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STRONGLY RIGHT FBN RINGS

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The annihilator of a finite generated β-critical module is called a *β-coprimative ideal.* A prime ideal P is called *β-prime* if the Krull dimension of R/P is β. This paper is concerned with the relationship between the set of β-prime ideals and the set of minimal β-coprimitive ideals over a strongly right FBN ring. It is shown that there exists a one-to-one correspondence between the set of β-prime ideals and the set on minimal β-coprimitive ideals over a strongly right FBN ring R for $-1 < \beta \leqslant \alpha$, where α is the Krull dimension of R.

1. INTRODUCTION

Jategaonkar has shown in **[9]** that the coprimitive ideals in an FBN ring are prime. But in general this is not true even for right FBN rings. In this paper, we are interested in the lattice of these coprimitive ideals and in particular the question as to when there exists a unique minimal β-coprimitive ideal for each β-prime. This question was considered by Boyle and Feller in **[4]**. They showed that if R is strongly right FBN, then there exists a $1-1$ correspondence between the isomorphism classes of indecomposable injectives and the minimal β-coprimitive ideals for $-1 < \beta \leqslant \alpha$, where α is the Krull dimension of the ring R. This paper was motivated by an effort to obtain a converse of Boyle and Feller's Theorem.

A right FBN ring R has the property that given a finitely generated module M, R satisfies the descending chain condition on annihilators of subsets of M. A module with this property is said to be a *$\triangle$-module*. For a right noetherian ring R with Krull dimension α, the set of all right ideals H of R such that the Krull dimension of R/H is strictly less than β forms a topology M_β for each β, $-1 < \beta \leqslant \alpha$. A *strongly right FBN ring* is defined to be a ring such that every M_β-critical module is a $\triangle$-module.

In Section 2, we consider the β-coprimitive ideals which are the annihilators of finitely generated β-critical modules. We prove that there exists a $1-1$ correspondence between the β-prime ideals and the minimal β-coprimitie ideals over a strongly right FBN ring. We also characterise strongly right FBN rings. From this characterisation we determine a converse of Boyle and Feller's Theorem **[4**, 3.4**]**.

Throughout this paper R denotes an associative ring with identity. All modules are right unital. If L is a subset of a module M, then the *annihilator* of L in R is

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$\operatorname{ann}(L) = \{r \in R \mid Lr = 0\}$. The notation $N \leqslant_e M$ means that N is an essential submodule of M. The *injective hull* of M is denoted by $E(M)$.

For a module M, *the Krull dimension* of M will be denoted by $|M|$. The definition of Krull dimension and some related results can be found in [**8**]. If U is a uniform module that contains a critical submodule C with $|C| = \beta$, then we write $cr\,|U| = \beta$. If I is an indecomposable injective module with $cr\,|I| = \beta$, then I is called *β-indecomposable injective.* A module M is said to be *β-smooth* if for any nonzero submodule N having Krull dimension, $|N| = \beta$. A module M is said to satisfy the *large condition* if $|M/N| < \beta$ for all essential submodules N of M.

If M is a uniform module, then the assassinator of M in R is denoted by $\operatorname{ass}(M) = \{r \in R \mid (\exists 0 \neq N \leqslant M)(Nr = 0)\}$. If a ring R has Krull dimension, then $\operatorname{ass}(M) = \operatorname{ann}(C)$ for some critical submodule C of M.

2. Strongly right FBN rings

Throughout this section we assume that R is a right noetherian ring with Krull dimension α. Since R is right noetherian, the set of right ideals, $M_\beta = \{H_R \leqslant R \mid |R/H| < \beta\}$ where $-1 < \beta \leqslant \alpha$, forms a topology in the sense of [**11**]. Using this topology, we can define a torsion theory. A module M is called *M_β-torsionfree* if $\operatorname{ann}(x) \notin M_\beta$ for all nonzero elements x of M, and *M_β-torsion* if $\operatorname{ann}(x) \in M_\beta$ for all x in M. Then a module M is called *M_β-critical* if M is M_β-torsionfree and if M/N is M_β-torsion for all nonzero submodules $N \leqslant M$. If M is M_β-critical, then M is uniform and β-smooth.

