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ON A THEOREM OF SULLIVAN

BY
MICHAEL A. PENNA

Introduction. The purpose of this note is to give an elementary geometric proof of the following result stated by Sullivan (see (4)).

THEOREM 1 (Sullivan). *Let K be a finite simplicial complex with vertices $v_1, \ldots, v_N$ and corresponding barycentric coordinates $b_1, \ldots, b_N$. Then the algebra of rational PL forms on K*

$$E^*(K) = (Q[b_1, \ldots, b_N] \otimes \wedge(db_1, \ldots, db_N))/I$$

where $Q[b_1, \ldots, b_N]$ is the ring of rational polynomials in $b_1, \ldots, b_N$, where $\wedge(db_1, \ldots, db_N)$ is the exterior algebra on $db_1, \ldots, db_N$, and where I is the ideal generated by

$$(*) \qquad \begin{gathered} (b_1 + \cdots + b_N) - 1 \\ db_1 + \cdots + db_N \\ b_{i_1} \cdots b_{i_p}\, db_{j_1} \cdots db_{j_q} \end{gathered}$$

if there is no $(p+q)$-simplex of K with vertices $v_{i_1}, \ldots, v_{i_p}, v_{j_1}, \ldots, v_{j_q}$. Furthermore the differential

$$d : E^q(K) \to E^{q+1}(K)$$

$$\Sigma_j f_j\, db_{j_1} \cdots db_{j_q} \mapsto \Sigma_{j, j_0} (\partial f_j / db_{j_0})\, db_{j_0}\, db_{j_1} \cdots db_{j_q}$$

where $\partial f_j / \partial b_{j_0}$ is the standard partial derivative of f_j with respect to b_{j_0}.

This result is useful for computations since it gives a simple canonical global representation for rational *PL* forms; such a representation is, of course, not available for smooth forms on smooth manifolds. The proof of this result presented below is also useful since it makes completely transparent how this global representation arises geometrically; or, from a different point of view, this proof makes completely transparent how natural the definition of *PL* de Rham complexes is.

The proof of the theorem has two parts. The first part is to show that any rational *PL* q-form θ can be written in the form $\theta = \Sigma_j f_j\, db_{j_1} \cdots db_{j_q}$ for $f_j \in Q[b_1, \ldots, b_N]$. This is elementary: one simply uses a "partition of unity"-type argument involving the barycentric coordinates and the (open) vertex

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stars; this works despite the fact that the barycentric coordinate functions do not form an honest partition of unity subordinate to the cover by vertex stars. The second part of the proof is to show that the relations (*) are the only relations in $E^*(K)$.

A more algebraic proof of Theorem 1 can be given as follows: Every finite simplicial complex K can be considered as a subcomplex of a simplex Δ^{N-1}, and by definition $E^*(K)=\varprojlim E^*(\sigma)$, the inverse limit taken over the partially ordered set of simplices σ of K. Thus there is a well defined map (restriction) $\rho: E^*(\Delta^{N-1})\to E^*(K)$; as before ρ is surjective. For each simplex σ of K there is a restriction map $\rho_\sigma: E^*(\Delta^{N-1})\to E^*(\sigma)$, and it is easy to compute the kernel ker ρ_σ of ρ_σ. The proof of Theorem 1 follows from the fact that ker $\rho=\varprojlim$ ker $\rho_\sigma=\bigcap_\sigma$ ker ρ_σ.

There are several ways to define de Rham complexes for locally finite, finite dimensional simplicial complexes, and this paper originated in an attempt to relate two such definitions given in (3) and (4). (The latter definition will be given below for completeness of exposition.) It is not difficult to prove a piecewise smooth version of Theorem 1, and in doing so show that the de Rham complex of rational *PL* forms on a simplicial complex is a subcomplex of the de Rham complex of piecewise smooth forms.

The Sullivan de Rham theory has been extended and developed considerably since (4) appeared (see, for example, (1) and references given there). The fact that the algebra $E^0(K)$ of 0-forms on K carries so much information (the key to the proof of Theorem 1) is treated in a more general setting in (2).

I am indebted to the referee for making several valuable suggestions concerning the presentation of this paper. In particular, he pointed out to me the more algebraic proof of Theorem 1.

