POST-NEWTONIAN TREATISE ON THE ROTATIONAL MOTION OF A FINITE BODY

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ABSTRACT. The definition of the angular momentum of a finite body is given in the post-Newtonian framework. The non-rotating and the rigidly rotating proper reference frame(PRF)s attached to the body are introduced as the basic coordinate systems. The rigid body in the post-Newtonian framework is defined as the body resting in a rigidly rotating PRF of the body. The feasibility of this rigidity is assured by assuming suitable functional forms of the density and the stress tensor of the body. The evaluation of the time variation of the angular momentum in the above two coordinate systems leads to the post-Newtonian Euler's equation of motion of a rigid body. The distinctive feature of this equation is that both the moment of inertia and the torque are functions of the angular velocity and the angular acceleration. The obtained equation is solved for a homogeneous spheroid suffering no torque. The post-Newtonian correction to the Newtonian free precession is a linear combination of the second, fourth and sixth harmonics of the precessional frequency. The relative magnitude of the correction is so small as of order of 10^{-23} in the case of the Earth.

1. INTRODUCTION

The theory on the motion of bodies in the gravitational field is not completed until it is formulated in the framework of the general relativity. The post-Newtonian formalism (Misner et al., 1970; Will, 1981) is a suitable approach to deal with slow motions in a weak gravitational field such as the rotation of the Earth. In this paper, we construct a post-Newtonian theory of the rotational motion of a finite body with use of the concept of the proper reference frame (PRF) of a massive body (Fukushima et al., 1986) and a definition of the rigid body in the post-Newtonian framework.

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2. ANGULAR MOMENTUM OF A FINITE BODY

In the relativistic theory, the sub-spacetime which a finite body (hereafter named the central body) occupies is a curved 4-dimensional rod called a world tube. A 3-dimensional shape of the central body seen at an instance is a cross section of the world tube cut by a 3-dimensional plane, which is called the equal-time plane.

If the non-rotating PRF of the central body is chosen as the basic coordinate system, the post-Newtonian expression of the angular momentum vector becomes as

$$J = \int_S (1 + \alpha / c^2) x \times (\rho v + \Delta \underset{\sim}{p} v / c^2) d^3x \qquad (2\text{-}1)$$

where

$$\alpha = \Pi + 2 \,\mathrm{Tr}\, \underset{\sim}{\chi} + v^2 + p / \rho$$

$$p = \mathrm{Tr}\, \underset{\sim}{p} / 3 \quad , \qquad \Delta \underset{\sim}{p} = \underset{\sim}{p} - p \underset{\sim}{1}$$

Here the integration is done on the equal-time plane S, **x** and **v** are the position and the coordinate velocity in the non-rotating PRF, respectively, ρ is the rest mass density, Π is the specific energy density, $\underset{\sim}{\chi}$ is the tensor force function, $\underset{\sim}{p}$ is the stress tensor in the non-rotating PRF, c is the speed of light, and $\underset{\sim}{1}$ is the 3-dimensional unit tensor.

In the non-rotating PRF, the expressions of $\underset{\sim}{\chi}$ and the Newtonian force function ϕ become simple as

$$\phi = \phi^{+} + [\phi^{*} - \phi^{*}_0 - x\cdot(dv/dt)_0] \qquad (2\text{-}2)$$

$$\underset{\sim}{\chi} = \gamma \phi^{+} \underset{\sim}{1} \qquad (2\text{-}3)$$

when the background metric is the PPN metric (Fukushima et al., 1986). Here the superfix + denotes the contribution of the central body itself, the superfix * denotes the contribution of external bodies, the suffix 0 denotes the values at the barycenter of the central body, and $(dv/dt)_0$ is the acceleration of the central body. The simpleness of expressions (2-2) and (2-3) is the reason why we prefer the non-rotating PRF to other comoving coordinate systems.