LEMMA 2.1. *Let M be a module. Then the following statements are equivalent:*

(1) *M is M_β-critical;*
(2) *every submodule of M with Krull dimension is β-critical;*
(3) *every finitely generated submodule of M is β-critical.*

R is *right bounded* if every essential right ideal of R contains a nonzero ideal of R. A right noetherian ring R is said to be *right fully bounded* (right FBN) if each prime factor ring of R is called an FBN ring. It was shown by Amitsur in [**1**] that a prime ring which satisfies a polynomial identity is right bounded. Cauchon has shown in [**5**] that a right noetherian ring R is right FBN if and only if every finitely generated module is a $\triangle$-module.

A right noetherian ring is defined in [**4**] to be *strongly right FBN* if every M_β-critical module is a $\triangle$-module. Since $\triangle$-modules play an important role in the study of strongly right FBN rings, we include the following theorem that summaries some of the known properties of $\triangle$-modules.

THEOREM 2.2. [**10**]*: Let M be β-smooth. The following statements are equiva-*

lent:

(1) $|R/\operatorname{ann}(M)|$ *is* β;
(2) $R/\operatorname{ann}(M)$ *is* β*-smooth;*
(3) M *is a* $\triangle$*-module;*
(4) M *is finitely annihilated.*

If R is a strongly right FBN ring, then it is easy to show that R is right FBN by Theorem 2.2.

Over an FBN ring, every finitely generated critical module is a uniform prime $\triangle$-module by [**5**] and [**9**]. This enables us to show that every M_β-critical module M has a prime annihilator and hence $\operatorname{ann}(M) = \operatorname{ann}(C)$ for every finitely generated β-critical submodule C of M. Since C is a $\triangle$-module, $|R/\operatorname{ann}(M)| = |R/\operatorname{ann}(C)| = |C| = \beta$. Therefore M is a $\triangle$-module by Theorem 2.2. This argument shows that an FBN ring is strongly right FBM.

In particular a strongly right FBN ring is a class of rings between FBN-rings and right FBN-rings.

PROPOSITION 2.3. *If R is a right noetherian PI-ring, then R is strongly right FBN.*

PROOF: It is known [**1**] that a noetherian PI-ring is FBN, and Cauchon has shown in [**6**] that a right noetherian prime PI-ring is left noetherian. Also Boyle and Feller have shown in [**4**] that R is strongly right FBN if and only if R/P is strongly right FBN for all minimal prime ideals P. Hence if P is a minimal prime ideal of R, then R/P is a right noetherian prime PI-ring, and thus R/P is a noetherian PI-ring. Therefore R/P is FBN. By the above remark, R/P is strongly right FBN for all minimal prime P, and hence so is R. ∎

For FBN rings the annihilator of a critical module is prime, but this is not true for strongly right FBN rings. For example, if $R = \begin{bmatrix} F & F[x] \\ 0 & F[x] \end{bmatrix}$ and $M_R = \begin{bmatrix} F & f[x] \\ 0 & 0 \end{bmatrix}$, then R is a strongly right FBN ring and M is a 1-critical module. However $\operatorname{ann}(M) = 0$ is not a prime ideal. We introduce the definition of the annihilator of a critical module and examine the relationship between prime ideals and these annihilator ideals over a strongly right FBN ring.

An ideal D is called a *β-coprimitive ideal* if $D = \operatorname{ann}(C)$ for some finitely generated β-critical module. A β-coprimitive ideal will be called a *minimal β-coprimitive ideal* if it is minimal in the collection of β-coprimitive ideals. A prime ideal is called *β-prime* if $|R/P| = \beta$. Every β-prime is a β-coprimitive ideal.

The *critical socle* of a module M is the sum of the critical submodules and is denoted by SM. If U is a uniform module with $cr\,|U| = \beta$, then every finitely generated submodule of SU is critical by [**2**, 3.1]. Hence SU is M_β-critical by *Lemma 2.1*. Therefore if R is strongly right FBN, then SU is a $\triangle$-module. This gives the first implication in the following lemma.

LEMMA 2.4. *The following statements are equivalent for a right noetherian ring* R:

(1) *R is strongly right FBN,*
(2) *SU is a $\triangle$-module for all uniform modules U,*
(3) *SI is a $\triangle$ for all indecomposable injectives I. Moreover, in this situation,* $\operatorname{ann}(SI)$ *is a minimal coprimitive ideal.*

PROOF: It is enough to prove that (3) implies (1). Let M be M_β-critical. Since $SE(M)$ is a $\triangle$-module and $M \subseteq SE(M)$, then M is a $\triangle$-module.