1. **Proof of the Theorem.** First recall that the cochain complex $(E^*(K), d)$ of rational *PL* forms on K is defined as follows: For each n-simplex σ in K with vertices $v_{k_0},\ldots,v_{k_n}$ we consider differential forms $\theta_\sigma=\Sigma_j f_j\, db_{j_1}\cdots db_{j_q}$ where

$$j=\{j_1,\ldots,j_q\}\subseteq\{k_0,\ldots,k_n\},$$

$$b_{k_0}+\cdots+b_{k_n}=1,$$

$$db_{k_0}+\cdots+db_{k_n}=0,\qquad\text{and}$$

$$f_j\in Q[b_{k_0},\ldots,b_{k_n}]\qquad\text{for each } j.$$

A *rational PL q-form* on K is a collection $\theta=\{\theta_\sigma\}$ of such differential q-forms defined on the simplices of K for which the following compatibility condition holds: if σ and τ are two adjacent simplices of K then $\theta_{\sigma|\sigma\cap\tau}=\theta_{\tau|\sigma\cap\tau}$. The differential is the standard differential defined simplexwise.

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PROPOSITION 2. *Every* $\theta \in E^q(K)$ *may be written* $\theta = \Sigma_j f_j \, db_{j_1} \cdots db_{j_q}$ *where each* $f_j \in Q[b_1, \ldots, b_N]$.

Proof. Let v be a vertex of K and σ an n-simplex of K with vertices v, $v_{k_1}, \ldots, v_{k_n}$. Using the equations

$$b = 1 - (b_{k_1} + \cdots + b_{k_n})$$
$$db = -(db_{k_1} + \cdots + db_{k_n})$$

the restriction $\theta_{\sigma|\mathrm{St}\, v}$ of θ_σ to the star of v may be written

$$\theta_{\sigma|\mathrm{St}\, v} = \Sigma_j f_j^\sigma \, db_{j_1} \cdots db_{j_q}$$

where the summation is taken over all distinct q-tuples $j \subseteq \{k_1, \ldots, k_n\}$ and where $f_j^\sigma \in Q[b_{k_1}, \ldots, b_{k_n}]$. Compatibility of the forms θ_σ means that if τ is a simplex of K containing v, $v_{j_1}, \ldots, v_{j_q}$, then $f^\sigma_{j|\sigma\cap\tau} = f^\tau_{j|\sigma\cap\tau}$; it follows that the restriction $\theta_{|\mathrm{St}\, v}$ of θ to $\mathrm{St}\, v$ may be written

$$\theta_{|\mathrm{St}\, v} = \Sigma_j f_j \, db_{j_1} \cdots db_{j_q}$$

where, for each j, $v_{j_1}, \ldots, v_{j_q}$ are vertices in the link of v, and f_j is a rational polynomial in those b_j for which v_j is a vertex in the link of v. It is not difficult to show that

$$b(\theta|_{\mathrm{St}\, v}) = \Sigma_j bf \, db_{j_1} \cdots db_{j_q} \in E^q(K)$$

and consequently that

$$\theta = \Sigma_j b_j(\theta|_{\mathrm{St}\, v_j}) \in E^q(K). \quad \text{QED}$$

It is easy to show that for $f \in E^0(K)$, $\theta = \Sigma_j f_j \, db_{j_1} \cdots db_{j_q}$ and $\varphi = \Sigma_j g_j \, db_{j_1} \cdots db_{j_q}$ in $E^q(K)$, and $\psi = \Sigma_k h_k \, db_{k_1} \cdots db_{k_r}$ in $E^r(K)$ we have

$$f \cdot \theta = \Sigma_j f f_j \, db_{j_1} \cdots db_{j_q},$$
$$\theta + \varphi = \Sigma_j (f_j + g_j) \, db_{j_1} \cdots db_{j_q}, \quad \text{and}$$
$$\theta \wedge \psi = \Sigma_{j,k} f_j h_k \, db_{j_1} \cdots db_{j_q} \, db_{k_1} \cdots db_{k_r}.$$

Global representations of rational PL forms are not unique: First we clearly have the relations in $E^*(K)$ generated by the first two relations of (*). Second, since the differential d is support non-increasing, we have the relation $f \, db_{j_1} \cdots db_{j_q} = 0$ whenever $f = 0$ on $\bigcap_{\ell=1}^q \mathrm{St}\, v_\ell$; in particular $b_{i_1} \cdots b_{i_p} \, db_{j_1} \cdots db_{j_q} = 0$ if there is no $(p+q)$-simplex of K with vertices $v_{i_1}, \ldots, v_{i_p}, v_{j_1}, \ldots, v_{j_q}$. Thus the relations of (*) are relations in $E^*(K)$.

