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AN EXAMPLE OF RANK TWO SYMMETRIC OSSERMAN SPACE

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Recently, Blažić, Bokan and Rakić, obtained some classes of 4-dimensional Osserman pseudo-Riemannian manifolds. One of these is the class of rank 2 locally symmetric space endowed with an integrable para-quaternionic structure. In this paper we give an explicit construction of an example of a space of that kind.

0. INTRODUCTION

Let (M,g) be a 4-dimensional pseudo-Riemannian manifold of signature $(2,2)$. Let S_p^- (respectively, S_p^+) be the set of all unit timelike (spacelike) vectors in the tangent space T_pM. The curvature or Jacobi operator $R_X : Y \longmapsto R(Y,X)X$ is a symmetric endomorphism of T_pM which restricts to the endomorphism $\mathcal{K}_X$ of the orthogonal complement, $T_X S_p^\varepsilon$, of $X \in S_p^\varepsilon$ (where $\varepsilon = \pm$).

DEFINITION 0.1: M is timelike (respectively, spacelike) Osserman if the Jordan form of $\mathcal{K}_X$ is independent of $X \in S_p^-$ (respectively, $X \in S_p^+$) and of $p \in M$.

For Riemannian manifolds, Osserman [**5**] made the following conjecture:

CONJECTURE (OSSERMAN). If the eigenvalues of the Jacobi operator $\mathcal{K}_X$ are independent of the choice of unit vectors $X \in T_pM$ and of the choice $p \in M$, then either M is locally a rank-one symmetric space or M is flat.

Chi [**2**] proved the conjecture for $n \neq 4k$, $k > 1$. He has obtained some related results [**3**]. The Osserman conjecture and related topics were studied by Gilkey, Swann and Vanhecke [**4**].

Definition 0.1 is the natural generalisation of the Osserman condition in the pseudo-Riemannian case. If M is 4-dimensional pseudo-Riemannian manifold of signature (2,2), the Osserman condition is equivalent to the independency of the minimal polynomial of the Jacobi operator $\mathcal{K}_X$ of X and $p \in M$.

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The main result of this paper is the explicit construction of an example of a timelike Osserman rank two symmetric space which is given in the proof of Theorem 1.1. This example shows the difference between of the Osserman manifolds in Riemannian and pseudo-Riemannian geometry. In the paper [**1**] we proved the following theorem, on which we base the present construction.

THEOREM 0.2. *(i) There exists a symmetric pseudo-Riemannian space M with a metric of signature (2,2) such that the matrix $\mathcal{K}_{E_1}$ of its Jacobi operator in the orthonormal basis, $E_V = \{E_1, E_2, E_3, E_4\}$, where E_1 and E_2 are timelike vectors, and E_3 and E_4 are spacelike vectors, is :*

$$\mathcal{K}_{E_1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(ii) The nonzero components of the curvature tensor R in the basis E_V are :

$$(0.1) \qquad \begin{aligned} \frac{1}{2} &= R_{1221} = R_{4334} = R_{1331} = R_{4224} = R_{1224} = R_{1334} = R_{1342}, \\ -\frac{1}{2} &= R_{2113} = R_{2443} = R_{1234}, \end{aligned}$$

(iii) The holonomy algebra $\mathfrak{h}$ of M is 1-dimensional generated by:

$$(0.2) \qquad m = \begin{bmatrix} 0 & -1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & -1/2 & 0 \end{bmatrix}.$$

(iv) M can be endowed with an integrable para-quaternionic structure.

For the proof of this theorem see [**1**, Section 9.1].

REMARK A. From (0.1) one can see that the sectional curvature of the plane $E_1 \wedge E_4$ is vanishing and can easily verify that M is a rank two symmetric space.

