Canad. Math. Bull. Vol. **50** (2), 2007 pp. 161–171

Functoriality of the Coniveau Filtration

Donu Arapura and Su-Jeong Kang

Abstract. It is shown that the coniveau filtration on the cohomology of smooth projective varieties is preserved up to shift by pushforwards, pullbacks and products.

There is a natural descending filtration on the singular cohomology of a complex smooth projective variety called the coniveau filtration. A class lies in the p-th level of coniveau if it is supported on a closed subvariety of codimension at least p, that is, if it vanishes on the complement of such a subvariety. The generalized Hodge conjecture would imply, rather trivially, that the coniveau filtration is compatible with pushforwards, pullbacks and products. The purpose of this paper is to prove this statement unconditionally.

The main difficulty is proving the compatibility of coniveau with respect to pullbacks along closed immersions. Given a closed immersion $i\colon X \to Y$ and a class $\alpha \in H^k(Y)$ supported on subvariety S, $i^*\alpha$ is supported on $S \cap X$. So the only issue is what to do when S and X do not intersect properly. The classical solution to this kind of problem is to prove a moving lemma. However, our situation is fairly rigid which makes this approach difficult. We solve the problem by blowing up Y and choosing suitable lifts of X and S in general position. Much of the work in this paper involves showing that this suffices.

1 Main Theorem

We use the symbol $\subset$ for nonstrict inclusion. All our varieties will be defined over $\mathbb{C}$. Given a variety X, we write $H^i(X)$ for its singular cohomology with rational coefficients. We will say that a class $\alpha \in H^i(X)$ is supported on a closed subvariety S if it lies in the kernel

$$\operatorname{Ker}[H^i(X) \to H^i(X - S)].$$

By [D, Proposition 8.2.8], this is equivalent to α lying in the image of a Gysin homomorphism. More precisely, that

$$\alpha \in \operatorname{Im}[H^{i-2q}(\tilde{S}) \to H^i(X)],$$

where $q = \operatorname{codim}(S, X)$ and $\tilde{S} \to S$ is a desingularization. The *coniveau* filtration on $H^i(X)$ is given by

$$N^p H^i(X) = \sum_{\operatorname{codim} S \geq p} \operatorname{Ker}[H^i(X) \to H^i(X - S)].$$

Received by the editors January 22, 2005; revised February 3, 2005.
The first author was partially supported by the NSF.
AMS subject classification: 14C30.

https://doi.org/10.4153/CMB-2007-017-5 Published online by Cambridge University Press

We call an element $\alpha \in N^pH^i(X)$ *irreducible* if it is supported on one irreducible subvariety S. Note that every class in $N^pH^i(X)$ is a finite sum of irreducible elements.

The generalized Hodge conjecture (GHC) predicts that $N^pH^i(X)$ is the maximal Hodge substructure of $F^pH^i(X)$ [G2, L]. The filtration on $H^i(X)$ by maximal Hodge structures in $F^\bullet H^i(X)$ is easily seen to have good functorial properties. Our main result is that this so for coniveau without assuming GHC.

Theorem 1.1 *The coniveau filtration $N^\bullet$ is preserved (up to shift) by pushforwards, exterior products and pullbacks. More precisely:*

(i) *If $f\colon X \to Y$ is a map of smooth projective varieties of dimensions n and m respectively, then*

$$f_*(N^pH^i(X)) \subset N^{p+m-n}(H^{i+2(m-n)}(Y)).$$

(ii)

$$N^p(H^i(X)) \otimes N^q(H^j(Y)) \subset N^{p+q}H^{i+j}(X \times Y).$$

(iii) *If f is as above, then*

$$f^*(N^pH^i(Y)) \subset N^pH^i(X).$$

The following was stated in [A], but the proof there was incomplete.

Corollary 1.2

$$N^pH^i(X) \cup N^qH^j(X) \subset N^{p+q}H^{i+j}(X).$$

Proof The cup product is a composition of exterior product and restriction to the diagonal. ■

Proof of (i) and (ii) We give the proof of the first two parts of the theorem now, since we will need them. The proof of (iii) will be postponed until the final section.