We will now show that relations (*) generate all other relations in $E^*(K)$. We will do this first in the special case of $E^0(K)$, and then, using this result, in the general case.

If Δ^{N-1} is the $(N-1)$-simplex with vertices $v_1, \ldots, v_N$, then K can naturally

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be considered as a subcomplex of Δ^{N-1}. By definition

$$E^0(\Delta^{N-1}) = Q[b_1, \ldots, b_N]/(b_1 + \cdots + b_N) - 1.$$

PROPOSITION 3: *The kernel ker ρ of the surjection $\rho: E^0(\Delta^{N-1}) \to E^0(K)$ given by restriction is the ideal I generated by all products of the form $b_{i_0} \cdots b_{i_p}$ for which there is no p-simplex of K with vertices $v_{i_0}, \ldots, v_{i_p}$.*

Proof. Clearly $I \subseteq \ker \rho$. If $f \in \ker \rho$, then $f = \Sigma_i b_i(f|_{\mathrm{St}\, v_i})$, and it suffices to show that $b_i(f|_{\mathrm{St}\, v_i}) \in I$ for each i.

To do this, use the identity

$$b_i = 1 - (b_1 + \cdots + \hat{b}_i + \cdots + b_N)$$

to eliminate b_i from $f|_{\mathrm{St}\, v_i}$ and uniquely write

$$f|_{\mathrm{St}\, v_i} = \sum_{p=0}^{N} (\Sigma_j b_{j_1} \cdots b_{j_p} f_j)$$

where, for each p, the second summation is taken over all distinct p-tuples j, and where each $f_j \in Q[b_{j_1}, \ldots, b_{j_p}]$. By induction on p one can show that $f_j = 0$ if $v_i, v_{j_1}, \ldots, v_{j_p}$ are the vertices of a p-simplex of K: The induction step involves evaluating f on the p-simplex σ in K with vertices $v_i, v_{j_1}, \ldots, v_{j_p}$. It follows that

$$f|_{\sigma \cap \mathrm{St}\, v_i} = (b_{j_1} \cdots b_{j_p} f_j)|_\sigma = 0$$

so that $f_{j|\sigma} = 0$ and consequently $f_j = 0$. QED

Observe that by definition we also have

$$E^*(\Delta^{N-1}) = (Q[b_1, \ldots, b_N] \otimes \wedge (db_1, \ldots, db_N))/I$$

where I is the ideal generated by

$$(b_1 + \cdots + b_N) - 1$$
$$db_1 + \cdots + db_N.$$

PROPOSITION 4. *The kernel ker ρ^q of the surjection $\rho^q: E^q(\Delta^{N-1}) \to E^q(K)$ given by restriction (considered as a map of modules with respect to $\rho: E^0(\Delta^{N-1}) \to E^0(K)$) is the submodule I^q generated by all forms $b_{i_1} \cdots b_{i_p} db_{j_1} \cdots db_{j_q}$ for which there is no $(p+q)$-simplex of K with vertices $v_{i_1}, \ldots, v_{i_p}, v_{j_1}, \ldots, v_{j_q}$.*

Proof. Clearly $I^q \subseteq \ker \rho^q$. If $\theta \in \ker \rho^q$, then $\theta = \Sigma_i b_i(\theta|_{\mathrm{St}\, v_i})$ and it again suffices to show that $b_i(\theta|_{\mathrm{St}\, v_i}) \in I^q$ for each i.

