Adv. Appl. Prob. **42,** 739–760 (2010)
Printed in Northern Ireland

SELECTING THE LAST CONSECUTIVE RECORD IN A RECORD PROCESS

SHOOU-REN HSIAU,* *National Changhua University of Education*

Abstract

Suppose that $I_1, I_2, \dots$ is a sequence of independent Bernoulli random variables with $\mathrm{E}(I_n) = \lambda/(\lambda + n - 1)$, $n = 1, 2, \dots$. If λ is a positive integer k, $\{I_n\}_{n\geq 1}$ can be interpreted as a k-record process of a sequence of independent and identically distributed random variables with a common continuous distribution. When $I_{n-1}I_n = 1$, we say that a consecutive k-record occurs at time n. It is known that the total number of consecutive k-records is Poisson distributed with mean k. In fact, for general $\lambda > 0$, $\sum_{n=2}^{\infty} I_{n-1}I_n$ is Poisson distributed with mean λ. In this paper, we want to find an optimal stopping time τ_λ which maximizes the probability of stopping at the last n such that $I_{n-1}I_n = 1$. We prove that τ_λ is of threshold type, i.e. there exists a $t_\lambda \in \mathbb{N}$ such that $\tau_\lambda = \min\{n \mid n \geq t_\lambda,\ I_{n-1}I_n = 1\}$. We show that t_λ is increasing in λ and derive an explicit expression for t_λ. We also compute the maximum probability Q_λ of stopping at the last consecutive record and study the asymptotic behavior of Q_λ as $\lambda \to \infty$.

Keywords: Optimal stopping; threshold type; consecutive record; monotone stopping rule; record process

2010 Mathematics Subject Classification: Primary 60G40
Secondary 62L15; 60K99

1. Introduction

Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed (i.i.d.) random variables with common continuous distribution function F. Observing $X_1, X_2, \dots$ sequentially, we say that a record occurs at time n if $X_n > \max_{1\leq i\leq n-1} X_i$. Set $I_1 = 1$ and $I_n = \mathbf{1}_{\{X_n > \max_{1\leq i\leq n-1} X_i\}}$ for $n > 1$. Then $I_n = 1$ if a record occurs at time n, and $I_n = 0$ otherwise. We call $\{I_n\}_{n\geq 1}$ the record process of $\{X_n\}_{n\geq 1}$. For the sequence $I_1, I_2, \dots,$ it is well known that they are independent Bernoulli random variables with $\mathrm{E}(I_n) = 1/n$. Moreover, since $\sum_{n=1}^{\infty} \mathrm{P}(I_n = 1) = \sum_{n=1}^{\infty}(1/n) = \infty$, we have, by the Borel–Cantelli lemma, $\mathrm{P}(I_n = 1 \text{ infinitely often}) = 1$ and, therefore, $\sum_{n=1}^{\infty} I_n = \infty$ almost surely (a.s.). This means that, with probability 1, there are infinitely many records in the sequence $I_1, I_2, \dots$. However, the number of consecutive records in $I_1, I_2, \dots$ can be shown to be finite and Poisson distributed with mean 1. More precisely, we say that a consecutive record occurs at time n if $I_{n-1}I_n = 1$. Since $\sum_{n=1}^{\infty} \mathrm{E}(I_n I_{n+1}) = \sum_{n=1}^{\infty} 1/[n(n+1)] = 1$, $\sum_{n=1}^{\infty} I_n I_{n+1} < \infty$ a.s. In fact, Hahlin (1995) first proved that $\sum_{n=1}^{\infty} I_n I_{n+1}$ is Poisson distributed with mean 1. Around 1996, Persi Diaconis also gave an unpublished proof, and later a number of generalizations have been studied in the literature; see Csörgö and Wu (2000), Chern *et al.* (2000), Joffe *et al.* (2004), Mori (2001), Sethuraman and Sethuraman (2004), and Holst (2007).

