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PULLBACKS IN REGULAR CATEGORIES

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Given a pair of maps

$$A \xrightarrow{g} B \xrightarrow{f} C$$

in a category, we would like to know whether they form part of a pullback diagram as follows:

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g'} \\ A & \xrightarrow[f']{} & D \end{array}$$

I am indebted to Basil Rattray for mentioning the solution of this problem for the category of sets. Here we shall solve it for any regular category in the sense of Barr [1].

It will be useful to make the following definition.

Given three maps as follows:

$$A \xrightarrow{g} B \underset{v}{\overset{u}{\leftleftarrows}} K,$$

we say that they have a common pullback

$$A \underset{v'}{\overset{u'}{\leftleftarrows}} P \xrightarrow{h} K$$

provided both

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u} \\ P & \xrightarrow[h]{} & K \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle v'}\uparrow & & \uparrow{\scriptstyle v} \\ P & \xrightarrow[h]{} & K \end{array}$$

are pullback squares.

PROPOSITION 1. *Let $\underline{A}$ be a regular category. A pair of maps $A \xrightarrow{g} B \xrightarrow{f} C$ is part of a pullback if and only if the maps $A \xrightarrow{g} B \underset{v}{\overset{u}{\leftleftarrows}} K$, with $K \underset{v}{\overset{u}{\rightrightarrows}} B$ being*

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the kernel pair of $B \xrightarrow{f} C$, *have a common pullback such that* (u', v') *is a kernel pair.*

We shall use the following properties of a regular category [1]:

(1) Every map has a kernel pair.

(2) Every pair of maps has a coequalizer.

(3) Every map can be factored into a mono followed by a regular epi.*

(4) In the commutative diagram

$$\begin{array}{ccccc} X_1 & \overset{d^0}{\underset{d^1}{\rightrightarrows}} & X_0 & \xrightarrow{d} & X \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ Y_1 & \overset{e^0}{\underset{e^1}{\rightrightarrows}} & Y_0 & \xrightarrow{e} & Y \end{array}$$

let top and bottom rows be exact (that is, at the same time a kernel pair and a coequalizer). Then, if the square

$$\begin{array}{ccc} X_1 & \xrightarrow{d^0} & X_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ Y_1 & \xrightarrow{e^0} & Y_0 \end{array}$$

is a pullback, then so is

$$\begin{array}{ccc} X_0 & \xrightarrow{d} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y_0 & \xrightarrow{e} & Y \end{array}$$

Proof. (i) Suppose $A \xrightarrow{g} B \xrightarrow{f} C$ is part of a pullback square

$$\begin{array}{ccc} D & \xrightarrow{g'} & C \\ {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle f} \\ A & \xrightarrow[g]{} & B \end{array}$$

Let (u, v) be a kernel pair of f, $A \xleftarrow{u'} P_u \xrightarrow{h} K$ and $A \xleftarrow{v'} P_v \xrightarrow{h'} K$ be pullbacks of $A \xrightarrow{g} B \xleftarrow{u} K$ and $A \xrightarrow{g} B \xleftarrow{v} K$, respectively. Then we obtain the following

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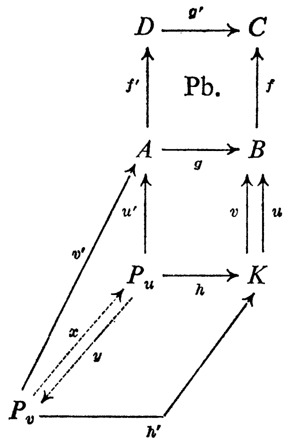
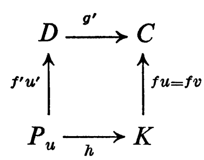
* A *regular* epi is a coequalizer of some pair of maps.

diagram:



Hence



is a pullback.

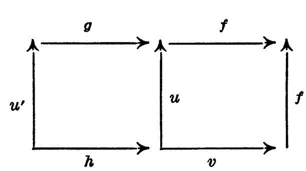
Since $g'f'v'=(fv)h'$, $\exists!x \ni f'u'x=f'v'$ and $hx=h'$. Similarly $\exists!y \ni f'v'y=f'u'$ and $h'y=h$. Therefore

$$f'u'xy=f'v'y=f'u' \quad \text{and} \quad hxy=h.$$

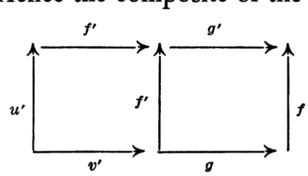
By the uniqueness property of pullbacks $xy=1$.