(i) An irreducible element $\alpha \in N^pH^i(X)$ lies in the image of a map $k_*(H^{i-2q}(T))$ where $k\colon T \to X$ is a morphism from a smooth projective variety of dimension $n - q \leq n - p$. Therefore

$$f_*(\alpha) \in (f \circ k)_*H^{i-2q}(T) \subset N^{p+m-n}(H^{i+2(m-n)}(Y)).$$

This proves (i).

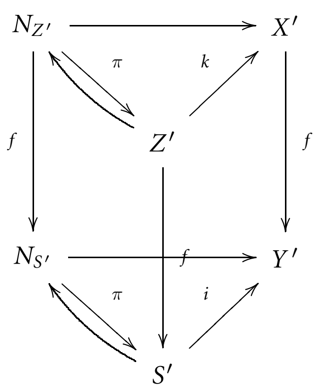
(ii) Let $\alpha \in N^pH^i(X)$ and $\beta \in N^qH^j(X)$ be irreducible classes. Then α and β are supported on subvarieties $S \subset X$ and $T \subset Y$ such that $\dim S \leq \dim X - p$ and $\dim T \leq \dim Y - q$. Then $\dim(S \times T) \leq \dim(X \times Y) - p - q$. It follows that $\alpha \times \beta$, which is supported on $S \times T$, lies in $N^{p+q}(H^{i+j}(X \times Y))$ as expected. ■

2 Lemmas

In this section, we give the basic lemmas needed to finish the proof of the theorem. Many of these lemmas are nothing but special cases of it.

Lemma 2.1 *Let $f: X \to Y$ be a surjective map of smooth projective varieties, then $f^*(N^pH^i(Y)) \subset N^pH^i(X)$.*

Proof Suppose that $\alpha \in N^pH^i(Y)$ is an irreducible class supported on an irreducible variety $S \subset Y$ of codimension $q \geq p$. The preimage $f^{-1}S$ will have codimension less than or equal to q. By taking general hyperplane sections, we can find a cycle $Z \subset f^{-1}S$ of codimension exactly q surjecting onto S. By stratification theory [GM, pp. 33–43], we can find a proper Zariski closed set $Z'' \subset Z$ containing the union of singular loci $Z_{\text{sing}} \cup f^{-1}S_{\text{sing}}$, such that the map $f: X - Z'' \to Y - f(Z'')$ is locally trivial along tubular neighborhoods of $Z' = Z - Z''$ and $S' = S - f(Z'')$. To make the last condition precise, consider the diagram



where $X' = X - Z''$, $Y' = Y - f(Z'')$, and $N_{Z'}$ $N_{S'}$ denotes appropriately chosen tubular neighbourhoods of Z' and S' respectively. The above condition is that $N_{Z'} \to f^*N_{S'}$ is a locally trivial map of locally trivial fibre bundles (for the classical topology) over Z'. Fibrewise, we have an open immersion of $2q$ real dimensional oriented manifolds, and this induces an isomorphism of compactly supported $2q$ dimensional cohomologies. Thus the Thom class $\tau_{Z'}$ of $N_{Z'}$, which can be viewed as an element of relative cohomology $H^{2q}_{Z'}(N_{Z'}) = H^{2q}(N_{Z'}, N_{Z'} - Z')$, coincides with the pullback of the Thom class τ'_S on $f^*N_{S'}$. The Gysin map $H^{i-2q}(Z') \to H^i(X')$ is given by $\alpha \mapsto \pi^*\alpha \cup \tau_{Z'}$ extended by 0 to X'. A similar description holds for (S', Y'). It follows that we have a commutative diagram

$$\begin{array}{ccc} H^{i-2q}(S') & \longrightarrow & H^i(Y') \\ \downarrow & & \downarrow \\ H^{i-2q}(Z') & \longrightarrow & H^i(X') \end{array}$$

https://doi.org/10.4153/CMB-2007-017-5 Published online by Cambridge University Press

Therefore, since $\alpha \in \operatorname{Ker}[H^i(Y) \to H^i(Y-S) = H^i(Y'-S')]$, its image in $H^i(X)$ maps to $\operatorname{Ker}[H^i(X) \to H^i(X-Z) = H^i(X'-Z')]$. This implies that f^* preserves N^p. ■

In the special case where f is also flat, there is a much simpler proof, which we feel compelled to give.