In the proof of Theorem 0.2 we use Wu's theory on symmetric holonomy systems. We denote by H the 1-dimensional connected Lie subgroup of $GL(V)$, the Lie algebra of which is generated by endomorphism m. Wu has proved in [**7**], that every such H can be realised as the holonomy group of a simply connected symmetric space M whose tangent space at a point can be indentified with V (in our case $V = (\mathbb{R}^4, g)$ of signature (2,2)) and with curvature tensor R. For Wu's construction one has to calculate the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus V$,(as a direct sum of vector spaces and not as the direct sum of Lie algebras) where the Lie brackets $[\cdot,\cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ are defined by

$$(0.3) \qquad \begin{aligned} [h_1, h_2] &= [h_1, h_2]_{\mathfrak{h}} \quad \text{if} \quad h_1, h_2 \in \mathfrak{h}, \\ [h_1, x] &= h_1(x) \quad \text{if} \quad h_1 \in \mathfrak{h}, x \in V, \\ [x, y] &= R(x, y) \quad \text{if} \quad x, y \in V. \end{aligned}$$

Then if we show the solvability of G, the corresponding Lie group of $\mathfrak{g}$, we know that the homogeneous manifold $M = G/H$ is diffeomorphic to a Euclidean space (in our case $\mathbb{R}^4$).

1. Construction

In this section using Theorem 0.2 we construct a pseudo-Riemannian manifold of signature (2,2) on $\mathbb{R}^4$ such that this manifold satisfies the Osserman timelike condition with curvature tensor given by the formulas (0.1).

Theorem 1.1. *Let $M = \mathbb{R}^4$, let (u_1, u_2, u_3, u_4) be the Cartesian coordinates and*

$$g = \frac{1}{6}\big(v_2^2 dv_1 \otimes dv_1 + v_1^2 dv_2 \otimes dv_2 - v_1 v_2[dv_1 \otimes dv_2 + dv_2 \otimes dv_1]\big) - \frac{1}{2}\big([dv_1 \otimes dv_4 + dv_4 \otimes dv_1 + dv_2 \otimes dv_3 + dv_3 \otimes dv_2]\big). \tag{1.1}$$

Then $\big(\mathbb{R}^4, g\big)$ is a timelike Osserman rank two symmetric space.

Proof: Let $\mathfrak{g}$ be the 5-dimensional Lie algebra defined by relations (0.1)-(0.3). If we change basis of V and take the new basis $F = \{m, F_1, F_2, F_3, F_4\}$ where :

$$F_1 = \frac{(E_1 + E_4)}{2}, \quad F_2 = \frac{(E_2 - E_3)}{2}, \quad F_3 = \frac{(E_2 + E_3)}{2}, \quad F_4 = \frac{(E_1 - E_4)}{2}, \tag{1.2}$$

then the only nonvanishing comutators in the Lie algebra $\mathfrak{g}$ are :

$$[F_1, F_2] = m, \quad [m, F_1] = F_3, \quad \text{and} \quad [m, F_2] = -F_4. \tag{1.3}$$

The algebra $\mathfrak{g}$ defined by formulas (0.3) and (1.3), is nilpotent because $\mathcal{D}^4\mathfrak{g} = \{0\}$ (see [**6**]), and so it is solvable. Now, the Campbell-Hausdorff series and the formulas (1.3) enables us to express the group multiplication in terms of coordinates. More precisely, let X, Y be the elements of $\mathfrak{g}$, and let $X = (x_i)$, and $Y = (y_i), i = 0, \cdots, 4$, be their coordinates in basis F. Then we have

$$\begin{aligned}
&Z = X \cdot Y = Z(X,Y) = (z_0, z_1, \ldots, z_4), \text{ where:}\\
&z_0 = z_0(X,Y) = x_0 + y_0 + \frac{1}{2}(x_1 y_2 - x_2 y_1),\\
&z_1 = z_1(X,Y) = x_1 + y_1, \qquad z_2 = z_2(X,Y) = x_2 + y_2,\\
&z_3 = z_3(X,Y) = x_3 + y_3 + \frac{1}{2}(x_0 y_1 - x_1 y_0) + \frac{1}{12}(y_1 - x_1)(x_1 y_2 - x_2 y_1),\\
&z_4 = z_4(X,Y) = x_4 + y_4 - \frac{1}{2}(x_0 y_2 - x_2 y_0) - \frac{1}{12}(y_2 - x_2)(x_1 y_2 - x_2 y_1).
\end{aligned} \tag{1.4}$$