Received 1 June 2009; revision received 21 March 2010.
* Postal address: Department of Mathematics, National Changhua University of Education, No. 1, Jin-De Road, Changhua 500, Taiwan, R.O.C. Email address: srhsiau@cc.ncue.edu.tw

https://doi.org/10.1239/aap/1282924061 Published online by Cambridge University Press

Besides the generalizations mentioned above, Arratia *et al.* (1992) applied the Ewens sampling formula to the permutations of $\{1, 2, \dots, n\}$ to obtain some Poisson process approximation theorems, which imply the following interesting result. If $I_1, I_2, \dots$ is a sequence of independent Bernoulli random variables with $\mathrm{E}(I_n) = \lambda/(\lambda + n - 1)$, $n = 1, 2, \dots$ ($\lambda > 0$), then $\sum_{n=1}^{\infty} I_n I_{n+1}$ is Poisson distributed with mean λ. For $\lambda = 1$, this result reduces to the previous result. Here we note that when λ is a positive integer k, $\{I_n\}_{n\geq 1}$ can be interpreted as the k-record process of $\{X_n\}_{n\geq 1}$ mentioned previously. In fact, if we let A_n denote the event that at most $k-1$ of $X_1, X_2, \dots, X_{n-1}$ are greater than X_n, i.e. X_n is a k-record in $X_1, X_2, \dots, X_n$, then $I_{A_k}, I_{A_{k+1}}, \dots$ form an independent Bernoulli sequence with $\mathrm{E}(I_{A_{k+n-1}}) = k/(k + n - 1)$, $n \geq 1$, and so $\{I_n\}_{n\geq 1}$ has the same distribution as $\{I_{A_{k+n-1}}\}_{n\geq 1}$.

Inspired by the above result, in this paper we study the following problem. Find an optimal strategy to maximize the probability of selecting the last consecutive record in $I_1, I_2, \dots$, i.e. the last n with $I_{n-1}I_n = 1$. Essentially, this is a kind of optimal stopping problem that we can state formally as follows. For a fixed $\lambda > 0$, let $I_1, I_2, \dots$ be a sequence of independent Bernoulli random variables with $\mathrm{E}(I_n) = \lambda/(\lambda + n - 1)$, $n \geq 1$. Let $S_n = I_1I_2 + I_2I_3 + \dots + I_{n-1}I_n$ and $S = S_\infty = \sum_{i=1}^{\infty} I_i I_{i+1}$. For each $n = 1, 2, \dots$, let $\mathcal{F}_n = \sigma(I_1, I_2, \dots, I_n)$ be the σ-field generated by $I_1, I_2, \dots, I_n$, let $\mathcal{F}_\infty = \sigma(I_1, I_2, \dots)$, and let C be the class of all stopping times adapted to $\{\mathcal{F}_n\}_{n=1}^{n=\infty}$. We want to find an optimal stopping time $\tau_\lambda \in C$ such that

$$\mathrm{P}(I_{\tau_\lambda - 1} I_{\tau_\lambda} = 1 \text{ and } S_{\tau_\lambda} = S) = \sup_{\tau \in C} \mathrm{P}(I_{\tau-1} I_\tau = 1 \text{ and } S_\tau = S).$$

Note that, since $\mathrm{P}(S = 0) = \lambda > 0$, a stopping time τ with $\mathrm{P}(\tau = \infty) > 0$ is allowed. But, we define $I_\infty = 0$ so that $\mathrm{P}(I_{\tau-1}I_\tau = 1 \text{ and } S_\tau = S \mid \tau = \infty) = 0$ for $\tau \in C$.

Problems of selecting the last event in a stochastic process have been studied by many authors; see, for example, Bruss (2000), Bruss and Paindaveine (2000), Hsiau and Yang (2002), Bruss and Louchard (2003), and Hsiau (2007). While infinite-horizon problems are typically much more involved than finite-horizon problems, fortunately the infinite-horizon problem addressed in this paper can be explicitly solved using the optimal stopping theory developed in Chow *et al.* (1971). In particular, the notion of the monotone case due to Chow and Robbins (1961) is very useful for solving our problem. In fact, by adopting a technique used in Dynkin (1963) to treat the secretary problem, we reformulated the problem in such a way that it is in the monotone case and so the optimal stopping time is of threshold type (see Section 2). We now present our main result.