Proof for Flat Maps Let $\alpha \in N^pH^i(Y)$ be irreducible. Then it is supported on an irreducible subvariety S of Y with $\operatorname{codim}(S,Y) = q \geq p$. Let $T = X \times_Y S$. Then $\operatorname{codim}(T,X) = q \geq p$ and we have a commutative diagram

$$\begin{array}{ccc} H^i(Y) & \xrightarrow{\iota^*} & H^i(Y-S) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ H^i(X) & \xrightarrow[\iota_1^*]{} & H^i(X-T) \end{array}$$

where $\iota\colon Y - S \hookrightarrow Y$, $\iota_1\colon X - T \hookrightarrow X$ inclusions. Since $\alpha \in \operatorname{Ker}(\iota^*)$, we have

$$0 = \pi^* \circ \iota^*(\alpha) = \iota_1^* \circ \pi^*(\alpha).$$

Hence, $\pi^*(\alpha) \in \operatorname{Ker}[H^i(X) \to H^i(X-T)] \subset N^pH^i(X)$. ■

Lemma 2.2 *Let $f\colon X \to Y$ be a generically finite map of smooth projective varieties, then $f^*\colon H^i(Y) \to H^i(X)$ is injective. Furthermore,*

$$f^*(H^i(Y)) \cap N^pH^i(X) = f^*(N^pH^i(Y)).$$

Proof Let $d = \deg f$. Then the first statement follows from $f_*f^* = d \cdot \mathrm{id}$, where $\mathrm{id}\colon H^i(Y) \to H^i(Y)$ the identity map. For the second statement, first note that f^* preserves coniveau by Lemma 2.1, so it is enough to show

$$f^*(H^i(Y)) \cap N^pH^i(X) \subset f^*(N^pH^i(Y)).$$

Suppose that $\alpha \in H^i(Y)$ satisfies $f^*(\alpha) \in f^*(H^i(Y)) \cap N^pH^i(X)$. Then, by Theorem 1.1(i), we have $\alpha = \frac{1}{d} f_*f^*(\alpha) \in f_*(N^pH^i(X)) \subset N^pH^i(Y)$, since $\dim X = \dim Y$. Hence $f^*(\alpha) \in f^*(N^pH^i(Y))$. ■

The next result is surely known, but we give a proof for lack of a reference.

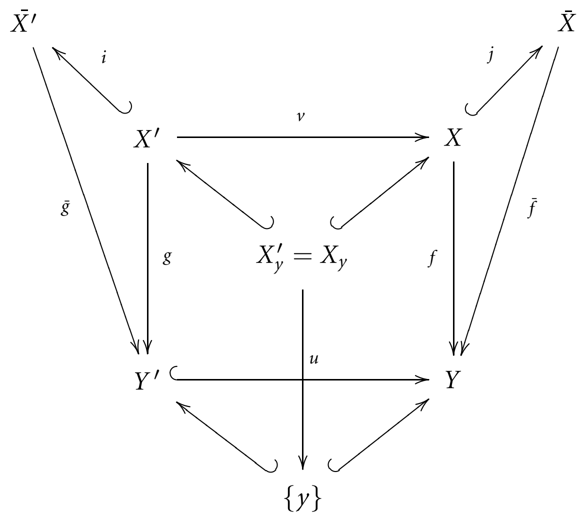
Lemma 2.3 *Consider a fibered square of varieties*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

https://doi.org/10.4153/CMB-2007-017-5 Published online by Cambridge University Press

where $u\colon Y' \to Y$ is an embedding. Then, we have $u^* R^k f_! \mathbb{Q} \simeq R^k g_! \mathbb{Q}$ for any k, where $R^k f_!$, $R^k g_!$ are direct images with proper support.