Since $\mathfrak{g}$ is nilpotent, the Campbell-Hausdorff formula defines a global diffeomorphism between G and $\mathfrak{g}$. Let M be the image of V via the exponential mapping, so $M \cong V$ is a homogeneous submanifold of G. Then we identify M with $\mathbb{R}^4$, forgetting the first component, so $M = \{(x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 \mid x_0 = 0\}$. For each $x \in G$, and $v \in M$, we consider the left translation $L_x v = x \cdot v$. In general, we get some element of G, even we take x from M, because V is not a subalgebra of $\mathfrak{g}$. Then we take the projection on M in the direction of $\exp \mathfrak{h}$, and we consider the mapping $L_x^h = \pi_M^h \circ L_x : M \longmapsto M$. For geometrical reasons we know that this mapping is well defined, which means there exists a unique $h \in \exp \mathfrak{h}$, $h = h(x, v)$, such that $(L_x v) \cdot h \in M$. Obviously, this map is a diffeomorphism since it is a composition of two diffeomorphisms. If $x = (x_0, x_1, x_2, x_3, x_4) \in G$ and $v = (0, v_1, v_2, v_3, v_4) \in M$ then $L_x^h v = (0, u_1, u_2, u_3, u_4) \cong (u_1, u_2, u_3, u_4)$, where

$$
\begin{aligned}
& u_1 = x_1 + v_1, \qquad u_2 = x_2 + v_2, \\
(1.5) \qquad & u_3 = x_3 + v_3 + x_0 v_1 + \frac{1}{2} x_0 x_1 + \frac{1}{6}(x_1 + 2v_1)(x_1 v_2 - x_2 v_1), \\
& u_4 = x_4 + v_4 - x_0 v_2 - \frac{1}{2} x_0 x_2 - \frac{1}{6}(x_2 + 2v_2)(x_1 v_2 - x_2 v_1).
\end{aligned}
$$

It still remains to calculate explicitly the metric on the manifold M. We know $\mathfrak{g} = T_e G = \mathfrak{h} \oplus V$, and $V \cong M$. But we changed the basis of V, and in our metric all of vectors from the basis $F_V = \{F_i, i = 1, \ldots, 4\}$ are isotropic and

$$
(1.6) \qquad \langle F_1, F_4 \rangle = \langle F_3, F_2 \rangle = -\frac{1}{2}, \qquad \text{and} \qquad \langle F_i, F_j \rangle = 0 \text{ otherwise.}
$$

Now if we take $X = \sum x_i F_i, Y = \sum y_i F_i \in V$ then, using the formulas (1.6), we get the metric on $T_0 M$ in the coordinates: $g_0(X, Y) = -(x_1 y_4 + x_2 y_3 + x_3 y_2 + x_4 y_1)/2$.

To finish our construction we use the formula for the transport of the metric from $T_0 M$ to $T_v M$: $g_v(X, Y) = g_0\big((L^h_{v^{-1}})_*(v)X, (L^h_{v^{-1}})_*(v)Y\big)$. If $X = (x_i) \in G$, from relations (1.4) we find the coordinates of its group inverse $X^{-1} = (-x_i)$, $i = 0, \cdots, 4$. Now, we calculate $(L^h_{v^{-1}})_*(v)$ from (1.5) and then using the above formula for transporting the metric, we get the metric given by (1.1). We see from Remark A that the manifold M is of rank two. ∎

REMARK B. By standard calculation of the curvature tensor from the metric on M we get that the only nonzero component of the curvature tensor in the basis F_V is $R^{M_F}_{1221} = 1/2$. But the prescribed curvature tensor R, given by the components (0.1), is calculated in basis E_V. One can easily find the connection between the bases E_V and F_V using (1.2), and after that verify that all components of the tensor R^M are the same as those of the prescribed tensor R.

REFERENCES

[1] N. Blažić, N. Bokan and Z. Rakić, 'Characterization of 4-dimensional Osserman pseudo-Riemannian Manifolds', (preprint).

[2] Q.S. Chi, 'A curvature characterization of certain locally rank-one symmetric spaces', *J. Differential Geom.* **28** (1988), 187–202.

[3] Q.S. Chi, 'Curvature characterization and classification of rank-one symmetric spaces', *Pacific. J. Math.* **150** (1991), 31–42.

[4] P.B. Gilkey, A. Swann and L. Vanhecke, 'Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator', *Quart. J. Math. Oxford.* **46** (1995), 299–320.

[5] R. Osserman, 'Curvature in the eighties', *Amer. Math. Monthly* **97** (1990), 731–756.

[6] M.M. Postnikov, *Lectures in geometry; Lie groups and Lie algebras*, (English translation) (Mir Publishers, Moskva, 1986).

[7] H. Wu, 'Holonomy groups of indefinite metrics', *Pacific. J. Math.* **20** (1967), 351–392.

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