Since SI is a $\triangle$-module, $\operatorname{ann}(SI) = \bigcap_{i=1}^{n} \operatorname{ann}(x_i)$ for some nonzero elements x_i of SI. Since $x_1R + \ldots + x_nR$ is a β-critical submodule of M by Lemma 2.1, then $\operatorname{ann}(SI) = \operatorname{ann}(x_1R + \ldots + x_nR)$ is a β-coprimitive ideal. Now suppose that D is a β-coprimitive ideal contained in $\operatorname{ann}(SI)$. Then $D = \operatorname{ann}(C)$ for some finitely generated β-critical module C and $(SI)D = 0$. By [**4**, 2.2], $EE(C) \simeq E(SI) = I$. Therefore, $\operatorname{ann}(SI) = D$ and hence $\operatorname{ann}(SI)$ is a minimal β-coprimitive ideal. ∎

In [**3**], it is shown that there is a $1-1$ correspondence between the isomorphism classes of α-indecomposable injective modules and the minimal α-coprimitive ideals. However, in general, this is not true for $\beta < \alpha$. For example, consider the ring $R = \begin{bmatrix} F & A/xA \\ 0 & A \end{bmatrix}$, where $A = F[x, (')][z]$, F is a field with derivation $(')$ as in [**7**, p. 55] and z commutes with x. Note that $|R| = 2$. If $C_1 = \begin{bmatrix} F & A/xA \\ 0 & 0 \end{bmatrix}$, then C_1 is 1-critical and $\operatorname{ann}(C_1) = 0$. Hence 0 is a minimal 1-coprimitive ideal of R. If $C_2 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} / \begin{bmatrix} 0 & 0 \\ 0 & zA \end{bmatrix}$, then C_2 is also 1-critical and $E(C_1) \neq E(C_2)$. Since 0 is the only minimal 1-coprimitive ideal, there is no 1–1 correspondence between the isomorphism classes of 1-indecomposable injectives and the minimal 1-coprimitive ideals.

PROPOSITION 2.5. *Let R be a strongly right FBN ring.*

(1) *If D is a minimal β-coprimitive ideal of R, then $D = \operatorname{ann}(SI)$ for a β-indecomposable injective I, where $-1 < \beta \leqslant \alpha$.*
(2) *Every β-prime ideal contains exactly one minimal β-coprimitive ideal and every minimal β-coprimitive ideal is contained in a unique β-prime ideal, where $-1 < \beta \leqslant \alpha$.*

(3) *Let $P = \text{ass}(I)$ and $D = \text{ann}(SI)$ for some β-indecomposable injective I. Then P is the maximal β-coprimitive ideal containing D and is the unique β-prime ideal containing D.*

PROOF: **(1)** If D is a minimal β-coprimitive ideal, then $D = \text{ann}(C)$ for some finitely generated β-critical module C. On the other hand, $\text{ann}(SE(C))$, then $\text{ann}(SE(C)) = \text{ann}(C) = D$ by the minimality of D.

(2) Let P by a β-prime ideal. Then $P = \text{ass}(I)$ for some indecomposable injective I. Since R is right FBN, then $\beta = |R/P| = |R/\text{ass}(I)| = cr\,|I|$ by [**8**, 8.6]. Hence I is a β-indecomposable injective, and $\text{ann}(SI)$ is a minimal coprimitive ideal contained in P by Lemma 2.4. Let $D = \text{ann}(SI)$. Suppose that P contains a minimal β-coprimitive ideal $D' \neq D$. Then by (1), $D' = \text{ann}(SI')$ for some β-indecomposable injective I'. Now $P = \text{ann}(C)$ for some finitely generated critical submodule C of I. Since P contains D', then $C \cdot D' = 0$. By [4, 2.2], $I' \simeq E(C) = I$. Hence $D' = \text{ann}(SI') = \text{ann}(SI) = D$, which is a contradiction. Therefore P contains exactly one minimal β-coprimitive ideal, which is $\text{ann}(SI)$.

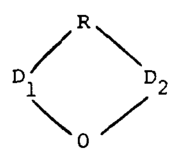
Conversely, let D be a minimal β-coprimitive ideal. Then, by (1), $D = \text{ann}(SI)$ for some β-indecomposable injective and thus $D \subseteq \text{ass}(I) = P$. Since R is right FBN, then $|R/P| = cr\,|I| = \beta$. Hence P is β-prime. Suppose that D is contained in a β-prime ideal $P' \neq P$. Then there exists a β-indecomposable injective I' such that $P' = \text{ass}(I')$. Note that $I' \not\simeq I$ and hence $\text{ann}(SI') \neq D$. Now P' contains two minimal β-coprimitive ideals, namely $\text{ann}(SI')$ and D, which is impossible. Therefore D is contained in a unique β-prime ideal P.