Proof Consider the diagram:



where $\bar{X}$ (resp., $\bar{X}'$) is a compactification of X (resp., X'), i, j are inclusions, and $\bar{f}, \bar{g}$ are extensions of f, g to compactifications respectively. X_y is the fibre of f at a point y in Y'. First, note that we have a natural map of sheaves on Y'

$$u^* R^k \bar{f}_*(j_! \mathbb{Q}) \longrightarrow R^k \bar{g}_*(i_! \mathbb{Q}).$$

Since $\bar{f}, \bar{g}$ are proper morphisms, by [I, Theorem 6.2], we have

$$(R^k \bar{f}_*(j_! \mathbb{Q}))_y \cong H^k(\bar{f}^{-1}(y), j_! \mathbb{Q}) = H^k_c(f^{-1}(y), \mathbb{Q}) \cong (R^k f_! \mathbb{Q})_y$$

and

$$(R^k \bar{g}_*(i_! \mathbb{Q}))_y \cong H^k(\bar{g}^{-1}(y), i_! \mathbb{Q}) = H^k_c(g^{-1}(y), \mathbb{Q}) \cong (R^k g_! \mathbb{Q})_y.$$

Hence we have

$$\begin{aligned}(u^* R^k f_! \mathbb{Q})_y &= (R^k f_! \mathbb{Q})_y = H^k_c(g^{-1}(y), \mathbb{Q}) \\ &\cong H^k_c(X_y, \mathbb{Q}) = (R^k g_! \mathbb{Q})_y\end{aligned}$$

for any $y \in Y'$ and this gives $u^* R^k f_! \mathbb{Q} \simeq R^k g_! \mathbb{Q}$. ■

It will follow, *a posteriori,* from the theorem that the next result holds for all H.

Lemma 2.4 *Let X be a smooth projective variety and $X \subset \mathbb{P}^N$ be a fixed embedding. There exists a countable union B of proper Zariski closed sets in the dual $(\mathbb{P}^N)^*$, such that for any hyperplane $H \notin B$, $H^i(X) \to H^i(X \cap H)$ preserves coniveau.*

https://doi.org/10.4153/CMB-2007-017-5 Published online by Cambridge University Press

Proof Let $\mathrm{Hilb}(X)$ be the Hilbert scheme of X [G1], $\mathcal{U}$ the universal family, and $\mathcal{T} = (X \times \mathrm{Hilb}(X)) - \mathcal{U}$ the complement of $\mathcal{U}$ in $X \times \mathrm{Hilb}(X)$. Let $p\colon \mathcal{T} \to \mathrm{Hilb}(X)$ the projection. The Hilbert scheme decomposes into a countable union $\mathrm{Hilb}(X) = \bigcup_{i=1}^{\infty} C_i$ of irreducible components. Consider the Cartesian diagram

$$\begin{array}{ccc} \mathcal{T}_i & \longrightarrow & \mathcal{T} \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p} \\ C_i & \xrightarrow{\iota_i} & \mathrm{Hilb}(X) \end{array}$$

where $\iota_i\colon C_i \to \mathrm{Hilb}(X)$ is the inclusion. The fibre $p_i^{-1}(c) \cong X - S_c$, where S_c is a closed subscheme of X corresponding to $c \in C_i$. Let $\mathcal{R}^k = R^k p_! \mathbb{Q}$ and let $\mathcal{R}_i^k = \iota_i^* \mathcal{R}^k$ be the pullback of $\mathcal{R}^k$ to C_i for each i. Then by Lemma 2.3, we have $\mathcal{R}_i^k = \iota_i^* \mathcal{R}^k \simeq R^k p_{i!} \mathbb{Q}$.

First we show that $\mathcal{R}_i^k$ is a constructible sheaf on C_i for each i, *i.e.,* there exists a decreasing sequence of Zariski closed sets $C_i = F_0^i \supset F_1^i \supset \cdots \supset F_{n_i}^i$ such that $\mathcal{R}_i^k|_{F_j^i - F_{j+1}^i}$ is a locally constant sheaf for each j. Consider the commutative diagram:

$$\begin{array}{ccccc} \mathcal{T}_i & \xhookrightarrow{\iota_1} & X \times C_i & \xhookleftarrow{\iota_2} & (X \times C_i) - \mathcal{T}_i = \mathcal{U}_i \\ & {\scriptstyle p_i}\searrow & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle h_i} & \\ & & C_i & & \end{array}$$

where $\mathcal{U}_i$ is the pullback of the universal family $\mathcal{U}$ to C_i, π is a projection and ι_1, ι_2 are inclusions. Note that π and h_i are proper and that π gives a fibrewise compactification of p_i. Hence we have an exact sequence of sheaves

$$\cdots \to R^{k-1}\pi_* \mathbb{Q} \xrightarrow{\rho_{k-1}} R^{k-1} h_{i*} \mathbb{Q} \xrightarrow{\delta} R^k p_{i!} \mathbb{Q} \xrightarrow{\gamma_k} R^k \pi_* \mathbb{Q} \xrightarrow{\rho_k} R^k h_{i*} \mathbb{Q} \to \cdots .$$