3. RIGID BODY IN THE POST-NEWTONIAN FRAMEWORK

The concept of the rigid body was the most important tool in developing the Newtonian theory of motion of a finite body (Goldstein, 1980). In the relativistic theory, the Euclidean distance on which the Newtonian rigidity is based changes easily when it is measured on another equal-time plane. Then we define the relativistic rigid body as the body that the space coordinates of all particles constituting it are time-independent in a certain rigidly rotating PRF.

Now a definition of the rigid body is obtained. But is it realizable? The answer is that the feasibility of this rigidity depends on the character of the matter constituting the body. In other words, the concept of the rigidity forces a constraint condition on the functional forms of the density distribution and the stress tensor of the body.

The equilibrium condition of a rigidly rotating body in the rotating PRF is expressed as

$$\nabla \cdot p / \rho = \nabla [\phi + (\omega \times x)^2/2] - \dot{\omega} \times x + \{ \text{post-Newtonian terms} \} \tag{3-1}$$

where ∇ is the 3-dimensional gradient operator, ω is the angular velocity, x is the space coordinates in the rotating PRF, and $\dot{}$ is the partial derivative with respect to the time coordinate. The post-Newtonian terms in equation (3-1) are all derived from the Newtonian solution of (3-1).

If (3-1) is regarded as a partial differential equation for ρ and $\underset{\sim}{p}$, it is satisfied by a number of set of $\rho(x)$ and $\underset{\sim}{p}(x)$. For example, the following set is one of the Newtonian solutions of it;

$$\begin{cases} \rho = \text{constant}, \\ p^n{}_m = \rho [\phi + (\omega \times x)^2/2] \delta^n{}_m \\ \quad - [nmk] \rho \omega_k [(x_n)^2 - (x_m)^2]/2 \end{cases} \tag{3-2}$$

where $[nmk]$ is the completely antisymmetric 3-dimensional symbol. The post-Newtonian correction of $\underset{\sim}{p}$ is not needed for the following discussion. Thus the rigid body is well-defined if the density and the stress tensor have suitable functional forms of space coordinates.

4. POST-NEWTONIAN EQUATION OF ROTATIONAL MOTION

From the definition of the rigidity in the previous section, the angular momentum of a rigid body is rewritten as

$$J = \underset{\sim}{I}\, \omega \tag{4-1}$$

Here $\underset{\sim}{I}$ is the post-Newtonian moment of inertia tensor defined as

$$\underset{\sim}{I} = - \int_S (1 + \alpha / c^2) \underset{\sim}{X} (\rho \underset{\sim}{1} + \Delta \underset{\sim}{p} / c^2) \underset{\sim}{X}\, d^3x \tag{4-2}$$

where

$$X^i{}_j = [ijm]\, x_m$$

and d^3x is the volume element in the rigidly rotating PRF. We note that the moment of inertia is not a constant but a function of ω, $\dot{\omega}$ and t in the post-Newtonian framework. This is because the velocity field and the external gravitational field increase the effective mass density.

Differentiating (4-1) with respect to t, we find

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$$\underset{\sim}{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \underset{\sim}{I}\boldsymbol{\omega} + [\, \{ \ddot{\boldsymbol{\omega}} \cdot (\partial / \partial \dot{\boldsymbol{\omega}}) \} + \{ \dot{\boldsymbol{\omega}} \cdot (\partial / \partial \boldsymbol{\omega}) \} + (\partial / \partial t) \underset{\sim}{I} \,] \, \boldsymbol{\omega} = \mathbf{N} \qquad (4\text{-}3)$$

This is the post-Newtonian Euler's equation of rotational motion and

$$\mathbf{N} = (d/dt) \int_S (1 + \alpha / c^2) \, x \times (\rho \, \mathbf{v} + \Delta \underset{\sim}{p} \, \mathbf{v}/c^2) \, d^3x \qquad (4\text{-}4)$$

is the expression of the torque in the rigidly rotating PRF. Here $\partial \underset{\sim}{I} / \partial t$ is due to the time variation of the external gravitational field.