Theorem 1. *The optimal stopping time τ_λ is of threshold type, that is, there exists a $t_\lambda \in \mathbb{N}$ such that*

$$\tau_\lambda = \min\{n \mid n \geq t_\lambda,\ I_{n-1}I_n = 1\}.$$

Moreover, the threshold t_λ can be described as follows:

(i) *if $\lambda \leq 1$ then $t_\lambda = 2$;*

(ii) *if $\lambda > 1$ then $t_\lambda = \lambda^2 - \lambda + 2$ when $\lambda^2 - \lambda \in \mathbb{N}$, and $t_\lambda \in \{\lfloor \lambda^2 - \lambda \rfloor + 2, \lfloor \lambda^2 - \lambda \rfloor + 3\}$ when $\lambda^2 - \lambda \notin \mathbb{N}$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.*

It seems quite surprising that the optimal threshold t_λ takes such a simple form. In Section 3 we first present several key lemmas and then use them to prove Theorem 1. The key lemmas are proved in Section 4. In Section 5 we prove that the threshold t_λ is increasing in λ. Finally, in Section 6 we compute the probability Q_λ of selecting the last consecutive record using the optimal stopping rule τ_λ. Moreover, as $\lambda \to \infty$, the asymptotic behavior of Q_λ is described analytically and numerically.

2. Monotone stopping rules

Because our goal is to select the last consecutive record, it is natural to focus our attention on the times at which consecutive records occur. Let T_1 denote the time at which the first consecutive record occurs, that is,

$$T_1 = \min\{n \mid n > 1,\ I_{n-1}I_n = 1\}.$$

Here we use the convention that $\min \varnothing = \infty$, which means that $T_1 = \infty$ if no consecutive record occurs. Similarly, we can define $T_2, T_3, \dots$ sequentially by

$$T_k = \min\{n \mid n > T_{k-1},\ I_{n-1}I_n = 1\}.$$

Note that $T_k = \infty$ if $T_{k-1} = \infty$ or no consecutive record occurs after time T_{k-1}. Moreover, let T denote the time at which the last consecutive record occurs, that is,

$$T = \max\{T_k \mid T_k < \infty\},$$

with the convention that $\max \varnothing = 0$, which means that $T = 0$ if $T_1 = \infty$ or, equivalently, no consecutive record occurs.

Since $I_1, I_2, \dots$ are independent, it is not difficult to see that $T_1, T_2, \dots$ form a Markov chain with state space $\{2, 3, \dots, \infty\}$. Hence, if we observed $T_1, T_2, \dots, T_{n-1}, T_n = t$ then the conditional probability that $T = t$ is

$$\mathrm{P}(T = t \mid T_n = t) = \begin{cases} \mathrm{P}\Bigl(\displaystyle\sum_{n=t}^{\infty} I_n I_{n+1} = 0 \,\Big|\, I_t = 1\Bigr) & \text{if } t < \infty, \\ 0 & \text{if } t = \infty. \end{cases} \tag{1}$$

Let $Y_n = \mathrm{P}(T = T_n \mid T_n)$ for $n = 1, 2, \dots$. Then our original optimal stopping problem is reduced to that for the process $\{Y_n, \mathcal{F}_{T_n}\}_{n\geq 1}$. More precisely, letting C' denote the class of all finite stopping times adapted to $\{\mathcal{F}_{T_n}\}_{n\geq 1}$, we want to find an optimal stopping time $\sigma_\lambda \in C'$ such that

$$\mathrm{E}(Y_{\sigma_\lambda}) = \sup_{\sigma \in C'} \mathrm{E}(Y_\sigma).$$

The idea of the above new version for our original problem comes from a technique used in Dynkin (1963) to reformulate the classical secretary problem so that it is monotone. In fact, an optimal stopping problem for $\{X_n, \mathcal{F}_n\}_{n\geq 1}$ is said to be monotone if the events $A_n = \{X_n \geq \mathrm{E}(X_{n+1} \mid \mathcal{F}_n)\}$, $n = 1, 2, \dots$, satisfy the following conditions:

$$A_1 \subseteq A_2 \subseteq \dots, \qquad \mathrm{P}\Bigl(\bigcup_{n=1}^{\infty} A_n\Bigr) = 1.$$

If the optimal stopping problem for $\{X_n, \mathcal{F}_n\}_{n\geq 1}$ is monotone then the stopping rule

$$\widetilde{\sigma} = \min\{n \mid X_n \geq \mathrm{E}(X_{n+1} \mid \mathcal{F}_n)\} \tag{2}$$

is important owing to the following result.

https://doi.org/10.1239/aap/1282924061 Published online by Cambridge University Press