From this we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Coker}(\rho_{k-1}) \longrightarrow \mathcal{R}_i^k \longrightarrow \mathrm{Ker}(\rho_k) \longrightarrow 0 .$$

Since for any k, $R^k \pi_* \mathbb{Q}$ and $R^k h_{i*} \mathbb{Q}$ are constructible by [V, Theorem 2.3.1], $\mathrm{Coker}(\rho_{k-1})$ and $\mathrm{Ker}(\rho_k)$ are also constructible, and consequently so is $\mathcal{R}_i^k$.

Now let $V_{ij} = F_j^i - F_{j+1}^i$ for each i, j. Let $\mathcal{W} = \{W_i \mid i = 0, 1, \dots\}$ be the set of irreducible components of V_{ij}. Then

(i) Each W_i is irreducible.
(ii) Each $w \in W_i$ corresponds to a closed subscheme S_w of X.
(iii) $\mathcal{R}^k|_{W_i}$ is a locally constant sheaf for each i.

https://doi.org/10.4153/CMB-2007-017-5 Published online by Cambridge University Press

Set

$$B_i = \Big\{ H \in (\mathbb{P}^N)^* \mid H \supset \bigcup_{w \in W_i} S_w \Big\},$$

$$B = \bigcup_{i=1}^{\infty} B_i.$$

Then B is a countable union of proper Zariski closed sets in $(\mathbb{P}^N)^*$.

Choose $H \in (\mathbb{P}^N)^* - B$. We show that $H^i(X) \to H^i(X \cap H)$ preserves coniveau. Let $\alpha \in N^p H^i(X)$. Then there is a subscheme S of X with $\operatorname{codim}(S, X) = q \geq p$ such that $\alpha \in \operatorname{Ker}[\psi\colon H^i(X) \to H^i(X - S)]$. Let $[s]$ be the element in $\operatorname{Hilb}(X)$ corresponding to S. Then $[s] \in W_j$ for some j. Since $H \notin B$, $H \notin B_j$, equivalently, there is $w \in W_j$ such that $S_w \nsubseteq H$. Set $k = 2\dim X - i$. Since $\mathcal{R}^k$ is a locally constant sheaf on the connected set W_j, we have an isomorphism

$$H_c^k(X - S_w, \mathbb{Q}) = \mathcal{R}_w^k \cong \mathcal{R}_{[s]}^k = H_c^k(X - S, \mathbb{Q})$$

given by parallel transport. Then by duality, we have

$$H^i(X - S_w, \mathbb{Q}) \cong H^i(X - S, \mathbb{Q}). \tag{1}$$

Note that S_w is a closed subscheme in general position with respect to H, and hence $S_w \cap H$ has codimension 1 in S_w. Therefore, we have a fibered square

$$\begin{array}{ccc} S_w \cap H & \longrightarrow & X \cap H \\ {\scriptstyle \tau_1}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ S_w & \longrightarrow & X \end{array}$$

(where all maps are inclusions) which induces a commutative diagram

$$\begin{array}{ccc} H^i(X \cap H) & \xrightarrow{\ \phi\ } & H^i((X \cap H) - (S_w \cap H)) \\ {\scriptstyle \tau^*}\big\uparrow & & \big\uparrow{\scriptstyle \tau_1^*} \\ H^i(X) & \xrightarrow[\ \psi_w\ ]{} & H^i(X - S_w) \end{array}$$

where ψ_w is the composition of ψ and the isomorphism (1). Now by the commutativity of the above diagram, we get $\phi \circ \tau^*(\alpha) = \tau_1^* \circ \psi_w(\alpha) = 0$. Since $\operatorname{codim}(S_w \cap H, X \cap H) = q \geq p$,

$$\tau^*(\alpha) \in \operatorname{Ker}[\phi\colon H^i(X \cap H) \to H^i((X \cap H) - (S_w \cap H))] \subset N^p H^i(X \cap H).$$