Although the net torque vanishes only when there is no external gravitational field, we assume that the torque caused by the central body itself vanishes even in the case there is an external gravitational field. If this assumption is accepted, the torque **N** is rewritten as

$$\begin{aligned} \mathbf{N} = & \int_S (1 + \alpha^+ / c^2) \, x \times (\rho \, \underset{\sim}{1} + \Delta \underset{\sim}{p}^+ / c^2) (dv/dt)^* \\ & + [\int_S x \times \underset{\sim}{p}^* \, (dv/dt) \, d^3x \\ & + \int_S x \times (d\underset{\sim}{p}/dt)^* \, \mathbf{v} \, d^3x \\ & + \int_S \mathbf{v} \times \Delta \underset{\sim}{p}^* \, \mathbf{v} \, d^3x \,] / c^2 \end{aligned} \qquad (4\text{-}5)$$

The post-Newtonian torque is a function of $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$ and t. Especially the torque is not always reduced to zero even if the body has a spherically symmetric density distribution.

Now we can compute the time development of the angular velocity of a finite body in the post-Newtonian framework if the orientation of the body and the motion of external bodies are given. Next we must know how the method to obtain the orientation of the body from its angular velocity will change in the post-Newtonian formalism. The introduction of the rigidly rotating and the non-rotating PRFs of the central body makes the answer to this question very simple, that is, no changes. In fact the orientation of the rigidly rotating central body in its non-rotating PRF is described by the usual Euler's angles in just the same manner as in the Newtonian mechanics. We remark that the well-known geodesic precession does not appear here (Fukushima et al., 1986).

5. FREE ROTATIONAL MOTION IN THE POST-NEWTONIAN FRAMEWORK

Equation (4-3) can be solved in the case of no torque if a suitable combination of ρ and $\underset{\sim}{p}$ satisfying the equilibrium condition (3-1) is given. In this section we assume ρ and $\underset{\sim}{p}$ are given by the set (3-2). Also we assume that the body has a spheroidal figure. We should keep it in our mind that the following solution is only one of many solutions of (4-3) and has no definite meanings.

Then the equation of motion is rewritten as

$$\left\{\begin{aligned} A\dot{\omega}_1 &= (B-C)\,\omega_2\omega_3 - \dot{A}\,\omega_1 - G\,[(\omega_2^2 - \omega_3^2)\,\dot{\omega}_1 \\ &\quad - \omega_1\omega_2\dot{\omega}_2 - \dot{\omega}_2\dot{\omega}_3 - \omega_3\ddot{\omega}_2\,]\,/\,c^2 \\ B\dot{\omega}_2 &= (C-A)\,\omega_3\omega_1 - \dot{B}\,\omega_2 - G\,[(\omega_1^2 - \omega_3^2)\,\dot{\omega}_2 \\ &\quad - \omega_1\omega_2\dot{\omega}_1 + \dot{\omega}_1\dot{\omega}_3 + \omega_3\ddot{\omega}_1\,]\,/\,c^2 \\ C\dot{\omega}_3 &= (A-B)\,\omega_1\omega_2 - \dot{C}\,\omega_3 - G\,[(\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2)\,\omega_3 \\ &\quad - (\omega_1\ddot{\omega}_2 - \omega_2\ddot{\omega}_1)\,]\,/\,c^2 \end{aligned}\right. \tag{5-1}$$

Here A, B, and C are the post-Newtonian principal moments of inertia defined as

$$\left\{\begin{aligned} A &= I_{11} + I_{33} + [\,A_1\,\omega_1^2 + A_2\,\omega_2^2 + A_3\,\omega_3^2\,]\,/\,c^2 \\ B &= I_{11} + I_{33} + [\,A_2\,\omega_1^2 + A_1\,\omega_2^2 + A_3\,\omega_3^2\,]\,/\,c^2 \\ C &= 2\,I_{11} + [\,A_3\,(\omega_1^2 + \omega_2^2) + 2A_3\,\omega_3^2\,]\,/\,c^2 \end{aligned}\right. \tag{5-2}$$