Theorem 2. (Chow and Robbins (1961).) *Suppose that the optimal stopping problem for* $\{X_n, \mathcal{F}_n\}_{n\ge 1}$ *is monotone. If the stopping rule* $\widetilde{\sigma}$ *defined as in (2) satisfies*

$$\liminf_n \int_{\{\widetilde{\sigma}>n\}} X_n^+ \,\mathrm{dP} = 0$$

then $\mathrm{E}(X_{\widetilde{\sigma}}) \ge \mathrm{E}(X_\sigma)$ *holds for all finite stopping times* σ *for which*

$$\liminf_n \int_{\{\sigma>n\}} X_n^- \,\mathrm{dP} = 0.$$

For the new version of our original problem, i.e. the optimal stopping problem for $\{Y_n, \mathcal{F}_{T_n}\}_{n\ge 1}$, we will show that it is monotone. To this end, we first introduce the following notation:

$${}_nP_0 = \mathrm{P}\biggl(\sum_{k=n}^{\infty} I_k I_{k+1} = 0 \biggm| I_n = 1\biggr)$$

and

$${}_nP_1 = \mathrm{P}\biggl(\sum_{k=n}^{\infty} I_k I_{k+1} = 1 \biggm| I_n = 1\biggr).$$

For ${}_nP_0$ and ${}_nP_1$, the following important property holds, which will be proved in Section 3.

Lemma 1. (i) ${}_nP_0 \to 1$ *and* ${}_nP_1 \to 0$ *as* $n \to \infty$*; hence,* ${}_nP_1/{}_nP_0 \to 0$ *as* $n \to \infty$.

(ii) ${}_nP_1/{}_nP_0$ *is decreasing in* n.

(iii) *There exists a positive integer* $\tilde{t}_\lambda$ *such that*

$$\tilde{t}_\lambda = \min\biggl\{n \biggm| \frac{{}_nP_1}{{}_nP_0} \le 1\biggr\}.$$

Consequently, ${}_nP_0 \ge {}_nP_1$ *if and only if* $n \ge \tilde{t}_\lambda$.

Now recall that $Y_n = \mathrm{P}(T = T_n \mid T_n)$, determined as in (1). Hence, $\mathrm{E}(Y_{n+1} \mid \mathcal{F}_{T_n}) = \mathrm{E}(Y_{n+1} \mid T_n) = \mathrm{P}(T = T_{n+1} \mid T_n)$. Moreover,

$$\mathrm{P}(T = T_{n+1} \mid T_n = t) = \begin{cases} \mathrm{P}\biggl(\displaystyle\sum_{k=t}^{\infty} I_k I_{k+1} = 1 \biggm| I_t = 1\biggr) & \text{if } t < \infty, \\ 0 & \text{if } t = \infty. \end{cases} \tag{3}$$

In view of (1), (3), and the definitions of ${}_nP_0$ and ${}_nP_1$, we see that, on $\{T_n = t\}$,

$$Y_n = \begin{cases} {}_tP_0 & \text{if } t < \infty, \\ 0 & \text{if } t = \infty, \end{cases} \tag{4}$$

and

$$\mathrm{E}(Y_{n+1} \mid \mathcal{F}_{T_n}) = \begin{cases} {}_tP_1 & \text{if } t < \infty, \\ 0 & \text{if } t = \infty. \end{cases} \tag{5}$$

https://doi.org/10.1239/aap/1282924061 Published online by Cambridge University Press

On the other hand, by the definition of $\tilde{t}_\lambda$, ${}_tP_0 \geq {}_tP_1$ if and only if $t \geq \tilde{t}_\lambda$. Hence, if $T_n = t$ and $Y_n \geq \mathrm{E}(Y_{n+1} \mid \mathcal{F}_{T_n})$, then $\infty > t \geq \tilde{t}_\lambda$ or $t = \infty$, and so $T_{n+1} = t' > \tilde{t}_\lambda$ or $T_{n+1} = \infty$, either of which implies that $Y_{n+1} \geq \mathrm{E}(Y_{n+2} \mid \mathcal{F}_{T_{n+1}})$. It turns out that

$$\{Y_n \geq \mathrm{E}(Y_{n+1} \mid \mathcal{F}_{T_n})\} \subseteq \{Y_{n+1} \geq \mathrm{E}(Y_{n+2} \mid \mathcal{F}_{T_{n+1}})\}. \tag{6}$$