Thus, $H^i(X) \to H^i(X \cap H)$ preserves coniveau. ∎

https://doi.org/10.4153/CMB-2007-017-5 Published online by Cambridge University Press

Corollary 2.5 *Let X be a smooth projective variety. There exist general hyperplanes $H_1, H_2, \dots, H_p$ such that for $T = X \cap H_1 \cap \cdots \cap H_p$, $H^i(X) \to H^i(T)$ preserves coniveau for any i.*

Lemma 2.6 *Let*

$$\begin{array}{ccc} T & \overset{f}{\hookrightarrow} & S \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ X & \underset{f}{\hookrightarrow} & Y \end{array}$$

be a fibered square of smooth projective varieties such that f, h are embeddings and $\operatorname{codim}(T, X) = \operatorname{codim}(S, Y) = c$. *Then the following diagram commutes:*

$$\begin{array}{ccc} H^{i-2c}(T) & \overset{f^*}{\longleftarrow} & H^{i-2c}(S) \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ H^i(X) & \underset{f^*}{\longleftarrow} & H^i(Y) \end{array}$$

Proof We use an argument very similar to the proof of Lemma 2.1. Let N_S be a tubular neighborhood of S in Y. Let $N_T = N_S \times_Y X \cong X \cap N_S$. Then $T \subset N_T$ and N_T is a tubular neighborhood of T in X, since $\operatorname{codim}(T, X) = \operatorname{codim}(S, Y) = c$. Then the Thom class τ_S of N_S restricts to the Thom class τ_T of N_T. The Gysin map for S (respectively, T) is given by a composition of pullback to N_S (respectively, N_T), cup product with τ_S (respectively, τ_T) followed by extension by 0. Commutativity of the diagram follows from these remarks. ■

3 Proof of Theorem 1.1(iii)

Now we give a proof of the third part of the Theorem 1.1

Let $\alpha \in N^pH^i(Y)$ be irreducible; then we must show that $f^*\alpha \in N^pH^i(X)$. By descending induction on p, we can assume that α is not supported on a variety of codimension greater than p, *i.e.*, $\alpha \notin N^{p+1}H^i(Y)$. Therefore α is supported on an irreducible subvariety $S \subset Y$ of codimension exactly p. Fix a desingularization $\tilde{S}$ of S. Then $\alpha = j_*(\beta)$ for some $\beta \in H^{i-2p}(\tilde{S})$, where j is the composition of $\tilde{S} \to S \hookrightarrow Y$.

If f is flat and surjective, then we are done by Lemma 2.1 (the special case suffices). Since any f can be factored as the inclusion of its graph followed by a projection $X \to X \times Y \to Y$, we can reduce to the case where f is a closed immersion.

For the remainder of the proof, assume that f is an inclusion. Let d be the codimension of X in Y. We consider three cases.

Case 1 (X is not contained in S.) We may assume that $S \cap X \neq \varnothing$, since otherwise the result is trivial. We may also assume that $\operatorname{codim}(S, Y) = p > 1$. To see this,

https://doi.org/10.4153/CMB-2007-017-5 Published online by Cambridge University Press

suppose that $p = 1$ and let $Z = X \times_Y S$. Then $\operatorname{codim}(Z, X) = 1$, since otherwise $X \subset S$. Since $f^*\alpha$ is supported on Z, we are done.

Let $\pi_1 \colon Y' = \mathrm{Bl}_S Y \to Y$ be the blow-up of Y along S, with E' the exceptional divisor. Choose a desingularization $\pi_2 \colon Y_1 \to Y'$ and let $\pi = \pi_1 \circ \pi_2$. We let E be the strict transform of E', X_1 be the strict transform of X in Y_1, and $E_1 = X_1 \times_{Y_1} E$. Then $\operatorname{codim}(E_1, X_1) = 1$ because X is not contained in S. First, note that by Lemma 2.2, we have $\pi^*(\alpha)$ is supported on S_1. So, we can find $\beta_1 \in H^{i-2p}(\tilde{S}_1)$ such that ${j_1}_*(\beta) = \pi^*(\alpha)$, where j_1 is the compostion $\tilde{S}_1 \to S_1 \to Y_1$ of a desingularization of S_1 and an inclusion. Now let H be a very ample divisor in Y_1 and set $S_1 = E \cap H_1 \cap \cdots \cap H_{p-1}$, where $H_i \in |H|$ are in general position. Note that $\pi(S_1) = S$. Let $T = S_1 \times_E E_1 = S_1 \cap E_1 = (E \cap H_1 \cap \cdots \cap H_{p-1}) \cap E_1 = E_1 \cap H_1 \cap \cdots \cap H_{p-1}$. Hence T is an iterated hyperplane section of E_1 of codimension $p - 1$. We have a commutative diagram