where

$$A_1 = 3\,(\;I_{1111} + 2\,I_{1133} + I_{3333}\,)\,/\,2$$

$$A_2 = 3\,(\,-\,I_{1122} + 2\,I_{1133} + I_{3333}\,)\,/\,2$$

$$A_3 = 3\,(\;I_{1111} + I_{1122}\,)\,/\,2$$

$$G = (\,I_{1122} - I_{1133}\,)\,/\,2$$

$$I_{nm} = \int_S \rho\,[\,1 + \{\,\Pi + (\,6\,\gamma + 1\,)\,\phi\,\}/c^2\,]\,x_n\,x_m\,d^3x$$

$$I_{nmpq} = \int_S \rho\,x_n\,x_m\,x_p\,x_q\,d^3x$$

The solution of equation (5-1) is obtained as

$$\left\{\begin{aligned} \omega_1 + i\,\omega_2 &= (\,\omega_\rho + \omega_{\rho 4}\cos 4\theta\,)\,e^{i\lambda} \\ \omega_3 &= \omega_z + \omega_{z4}\cos 4\theta \end{aligned}\right. \tag{5-3}$$

where

$$\lambda = \theta + \lambda_2 \sin 2\theta + \lambda_4 \sin 4\theta \tag{5-4}$$

$$\theta = \omega_p\,(\,t - t_o\,) \tag{5-5}$$

Here ω_ρ, ω_z and t_o are arbitrary constants and

$$\omega_p = [\,(\,C^* - A^*\,)/\,A^*\,]\,\omega_z$$

$$*\,[\,1 + (\,G\,/\,A^*\,)\,(\,C\,\omega_z^2\,/\,A^* - \omega_\rho^2\,) - \epsilon\,/\,4\,]$$

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$$\omega_{\rho 4} = (4 A^* - 3 C^*) \omega_\rho \epsilon / [48 (C^* - A^*)]$$
$$\omega_{z4} = - (A^* \omega_\rho)^2 \epsilon / [16 C^* (C^* - A^*) \omega_z]$$
$$\lambda_2 = A^* \epsilon / [4 (C^* - A^*)]$$
$$\lambda_4 = 3 \epsilon / 16 - \omega_{z4} / \omega_z$$
$$A^* = I_{11} + I_{33} + [(A_1 + A_3) \omega_\rho^2 / 2 + A_3 \omega_z^2]$$
$$C^* = 2 I_{11} + A_3 (\omega_\rho^2 + 2 \omega_z^2)$$
$$\epsilon = (A_1 - A_2) \omega_\rho^2 / [A^* c^2]$$

The post-Newtonian correction of the solution is graphically shown in the orbit of the polar motion as a superposition of the orbital deviations with two, four and six leaves on a circular orbit which represents the Newtonian free precession. Similarly the correction to the rotational velocity, which is constant in the Newtonian case, is the fourth higher harmonic of the frequency of the free precession. Also the frequency of the free precession is slightly shifted. However the shift of frequency will be cancelled by the adjustment of unknowns such as the principal moments of inertia. The relative magnitude of these corrections are of order of ϵ , which is very small as 10^{-23} in the case of the Earth.

References

Goldstein, H.:1980, Classical mechanics, 2nd ed., Addison-Wesley Publ. Co. Inc., Reading, Mass.

Fukushima, T., Fujimoto, M.-K., Kinoshita, H., and Aoki, S.:1986, 'Coordinate systems in the general relativistic framework', Proceedings of IAU Symp. No.114 held at Leningrad, May, 1985, D.Reidel Publ. Co., Dordrecht, Holland.

Misner, C.W., Thorne, K.S., and Wheeler, J.A.:1970, Gravitation, W.H.Freeman and Co., San Francisco.

Will, C.M.:1981, Theory and experiment in gravitational physics, Cambridge Univ. Press, Cambridge.

DISCUSSION

Vasiev : what can be said about the accuracy of the solution of the Fermi-Walker transportation ?

Fukushima : the equation of the F.-W. transportation was solved exactly in the post Newtonian approximation.

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