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A NON-ABELIAN NEAR RING IN WHICH $(-1)r=r$ IMPLIES $r=0$

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In this Bulletin Ligh [2] generalized to finite near rings with identity a theorem Zassenhaus [5] used to prove every finite near field has abelian addition. B. H. Neumann [4] extended Zassenhaus' result, using similar techniques and showed that all near fields are abelian. It has been an open question whether Ligh's generalization could be carried out to infinite near rings with identity. The purpose of this paper is to show that Ligh's theorem cannot be so extended. In particular, it cannot be extended even to distributively generated near rings, a type of near ring which has been useful in studying endomorphism rings of non-abelian groups [1, 3].

A triple $(R, +, \cdot)$ is a near ring if $(R, +)$ is a group, $(R, \cdot)$ is a semigroup and $r(s+t)=rs+rt$, for all r, s, t in R. R is a near field if further the non-zero elements form a group under multiplication. A near ring R is distributively generated if $(R, +)$ is generated by a set of right distributive elements that is closed under multiplication.

THEOREM. [*Ligh*, 2] *Let R be a finite near ring with identity* 1 *such that* $(-1)r=r$ *implies* $r=0$. *Then* $(R, +)$ *is abelian.*

THEOREM. *There exists an infinite non-abelian near ring with identity* 1 *in which* $(-1)r=r$ *implies* $r=0$.

Proof. Let G be the free (additive) group on two generators x and y. We at times use functional notation $g(x, y)$ for an element g in G; e.g., $g=g(x, y)=2x+y-x$.

For each pair of positive integers a and b define a map $Ta, b:G\to G$ by:

$$g(x, y)Ta, b = g(ax, by).$$

Let $T0, 0$ be the zero map and $g(-Ta, b)=-(gTa, b)=(-g)Ta, b$. Denote by R the set consisting of $T0, 0$ and all finite sums of maps of the form Ta, b or $-Ta, b$ and define $+$ and $\cdot$ on R as addition and composition of functions. It is clear that $(R, +)$ is a group. Verification of the following is routine:

(i) $$Ta, b \cdot Tc, d = (-Ta, b) \cdot (-Tc, d)$$

(ii) $$Ta, b \cdot (-Tc, d) = (-Ta, b) \cdot Tc, d = -Tac, bd$$

(iii) $$(\pm Ta, b \pm Tc, d) \cdot Tr, s = \pm Ta, b \cdot Tr, s \pm Tc, d \cdot Tr, s$$

(iv) $$(\pm Ta, b \pm Tc, d) \cdot (-Tr, s) = \mp Tc, d \cdot Tr, s \mp Ta, b \cdot Tr, s.$$

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Using the above one easily shows that $(R, +, \cdot)$ is a distributively generated near ring with identity $1=T1, 1$ and that R is additively generated by the maps of the form Ta, b. Since $(x+y)[T2, 1+T3, 1]\neq(x+y)[T3, 1+T2, 1]$, $(R, +)$ is not abelian. It remains to show that $(-1)W\neq W$ for all $W\neq T0, 0$ in R.

Let $W=\sum_{i=1}^{n} U_i$ be an arbitrary non-zero element of R where $U_i=Ta_i, b_i$, or $-Ta_i, b_i$, $i=1, \ldots, n$ and assume $U_i\neq -U_{i+1}$, $i=1-\ldots, n-1$. Consider the element $g(x, y)=x-y+x-y$ of G. We show that $gW\neq g(-1)W$ or equivalently $gW\neq(-g)W$ by induction on n in $W=\sum_{i=1}^{n} U_i$. Without loss of generality we assume that $U_1=Ta_1, b_1$.

First, $gU_1\neq(-g)U_1$. Since $U_2\neq -U_1$ we have four cases for U_2.