(3) Let L be a β-coprimitive ideal properly containing $P = \text{ass}(I)$. Then $L = \text{ann}(C)$ for some finitely generated β-critical module C. Since R is strongly right FBN, C is a $\triangle$-module and hence $|R/L| = \beta$ by Theorem 2.2. Since R/P satisfies the large condition, $\beta = |R/L| < |R/P| = \beta$, which is a contradiction. Therefore P is the maximal β-coprimitive ideal containing D. ∎

By Proposition 2.5(3), the lattice of β-coprimitive ideals over a strongly right FBN ring has a maximal element and a minimal element. However, the lattice can be more complicated between these elements. The following example shows that the lattice of β-coprimitive ideals need not be linearly ordered.

Example 2.6. Let $R = \begin{bmatrix} F & F[x] \oplus F[x] \\ 0 & F[x] \end{bmatrix}$. Then R is a strongly right FBN ring. Let $L_1 = \begin{bmatrix} 0 & F[x] \oplus 0 \\ 0 & F[x] \end{bmatrix}$ and $L_2 = \begin{bmatrix} 0 & 0 \oplus F[x] \\ 0 & F[x] \end{bmatrix}$. Then $C_1 = R/L_1$ and $C_2 = R/L_2$ are 1-critical. Hence $D_1 = \text{ann}(C_1) = \begin{bmatrix} 0 & F[x] \oplus 0 \\ 0 & 0 \end{bmatrix}$ and $D_2 = \text{ann}(C_2) =$

$\begin{bmatrix} 0 & 0 \oplus F[x] \\ 0 & 0 \end{bmatrix}$ are 1-coprimitive ideals. On the other hand, $\operatorname{ass}(C_1) = \operatorname{ass}(C_2) = P = \begin{bmatrix} F & F[x] \oplus F[x] \\ 0 & 0 \end{bmatrix}$ and $|R/P| = 1$. Thus P is 1-prime. Since R is also right FBN, then there is an isomorphism $f\colon E(C_1) \simeq E(C_2)$. Therefore $f(C_1) + C_2$ is 1-critical by [**2**, 3.1], and $\operatorname{ann}(f(C_1) + C_2) = D_1 \ D_2 = 0$ is a minimal 1-coprimitive ideal. Now we have the following diagram in the lattice of 1-coprimitive ideals of R.



We can now characterise strongly right FBN rings. Further, this provides a situation when the converse of Boyle and Feller's theorem [**4**, 3.4] holds.

THEOREM 2.7. *Let R a ring. Then the following statements are equivalent:*

(1) *R is a strongly right FBN ring;*

(2) *R is right FBN and R has the descending chain condition on β-coprimitive ideals, where β is an ordinal with $-1 < \beta \leqslant \alpha$,*

(3) *R is right FBN and $\operatorname{ann}(SI)$ is a minimal β-coprimitive ideal for all β-indecomposable injectives, for $-1 < \beta \leqslant \alpha$.*

(4) *R is right FBN and every β-prime ideal contains a minimal β-coprimitive ideal for $-1 < \beta \leqslant \alpha$;*

(5) *R is right FBN and the correspondence $\phi\colon P \to \operatorname{ann}(SI_p)$, where I_p is an indecomposable injective summand of $E(R/P)$, is a bijection between the set of β-prime ideals and the set of minimal β-coprimitive ideals for $-1 < \beta \leqslant \alpha$;*

(6) *R is right FBN and the correspondence $\psi\colon I \to \operatorname{ann}(SI)$ is a bijection between the isomorphism classes of β-indecomposable injectives and the set of minimal β-coprimitive ideals for $-1 < \beta \leqslant \alpha$.*

