frac{1}{C} V(0, x_0)$$

for $|x_0| < \delta_0$. This together with the stability of $x = 0$ for (2.1) proves the theorem.

APPLICATION. We shall consider the special case of (2.2):

$$x'(t) = Ax(t) + h(x(t)) + \int_0^t g(t, s, x(s))ds , \quad t \geq 0 , \tag{3.6}$$

where A is a real $n \times n$ constant matrix, $h(x)$ is continuous on R^n and satisfying

$$|h(x)| \leq \mu|x| \tag{3.7}$$

and $g(t, s, x)$ is continuous on $R^+ \times R^+ \times R^n$ for $0 \leq s \leq t < \infty$ and satisfies

$$|g(t, s, x)| \leq K(t, s)|x| \tag{3.8}$$

with $\int_0^t K(t, s)ds \to 0$ as $t \to \infty$.

THEOREM 3.2. *Suppose that*

(i) A *is a stable matrix,*

(ii) (3.7) *and* (3.8) *hold.*

Then for sufficiently small μ, $x = 0$ *of* (3.6) *is* L^2*-stable.*

The proof is analogous to the one given in [9, pp. 294-296] with a choice of $f(s) = q^2s$ where $q > 1$ and hence omitted.

EXAMPLE 3.3. Consider the scalar integro-differential equation

$$x'(t) = -2x(t) + \int_0^t e^{-5t+2s}x(s)ds .$$

Choose $V(x) = \frac{1}{4}x^2$ and $f(s) = q^2s$ where $q > 1$. Here $K(t, s) = e^{-5t+2s}$. Thus all the conditions of Theorem 3.2 are satisfied.

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References

[1] T.A. Burton, "Stability theory for Volterra equations", *J. Differential Equations* 32 (1979), 101-118.

[2] T.A. Burton, "Uniform stability for Volterra equations", *J. Differential Equations* 36 (1980), 40-53.

[3] Ronald Grimmer and George Seifert, "Stability properties of Volterra integrodifferential equations", *J. Differential Equations* 19 (1975), 142-166.

[4] J. Hale, *Functional differential equations* (Applied Mathematical Sciences, 3. Springer-Verlag, New York, Heidelberg, Berlin, 1971).

[5] Junji Kato, "On Liapunov-Razumikhin type theorems", *Japan-United States seminar on ordinary differential and functional equations*, Kyoto, Japan, 1971, 54-65 (Lecture Notes in Mathematics, 243. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[6] V. Lakshmikantham and M. Rama Mohana Rao, "Integro-differential equations and extension of Liapunov's method", *J. Math. Anal. Appl.* 30 (1970), 435-447.

[7] B.S. Razumihin, "The application of Liapunov's method to problems in the stability of systems with delay", *Automat. Remote Control* 21 (1960), 515-520.

[8] George Seifert, "Liapunov-Razumikhin conditions for stability and boundedness of functional differential equations of Volterra type", *J. Differential Equations* 14 (1973), 424-430.

[9] George Seifert, "Liapunov-Razumikhin conditions for asymptotic stability in functional differential equations of Volterra type", *J. Differential Equations* 16 (1974), 289-297.

[10] Taro Yoshizawa, *Stability theory by Liapunov's second method* (Publications of the Mathematical Society of Japan, 9. The Mathematical Society of Japan, Tokyo, 1966).

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