For each i we use the equations

$$b_i = 1 - (b_1 + \cdots + \hat{b}_i + \cdots + b_N)$$
$$db_i = -(db_1 + \cdots + \widehat{db_i} + \cdots + db_N)$$

to uniquely write the restriction $\theta|_{\mathrm{St}\, v_i}$ of θ to St v_i

$$\theta|_{\mathrm{St}\, v_i} = \Sigma_j f_j\, db_{j_1} \cdots db_{j_q}$$

where the summation is taken over all distinct q-tuples j, and where b_i and db_i have been eliminated. If $v_i, v_{j_1}, \ldots, v_{j_q}$ are the vertices of a q-simplex of K then for every simplex σ of K which contains $v_i, v_{j_1}, \ldots, v_{j_q}$, $(f_j\, db_{j_1} \cdots db_{j_q})|_\sigma = 0$ so that $f_j|_\sigma = 0$; thus $f_j = 0$ on $\mathrm{St}\, v_i \cap (\bigcap_{\ell=1}^q \mathrm{St}\, v_{k_\ell})$. Applying Proposition 3 in the case of $K = \mathrm{St}\, v_i \cap (\bigcap_{\ell=1}^q \mathrm{St}\, v_{k_\ell})$, we find that f_j can be written $f_j = \Sigma_i g_i b_{i_1} \cdots b_{i_p}$ where, for each i, there is no $(p-1)$-simplex of $\mathrm{St}\, v_i \cap (\bigcap_{\ell=1}^q \mathrm{St}\, v_{k_\ell})$ with vertices $v_{i_1}, \ldots, v_{i_p}$. Clearly, then,

$$b_i(\theta|_{\mathrm{St}\, v_i}) = \Sigma g_i b_i b_{i_1} \cdots b_{i_p}\, db_{j_1} \cdots db_{j_q}$$

where, for each for each $i, i_1, \ldots, i_p, j_1, \ldots, j_q$ there is no $(p+q)$-simplex of K with vertices $v_i, v_{i_1}, \ldots, v_{i_p}, v_{j_1}, \ldots, v_{j_q}$. QED

The key to this proof of Sullivan's result is that the 0-forms $E^0(K)$ on K carry so much information. In fact $E^0(K)$ completely determines K in the following sense: Given a finite simplicial complex K, there is a uniquely defined set of generators $b_1, \ldots, b_N$ for $E^0(K)$ for which the following sequence is exact

$$0 \to \ker \rho \to Q[b_1, \ldots, b_N]/(b_1 + \cdots + b_N) - 1 \to E^0(K) \to 0$$

where ker ρ is as in Proposition 3.

PROPOSITION 5: *Let R be any ring. For every representation of R by an exact sequence of the form*

$$0 \to \ker \tilde{\rho} \to Q[x_1, \ldots, x_N]/(x_1 + \cdots + x_N) - 1 \to R \to 0$$

where ker $\tilde{\rho}$ *is generated by products* $x_{i_0} \cdots x_{i_p}$, *for* $p = 0, 1, 2, \ldots,$ *there is an essentially unique finite simplicial complex K for which $R = E^0(K)$. (Here "essentially unique" means up to a possible renaming of vertices.)*

Proof. Construct K as a subcomplex of the $(N-1)$-simplex Δ^{N-1} with vertices $v_1, \ldots, v_N$ by saying that $v_{i_0}, \ldots, v_{i_p}$ is a p-simplex of K iff $x_{i_0} \cdots x_{i_p}$ is non-zero in R. The kernel ker ρ *of the surjection*

$$\rho: E^0(\Delta^{N-1}) = Q[x_1, \ldots, x_N]/(x_1 + \cdots + x_N) - 1 \to E^0(K)$$

is again the ideal I generated by all products of the form $x_{i_0} \cdots x_{i_p}$ for which there is no p-simplex of K with vertices $v_{i_0}, \ldots, v_{i_p}$. (This is Proposition 3 except now it is possible that some vertex v_i of Δ^{N-1} is not a vertex of K. The proof of Proposition 3 can be modified to cover this case simply by observing that if v_i is not a vertex of K then $x_i \in I$ so that $x_i(f|_{\mathrm{St}\, v_i}) \in I$.) There is a natural map from $E^0(K)$ to R for which the following diagram commutes

$$\begin{array}{ccccccccc} 0 \to & \ker \rho & \longrightarrow & E^0(\Delta^{N-1}) & \longrightarrow & E^0(K) & \to 0 \\ & \downarrow = & & \downarrow = & & \downarrow & \\ 0 \to & \ker \tilde{\rho} & \to & Q[x_1, \ldots, x_N]/(x_1 + \cdots + x_N) - 1 & \to & R & \to 0 \end{array},$$

so $R = E^0(K)$ by the 5-Lemma. QED

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Department of Mathematical Sciences
I.U.P.U.I.
Indianapolis, Indiana, U.S.A.
46205