Similarly $yx=1$. Thus $P_u \simeq P_v$. Without loss in generality we may assume $P_u=P_v$. So there is a common pullback $A \underset{v'}{\overset{u'}{\leftleftarrows}} P \xrightarrow{h} K$ of $A \xrightarrow{g} B \underset{v}{\overset{u}{\leftleftarrows}} K$. We note also that $f'u'=f'v'$.

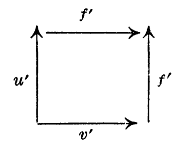
It remains to show that (u', v') is a kernel pair. In the following diagram all squares are pullbacks:



But $fg=g'f'$ and $vh=gv'$. Hence the composite of the following squares



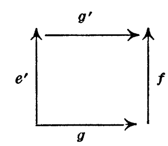
is a pullback. We recall that the right square is also a pullback. Therefore so is the left square, i.e.



is a pullback.

We deduce that (u', v') is a kernel pair of f'.

(ii) Let $A \overset{u'}{\underset{v'}{\leftleftarrows}} P \xrightarrow{h} K$ be the common pullback of $A \xrightarrow{g} B \overset{u}{\underset{v}{\leftleftarrows}} K$ and assume that (u', v') is a kernel pair. Let e' be the coequalizer of (u', v'). Since $fgu' = fuh = fvh = fgv'$, there exists a unique g' such that $g'e' = fg$. We claim that



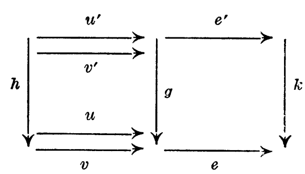
is a pullback.

Let $f = me$ be the factorization of f such that m is a mono and e is a regular epi. Hence e is a coequalizer of (u, v). Since $egu' = egv'$, there exists a unique map k such that $ke' = eg$. We obtain

$$mke' = meg = fg = g'e',$$

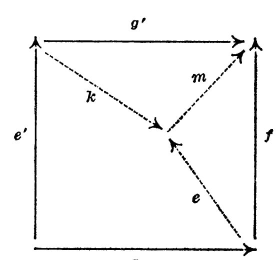
and therefore $mk = g'$.

We have exact top and bottom rows in the following diagram:



Also the left squares are pullbacks. Hence, by property 4 of a regular category, the right square is also a pullback.

It follows that



is a pullback, since $g' = mk$, $f = me$ and m is mono.

The proof is now complete.

It is of interest to know whether the pullback constructed in Proposition 1 is essentially unique. This is the case whenever f is a regular epi.

PROPOSITION 2. *Let $\underline{A}$ be a regular category. Given a pair of maps $A \xrightarrow{g} B \xrightarrow{f} C$ in $\underline{A}$, there is an essentially unique pair of maps $A \xrightarrow{f_0} D \xrightarrow{g_0} C$ such that*

$$\begin{array}{ccc} D & \xrightarrow{g_0} & C \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle f} \\ A & \xrightarrow[g]{} & B \end{array}$$

is a pullback, provided that f is a regular epi.

Proof. From Proposition 1, we constructed the pullback square

$$\begin{array}{ccc} \cdot & \xrightarrow{g'} & \cdot \\ {\scriptstyle e'}\uparrow & & \uparrow{\scriptstyle f} \\ \cdot & \xrightarrow[g]{} & \cdot \end{array}$$

where e' is the coequalizer of its kernel pair (u', v'). We also showed that if

$$\begin{array}{ccc} \cdot & \xrightarrow{g_0} & \cdot \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle f} \\ \cdot & \xrightarrow[g]{} & \cdot \end{array}$$

is a pullback, then (u', v') is also a kernel pair of f_0. Since in a regular category pullbacks preserve regular epis, f_0 is a regular epi. Then f_0 is a coequalizer of its own kernel pair (u', v'). Now both e' and f_0 are coequalizers of (u', v'). Hence there exists an iso i such that $ie'=f_0$. Since $g'e'=fg=g_0f_0$, we obtain $g'e'=g_0ie'$; and hence $g'=g_0i$. We have thus proved Proposition 2.

The following example shows that we cannot drop the condition that f is a regular epi.

EXAMPLE. Let f be a mono and take $g=1$. The following squares give two different pullbacks which have $\xrightarrow{1}\xrightarrow{f}$ as part of the squares:

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle f} \\ \cdot & \xrightarrow[1]{} & \cdot \end{array} \quad \text{and} \quad \begin{array}{ccc} \cdot & \xrightarrow{1} & \cdot \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ \cdot & \xrightarrow[1]{} & \cdot \end{array}$$

REFERENCE

1. M. Barr, *Algèbre des Catégories—Catégories exactes*, C.R. Acad. Sci. Paris, t.**272** (1971), 1501–1503.

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