Moreover, since $\sum_{n=1}^{\infty} I_n I_{n+1} < \infty$ with probability 1, we have $\mathrm{P}(T_n < \infty \text{ for all } n) = 0$ and, hence, $\mathrm{P}(T_n = \infty \text{ for some } n) = 1$. This implies that $\mathrm{P}(Y_n = \mathrm{E}(Y_{n+1} \mid \mathcal{F}_{T_n}) = 0$ $\text{for some } n) = 1$ and so

$$\mathrm{P}\biggl(\bigcup_{n=1}^{\infty}\{Y_n \geq \mathrm{E}(Y_{n+1} \mid \mathcal{F}_{T_n})\}\biggr) = 1. \tag{7}$$

By combining (6) and (7), it follows that the optimal stopping problem for $\{Y_n, \mathcal{F}_{T_n}\}_{n\geq 1}$ is monotone. Hence, the following stopping rule is a candidate of the optimal stopping rules:

$$\sigma_\lambda = \min\{n \mid Y_n \geq \mathrm{E}(Y_{n+1} \mid \mathcal{F}_{T_n})\}. \tag{8}$$

Note that $0 \leq Y_n \leq 1$ and $Y_n \to 0$ a.s., since $\mathrm{P}(T_n = \infty \text{ for some } n) = 1$. By the bounded convergence theorem we have

$$\lim_{n\to\infty} \int |Y_n| \, \mathrm{dP} = 0.$$

This implies that

$$\liminf_{n} \int_{\{\sigma_\lambda > n\}} Y_n^+ \, \mathrm{dP} = 0.$$

Moreover, since $0 \leq Y_n \leq 1$, it is true that $Y_n^- = 0$ and, hence,

$$\liminf_{n} \int_{\{\sigma > n\}} Y_n^- \, \mathrm{dP} = 0$$

holds for all finite stopping times σ. Now we can apply Theorem 2 to $\{Y_n, \mathcal{F}_{T_n}\}_{n\geq 1}$ to conclude that $\mathrm{E}(Y_{\sigma_\lambda}) \geq \mathrm{E}(Y_\sigma)$ holds for all finite stopping times σ, i.e.

$$\mathrm{E}(Y_{\sigma_\lambda}) = \sup_{\sigma \in C'} \mathrm{E}(Y_\sigma).$$

Note that, in view of (4), (5), (8), and Lemma 1, our original optimal stopping problem has the optimal stopping rule

$$\tau_\lambda = \min\{n \mid n \geq t_\lambda, I_{n-1} I_n = 1\},$$

where $t_\lambda = \max(\tilde{t}_\lambda, 2)$. We call t_λ the threshold of τ_λ. So far, we have proved the first half of Theorem 1, that is, the optimal stopping rule τ_λ is of threshold type. In Section 3 we investigate ${}_nP_0$ and ${}_nP_1$, and then express t_λ in terms of λ explicitly.

3. The threshold t_λ: proof of Theorem 1

In this section we first prove Lemma 1 and then describe the threshold t_λ in terms of λ. To this end, we need the exact expressions for ${}_nP_0$ and ${}_nP_1$.

Lemma 2. *For each $n = 1, 2, \dots,$ we have*

$$ {}_nP_0 = \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^j (\lambda+1)_{[j]}}{j!\,(\lambda+n)_{[j]}} \tag{9} $$

and

$$ {}_nP_1 = \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{j+1} (\lambda+1)_{[j+1]}}{j!\,(\lambda+n)_{[j+1]}}, \tag{10} $$

where $x_{[0]} = 1$ and $x_{[j]} = x(x+1)\cdots(x+j-1)$ for $j = 1, 2, \dots.$

Using the above expressions for ${}_nP_0$ and ${}_nP_1$, we can establish recurrence relations for ${}_nP_0$ and ${}_nP_1$, as well as for $r_n = {}_nP_1/{}_nP_0$.