PROOF: (1) $\Rightarrow$ (2) Let $D_1 \supseteq D_2 \supseteq \ldots$ be a descending chain of β-coprimitive ideals. Then for each i, $D_1 = \operatorname{ann}(C_i)$ for some finitely generated β-critical module C_i. Since R is right FBN and $\operatorname{ann}(C_i) \subseteq \operatorname{ann}(C_1)$ for all $i \geqslant 1$, then $E(C_1) = E(C_i)$ for all $i \geqslant 1$ by [**4**, 2.2]. Hence $\sum_{i=1}^{\infty} C_i$ can be considered as a submodule of SI_1, where $I_1 = E(C_1)$. Since R is strongly right FBN, $\operatorname{ann}(SI_1)$ is a minimal β-coprimitive ideal and is contained in D_i for all i. Let $D = \operatorname{ann}(SI_1)$ and consider the descending chain of $D_1/D \supseteq D_2/D \supseteq \ldots$. Since R is right FBN, then R/D is β-smooth for all i. Thus $|D_{i-1}/D_i| = \beta$ for $i \geqslant 2$. This contradicts to the fact that $|R/D| = \beta$. Therefore the chain is finite.

(2) $\Rightarrow$ (3) Let I be a β-indecomposable injective, and let C_1 be a critical submodule of I. Then $D_1 = \operatorname{ann}(C_1)$ is a β-coprimitive ideal. If $(SI) \cdot D_1 \neq 0$, then there exists a β-critical submodule C_2 of SI such that $D_2 = \operatorname{ann}(C_2)$ and $D_1 \not\subseteq D_2$. By [**2**, 3.1], $C_1 + C_2$ is a critical submodule of SI, and hence $D_1 \cap D_2 = \operatorname{ann}(C_1 + C_2)$ is a β-coprimitive ideal. Continuing in this manner, we form a descending chain of β-coprimitive ideals $D_1 \supseteq D_1 \cap D_2 \supseteq \ldots$. By hypothesis, this chain must stop with a β-coprimitive ideal D. By the construction, $D = \operatorname{ann}(SI)$. As in the proof of Lemma 2.4, $\operatorname{ann}(SI)$ is a minimal β-coprimitive ideal.

(3) $\Rightarrow$ (4) Let P be a β-prime ideal. Then $P = \operatorname{ass}(I)$ for some indecomposable injective I. Since R is right FBN, then $cr\,|I| = |R/P|$ by [**8**, 8.6]. However, P being β-prime implies that I is a β-indecomposable injective. Hence by hypothesis, $\operatorname{ann}(SI)$ is a minimal β-coprimitive ideal, and $\operatorname{ann}(SI) \subseteq \operatorname{ass}(I) = P$.

(4) $\Rightarrow$ (5) Let P be a β-prime ideal, and let D be a minimal β-coprimitive ideal contained in P. We claim that $\operatorname{ann}(SI_p) = D$, where I_p is an indecomposable injective summand of $E(R/P)$. Let $D = \operatorname{ann}(C)$ for some finitely generated β-critical module C. Since R is right FBN, then $E(C) \simeq I_p$, and thus $D = \operatorname{ann}(C')$ for some β-critical submodule C' of I_p. If $(SI_p) \cdot D \neq 0$, then there exists a β-critical submodule N' of I_p such that $N' \cdot D \neq 0$. Let $D' = \operatorname{ann}(N')$. Then by [**2**, 3.1], $C' + N'$ is a β-critical submodule of I_p and hence $D \cap D' = \operatorname{ann}(C' + N')$ is a β-coprimitive ideal contained in D. Hence $D = D'$ by the minimality of D. This is a contradiction. Therefore $D = \operatorname{ann}(SI_p)$. That the indicated map is a bijection now follows from Proposition 2.5(2).

(5) $\Rightarrow$ (6): Since R is right FBN, by [**8**, 8.6] there is a $1-1$ correspondence between the isomorphism classes of indecomposable injectives and prime ideals given by $I \to \operatorname{ass}(I)$. Therefore the result follows from (5).

(6) $\Rightarrow$ (1): Let I be indecomposable injective. Then $\operatorname{ann}(SI)$ is a minimal coprimitive ideal. Since R is right RBN, then $|R/\operatorname{ann}(SI)| = cr\,|SI|$. Therefore SI is a $\triangle$-module by Theorem 2.2. Hence by Lemma 2.4, R is strongly right FBN. ∎

COROLLARY 2.8. *If R is right FBN, then R is strongly right FBN if and only if there is a $1-1$ correspondence between the isomorphism classes of indecomposable injectives and the minimal β-coprimitive ideals for $-1 < \beta \leqslant \alpha$, given by $I \to \operatorname{ann}(SI)$.*

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