$$\begin{array}{ccc}
T = S_1 \times_E E_1 & \longrightarrow & S_1 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{\;f_1\;} & Y_1 \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
X & \xrightarrow{\;f\;} & Y
\end{array}$$

where the top square is Cartesian. We have $\operatorname{codim}(T, X_1) = \operatorname{codim}(S_1, Y_1) = p$. Therefore, by Lemma 2.6 we obtain the commutative diagram

$$\begin{array}{ccc}
H^{i-2p}(T) & \xleftarrow{\;h^*\;} & H^{i-2p}(S_1) \\
{\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle {j_1}_*} \\
H^i(X_1) & \xleftarrow[\;f_1^*\;]{} & H^i(Y_1)
\end{array}$$

So we have

$$\pi^* f^*(\alpha) = f_1^* \pi^*(\alpha) = f_1^* \circ {j_1}_*(\beta) = g_*(h^*(\beta)) \in \mathrm{Im}[H^{i-2p}(T) \to H^i(X_1)].$$

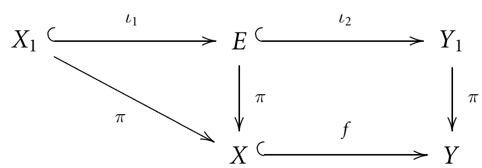
Hence $\pi^* f^*(\alpha) \in N^p H^i(X_1)$ and by Lemma 2.2, $f^*(\alpha) \in N^p H^i(X)$.

Case 2 *($X \subset S$ and $X \neq S$.)* Then $d = \operatorname{codim}(X, Y) > \operatorname{codim}(S, Y) \geq 1$.

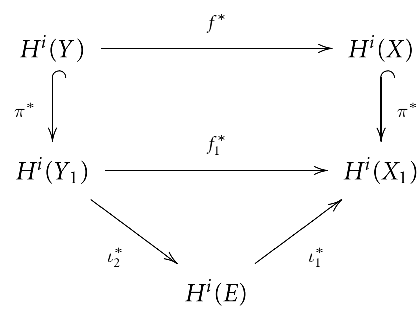
Let $\pi \colon Y' = \mathrm{Bl}_X Y \to Y$ be the blow-up of Y along X, E the exceptional divisor and S' the strict transform of S. Choose an embedded resolution $Y_1 \to Y'$ of singularities, so that the strict transform S_1 of S is nonsingular. Let E be the strict transform of E'. Let H be a very ample divisor on Y_1 and set $X_1 = E \cap H_1 \cap \cdots \cap H_{d-1}$, where $H_i \in |H|$

https://doi.org/10.4153/CMB-2007-017-5 Published online by Cambridge University Press

are in general position. Note that $\pi(X_1) = X$. Let $\iota_1 : X_1 \to E$ and $\iota_2 : E \to Y_1$ be inclusions. Then we have a commutative diagram



Let $f_1 = \iota_2 \circ \iota_1 : X_1 \to Y_1$ be the inclusion. Then the following diagram commutes.



Recall that we have to show that $f^*\alpha \in N^pH^i(X)$. By Lemma 2.2, it suffices to show that $\pi^*(f^*\alpha) \in N^pH^i(X_1)$. We now chase this element around the other side of the previous diagram,

$$\pi^*(f^*\alpha) = f_1^*(\pi^*\alpha) = \iota_1^* \circ \iota_2^*(\pi^*\alpha).$$

Since $\pi : Y_1 \to Y$ is a generically finite map, Lemma 2.2 implies that $\pi^*(\alpha) \in N^pH^i(Y_1)$. By our assumptions, $S_1 \times_{Y_1} E$ has codimension p in E, and $\iota_2^*(\pi^*\alpha)$ is supported on it by Lemma 2.6. Therefore $\iota_2^*(\pi^*\alpha) \in N^pH^i(E)$. Since X_1 is an iterated hyperplane section of E by general hyperplanes, ι_1^* preserves the coniveau filtration by Lemma 2.4. Hence, $\iota_1^* \circ \iota_2^*(\pi^*\alpha) \in N^pH^i(X_1)$ as required.