Lemma 3. *For each $n = 1, 2, \dots,$ we have*

$$ (\lambda+n)_nP_0 = (\lambda+n)_{n+1}P_0 - {}_{n+1}P_1, \tag{11} $$

$$ (\lambda+n)_nP_1 = \lambda(\lambda+1)_{n+1}P_0 - \lambda_{n+1}P_1. \tag{12} $$

Moreover, if we set $r_n = {}_nP_1/{}_nP_0$ then, for $n = 2, 3, \dots,$ we have $r_n \neq \lambda$ and

$$ r_{n+1} = \lambda + n + \frac{\lambda(n-1)}{r_n - \lambda}. \tag{13} $$

Here we have to note that, from (9) and (10), ${}_1P_0 = \mathrm{e}^{-\lambda}$ and ${}_1P_1 = \lambda\mathrm{e}^{-\lambda}$, and so $r_1 = \lambda$, which explains why relation (13) does not hold for $n = 1$. The proofs of Lemmas 2 and 3 need tedious computations, and so are deferred to Section 4. In the following, we use recurrence relation (13) to prove that r_n is decreasing in n and converges to 0. This work is much involved and we need to investigate the following sequence of functions:

$$ f_n(x) = \lambda + n + \frac{\lambda(n-1)}{x-\lambda}, \qquad x \neq \lambda, $$

for $n = 2, 3, \dots.$ Note that the sequence $r_2, r_3, \dots$ satisfies the relation $r_{n+1} = f_n(r_n)$.

Lemma 4. *For each $n = 2, 3, \dots,$ $f_n(x)$ satisfies the following properties.*

(i) *$x \leq 0$ implies that $f_n(x) > \lambda$ and $x > \lambda$ implies that $f_n(x) > \lambda + n$.*

(ii) *$x < y < \lambda$ implies that $f_n(x) > f_n(y)$.*

(iii) *$f_n(x) = x < \lambda$ if and only if $x = \frac{1}{2}(2\lambda + n - \sqrt{n^2 + 4(n-1)\lambda})$.*

(iv) *Let $x_n = \frac{1}{2}(2\lambda + n - \sqrt{n^2 + 4(n-1)\lambda})$. Then $f_n(x_n) = x_n$, $x_n \searrow 0$, $x_n - x_{n+1} > x_{n+1} - x_{n+2}$, and $f_{n+1}(x_n) < x_{n+2}$ for $n \geq 1$.*

(v) *$0 < x < \lambda$ and $0 < y < \lambda$ imply that $|f_n(x) - f_n(y)| \geq (n-1)|x-y|/\lambda$.*

Proof. If $x \leq 0$ then $\lambda/(x-\lambda) \geq -1$ and so

$$ \lambda + n + \frac{\lambda(n-1)}{x-\lambda} \geq \lambda + n - (n-1) > \lambda, $$

https://doi.org/10.1239/aap/1282924061 Published online by Cambridge University Press

which implies that $f_n(x) > \lambda$. Moreover, if $x > \lambda$ then $x - \lambda > 0$ and so

$$\lambda + n + \frac{\lambda(n-1)}{x-\lambda} > \lambda + n,$$

which implies that $f_n(x) > \lambda + n$. Hence, (i) follows.

If $x < y < \lambda$ then $x - \lambda < y - \lambda < 0$ and so

$$\lambda + n + \frac{\lambda(n-1)}{x-\lambda} > \lambda + n + \frac{\lambda(n-1)}{y-\lambda}.$$

Hence, (ii) follows.

If $f_n(x) = x$ then

$$\lambda + n + \frac{\lambda(n-1)}{x-\lambda} = x,$$

which yields the quadratic equation

$$x^2 - (2\lambda + n)x + \lambda(\lambda + 1) = 0.$$

It is easy to verify that this equation has just one root less than λ, that is,

$$x = \tfrac{1}{2}\big(2\lambda + n - \sqrt{n^2 + 4(n-1)\lambda}\big).$$

Conversely, if $x = (2\lambda + n - \sqrt{n^2 + 4(n-1)\lambda})/2$ then $f_n(x) = x < \lambda$. Hence, (iii) follows.

Now let $x_n = (2\lambda + n - \sqrt{n^2 + 4(n-1)\lambda})/2$, $n = 2, 3, \dots$. By (iii), $f_n(x_n) = x_n$. Observe that

$$\frac{1}{2}\big(2\lambda + n - \sqrt{n^2 + 4(n-1)\lambda}\big) = \frac{2\lambda(\lambda+1)}{2\lambda + n + \sqrt{n^2 + 4(n-1)\lambda}}.$$

It is clear that, as $n \to \infty$,

$$\frac{2\lambda(\lambda+1)}{2\lambda + n + \sqrt{n^2 + 4(n-1)\lambda}} \searrow 0,$$

which implies that $x_n \searrow 0$.