I.	$U_2 = Ta_2, b_2$		
II.	$U_2 = -Ta_2, b_2;$	$a_2 \neq a_1,$	$b_2 \neq b_1$
III.	$U_2 = -Ta_1, b_2;$		$b_2 \neq b_1$
IV.	$U_2 = -Ta_2, b_1;$	$a_2 \neq a_1.$	

Table A shows that in each of the cases I–IV the elements $g(U_1+U_2)$ and $(-g)(U_1+U_2)$ of G have the following properties for $k=2$:

TABLE A

	$F=$ U_1	$+$	U_2
Case I	$(g)F = a_1x-b_1y+a_1x-b_1y$	$+$	$a_2x-b_2y+a_2x-b_2y.$
	$(-g)F = b_1y-a_1x+b_1y-a_1x$	$+$	$b_2y-a_2x+b_2y-a_2x.$
II	$(g)F = a_1x-b_1y+a_1x-b_1y$	$+$	$b_2y-a_2x+b_2y-a_2x.$
	$(-g)F = b_1y-a_1x+b_1y-a_1x$	$+$	$a_2x-b_2y+a_2x-b_2y.$
III	$(g)F = a_1x-b_1y+a_1x-b_1y$	$+$	$b_2y+a_1x+b_2y-a_1x.$
	$(-g)F = b_1y-a_1x+b_1y-a_1x$	$+$	$a_1x-b_2y+a_1x-b_2y.$
IV	$(g)F = a_1x-b_1y+a_1x-b_1y$	$+$	$b_1y-a_2x+b_1y-a_2x.$
	$(-g)F = b_1y-a_1x+b_1y-a_1x$	$+$	$a_2x-b_1y+a_2x-b_1y.$

1. The elements are not equal.
2. Each element respectively has more terms than the corresponding element $(g)\sum_{i=1}^{k-1} U_i$ or $(-g)\sum_{i=1}^{k-1} U_i$.
3. The last two terms of one element are a_kx-b_ky and of the other are b_ky-a_kx. (In some cases $a_k=a_{k-1}$ or $b_k=b_{k-1}$).

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Further if we assume that 1, 2, and 3 are true for the elements $(g)\sum_{i=1}^{k} U_i$ and $(-g)\sum_{i=1}^{k} U_i$ with $k=m$, then Table B shows that 1, 2, 3 hold for $k=m+1$ in each case corresponding to I–IV. Thus $(-1)W=W$ only if $W=T0, 0$ and we see that Ligh's theorem cannot be generalized even to distributively generated infinite near rings.

TABLE B

	$F=$	$U_1+\cdots+U_m$	$+$	U_{m+1}
Case I	$(g)F=$	$a_1x-\cdots+a_mx-b_my$	$+$	$a_{m+1}x-b_{m+1}y+a_{m+1}x-b_{m+1}y.$
	$(-g)F=$	$b_1y-\cdots+b_my-a_mx$	$+$	$b_{m+1}y-a_{m+1}x+b_{m+1}y-a_{m+1}x.$
II	$(g)F=$	$a_1x-\cdots+a_mx-b_my$	$+$	$b_{m+1}y-a_{m+1}x+b_{m+1}y-a_{m+1}x.$
	$(-g)F=$	$b_1y-\cdots+b_my-a_mx$	$+$	$a_{m+1}x-b_{m+1}y+a_{m+1}x-b_{m+1}y.$
III	$(g)F=$	$a_1x-\cdots+a_mx-b_my$	$+$	$b_{m+1}y-a_mx+b_{m+1}y-a_mx.$
	$(-g)F=$	$b_1y-\cdots+b_my-a_mx$	$+$	$a_mx-b_{m+1}y+a_mx-b_{m+1}y.$
IV	$(g)F=$	$a_1x-\cdots+a_mx-b_my$	$+$	$b_my-a_{m+1}x+b_my-a_{m+1}x.$
	$(-g)F=$	$b_1y-\cdots+b_my-a_mx$	$+$	$a_{m+1}x-b_my+a_{m+1}x-b_my.$

(Note: In Table B we assume that $U_m=Ta_m, b_m$. If $U_m=-Ta_m, b_m$ a rearrangement of the cases yields the same result.)

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