Case 3 $(X = S.)$ Then codim $(X, Y) =$ codim $(S, Y) = p$.

In this case, we have $\beta \in H^{i-2p}(X)$ such that $f_*(\beta) = \alpha \in H^i(Y)$. We want to show that $f^*(\alpha) \in N^pH^i(X)$. Note that $f^*(\alpha) = f^*f_*(\beta) = [X]|_X \cup \beta$. By intersection theory [F1, Section 6.1], $[X]|_X$ is represented by the codimension p algebraic cycle $\gamma = [X \cdot X] \in N^pH^{2p}(X)$. So $\gamma = \sum_k n_k[V_k]$ where V_k is an irreducible subvariety of X of codimension p for each k. Since $V_k \times X$ is a subvariety of $X \times X$ of codimension p, we have a Gysin map

$$H^{i-2p}(\tilde{V}_k \times X) \xrightarrow{g_*} H^i(X \times X),$$

which sends $\beta \cong 1_k \otimes \beta$ to $[V_k] \otimes \beta$, where 1_k is a generator of $H^0(V_k)$ and $\tilde{V}_k \to V_k$ is a desingularization. Also note that

$$[V_k] \otimes \beta \in N^pH^{2p}(X) \otimes N^0H^{i-2p}(X) \subset N^pH^i(X \times X)$$

https://doi.org/10.4153/CMB-2007-017-5 Published online by Cambridge University Press

by part (ii) of Theorem 1.1. Let $\Delta\colon X \to X \times X$ be the diagonal map. Then, by letting $S_k = V_k \times X$, $Y_k = X \times X$, and $X_k = X$, for each k, we can reduce this to Case 1, *i.e.*, we have

$$\Delta^*\colon H^i(Y_k) \longrightarrow H^i(X_k),$$

with

$$[V_k] \otimes \beta \in \operatorname{Im}[H^{i-2p}(\tilde{S}_k) \to H^i(Y_k)] \subset N^p H^i(Y_k)$$

and $S_k \neq X_k$, where $\tilde{S}_k \to S_k$ is a desingularization of S_k. Then we have

$$\Delta^*\big([V_k] \otimes \beta\big) = [V_k] \cup \beta \in N^p H^i(X_k) = N^p H^i(X)$$

for each k. Therefore,

$$f^*(\alpha) = \sum_k n_k\big([V_k] \cup \beta\big) \in N^p H^i(X).$$

This completes the proof of the theorem. ■

References

[A] D. Arapura, *Hodge cycles on some moduli spaces.* preprint (2002)

[D] P. Deligne, *Théorie de Hodge. III.*, Inst. Hautes tudes Sci. Publ. Math. **44**(1974), 5–77.

[F1] W. Fulton, *Intersection Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete 2, Springer-Verlag, Berlin, 1984.

[GM] M. Goresky and R. Macpherson, *Stratified Morse Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete 14, Springer-Verlag, Berlin, 1988.

[G1] A. Grothendieck, *Fondements de Géometrie Algébrique.* Sminaire Bourbaki 7 , Soc. Math. France, Paris, pp. 297–307.

[G2] ———, *Hodge's general conjecture is false for trivial reasons.* Topology **8**(1969), 299–303.

[I] B. Iversen, *Cohomology of Sheaves.* Springer-Verlag, Berlin, 1986.

[L] J. Lewis, *A survey of the Hodge conjecture.* Second edition. CRM Monograph Series 10. American Mathematical Society, Providence, RI, 1999.

[V] J.-L. Verdier, *Classes d'homologie d'un cycle.* In: Séminaire de géométrie anaytique, Astérisque 36-37, Soc. Math. France, Paris, 1976, pp. 101–151.

Department of Mathematics
Purdue University
West Lafayette, IN 47907
U.S.A.
e-mail: (Arapura) *dvb@math.purdue.edu*
(Kang) *sjkang@math.purdue.edu*