Consider the function

$$g(t) = \tfrac{1}{2}\big(2\lambda + t - \sqrt{t^2 + 4(t-1)\lambda}\big), \qquad t \geq 1.$$

It is not difficult to verify that

$$g''(t) = \frac{2\lambda(\lambda+1)}{[t^2 + 4(t-1)\lambda]^{3/2}}, \qquad t \geq 1.$$

It is clear that $g''(t) > 0$ for all $t \geq 1$, and so $g(t)$ is a convex function in $t \geq 1$. Therefore,

$$\frac{g(t+2) - g(t+1)}{(t+2) - (t+1)} > \frac{g(t+1) - g(t)}{(t+1) - t}$$

holds for all $t \geq 1$. This implies that, for $n = 1, 2, \dots$, $g(n) - g(n+1) > g(n+1) - g(n+2)$, and so $x_n - x_{n+1} > x_{n+1} - x_{n+2}$.

To verify that $f_{n+1}(x_n) < x_{n+2}$, observe that

$$\begin{aligned} f_{n+1}(x_n) &= \lambda + n + 1 + \frac{n\lambda}{x_n - \lambda} \\ &= \lambda + n + 1 + \frac{n\lambda}{x_{n+1} - \lambda} + \left(\frac{n\lambda}{x_n - \lambda} - \frac{n\lambda}{x_{n+1} - \lambda}\right) \\ &= f_{n+1}(x_{n+1}) + \frac{n\lambda(x_{n+1} - x_n)}{(x_n - \lambda)(x_{n+1} - \lambda)} \\ &= x_{n+1} - \frac{n\lambda}{(x_n - \lambda)(x_{n+1} - \lambda)}(x_n - x_{n+1}). \end{aligned}$$

From this we see that

$$f_{n+1}(x_n) - x_{n+2} = (x_{n+1} - x_{n+2}) - \frac{n\lambda}{(x_n - \lambda)(x_{n+1} - \lambda)}(x_n - x_{n+1}).$$

Since $x_n - x_{n+1} > x_{n+1} - x_{n+2} > 0$, it is clear that $n\lambda/(x_n - \lambda)(x_{n+1} - \lambda) > 1$ implies that $f_{n+1}(x_n) - x_{n+2} < 0$. In the following, we prove that $n\lambda/(x_n - \lambda)(x_{n+1} - \lambda) > 1$. In fact, it is not difficult to show that $0 < \lambda - x_n < \sqrt{(n-1)\lambda}$ and $0 < \lambda - x_{n+1} < \sqrt{n\lambda}$, which imply that

$$\frac{n\lambda}{(x_n - \lambda)(x_{n+1} - \lambda)} > 1.$$

Hence, the above assertion is proved, and (iv) follows.

Finally, observe that

$$f_n(x) - f_n(y) = \frac{\lambda(n-1)(y-x)}{(x-\lambda)(y-\lambda)},$$

and so

$$|f_n(x) - f_n(y)| = \frac{\lambda(n-1)}{(x-\lambda)(y-\lambda)}|x - y|.$$

If $0 < x < \lambda$ and $0 < y < \lambda$, then

$$\frac{\lambda(n-1)}{|(x-\lambda)(y-\lambda)|} > \frac{\lambda(n-1)}{\lambda^2} = \frac{n-1}{\lambda},$$

from which (v) follows.

We are now in a position to prove Lemma 1.

Proof of Lemma 1. Because (iii) is an easy consequence of (i) and (ii), we just prove (i) and (ii). We first prove that ${}_nP_1/{}_nP_0$ converges to 0 as $n \to \infty$. By the definitions of ${}_nP_0$ and ${}_nP_1$, it is clear that $0 \le {}_nP_0 + {}_nP_1 \le 1$, from which we see that ${}_nP_0 \to 1$ implies that ${}_nP_1 \to 0$. Therefore, it suffices to prove that ${}_nP_0 \to 1$. Since $I_1, I_2, \dots$ are independent with $\mathrm{E}(I_k) = \lambda/(\lambda + k - 1)$, we have

$$\begin{aligned} \mathrm{P}\biggl(\sum_{k=n}^{\infty} I_k I_{k+1} = 0 \biggm| I_n = 1\biggr) &= 1 - \mathrm{P}\biggl(\sum_{k=n}^{\infty} I_k I_{k+1} \ge 1 \biggm| I_n = 1\biggr) \\ &= 1 - \mathrm{P}\biggl(I_{n+1} + \sum_{k=n+1}^{\infty} I_k I_{k+1} \ge 1\biggr) \\ &\ge 1 - \mathrm{E}\biggl(I_{n+1} + \sum_{k=n+1}^{\infty} I_k I_{k+1}\biggr) \end{aligned}$$

$$= 1 - \frac{\lambda}{\lambda + n} - \sum_{k=n+1}^{\infty} \frac{\lambda}{\lambda + k - 1} \frac{\lambda}{\lambda + k}$$
$$= 1 - \frac{\lambda(\lambda + 1)}{\lambda + n}.$$

Since $\lambda(\lambda + 1)/(\lambda + n) \to 0$ as $n \to \infty$, we see that ${}_nP_0 \to 1$ as $n \to \infty$. Hence, (i) follows.

Next, we want to prove that ${}_nP_1/{}_nP_0$ is decreasing in n. In fact, we have a more delicate result (see Lemma 5 below): $x_n < {}_nP_1/{}_nP_0 < x_{n-1}$. Since, by Lemma 4(iv), $\{x_n\}_{n\geq 2}$ is a decreasing sequence, we see that ${}_nP_1/{}_nP_0$ is a decreasing sequence in n.

As before, we write $r_n = {}_nP_1/{}_nP_0$ for $n = 1, 2, \ldots$. We know, from (9) and (10), that $r_1 = \lambda$. The following lemma describes the location of r_n for $n \geq 2$.

Lemma 5. *For each $n = 2, 3, \ldots$, we have $x_n < r_n < x_{n-1}$.*

Proof. Recall that $f_n(x_n) = x_n$ and $r_{n+1} = f_n(r_n)$ for $n \geq 2$. We first claim that $0 < r_n < \lambda$ for all $n \geq 2$. If not, there exists some r_k such that $r_k > \lambda$ (note that $r_k \neq \lambda$ by Lemma 3). Then, by Lemma 4(i) we have $r_{k+1} = f_k(r_k) > \lambda + k$, and in turn $r_{k+2} = f_{k+1}(r_{k+1}) > \lambda + k + 1, \ldots$. This yields the fact that $r_n \to \infty$, which contradicts the fact that $r_n \to 0$ (see Lemma 1(i)). Hence, $0 < r_n < \lambda$ for all $n \geq 2$.

Next, we prove that $x_n < r_n < x_{n-1}$ for all $n \geq 2$. Suppose that, for some r_k, $r_k \leq x_k < \lambda$. Then, by Lemma 4(ii), $r_{k+1} = f_k(r_k) \geq f_k(x_k) = x_k$. This states that if there is some r_n not satisfying $x_n < r_n < x_{n-1}$ then $r_n \geq x_{n-1}$ or $r_{n+1} \geq x_n$. We now proceed to prove the claim by contradiction. Suppose that $\lambda > r_k \geq x_{k-1}$ for some $k \geq 2$. By Lemma 4(iv), $r_{k+1} = f_k(r_k) \leq f_k(x_{k-1}) < x_{k+1}$, and so $r_{k+2} = f_{k+1}(r_{k+1}) > f_{k+1}(x_{k+1}) = x_{k+1}$. Furthermore, $r_{k+3} = f_{k+2}(r_{k+2}) < f_{k+2}(x_{k+1}) < x_{k+3}$, and so $r_{k+4} = f_{k+3}(r_{k+3}) > f_{k+3}(f_{k+2}(x_{k+1})) > f_{k+3}(x_{k+3}) = x_{k+3}$. In general, if we set $s_1 = x_{k+1}$, $s_2 = f_{k+2}(s_1)$, $s_3 = f_{k+3}(s_2)$, etc., then applying Lemma 4(iv) successively yields $r_{k+2} > s_1$, $r_{k+3} < s_2 < x_{k+3}$, $r_{k+4} > s_3 > x_{k+3}$, $r_{k+5} < s_4 < x_{k+5}$, $r_{k+6} > s_5 > x_{k+5}$, etc. Because $0 < r_n < \lambda$ and $0 < x_n < \lambda$ for all $n \geq 2$, the above inequalities imply that $0 < s_n < \lambda$ for all $n \geq 1$. Now applying Lemma 4(v) to the case $x = r_{k+2}$ and $y = s_1$ yields

$$|r_{k+3} - s_2| = |f_{k+2}(r_{k+2}) - f_{k+2}(s_1)| \geq \frac{k+1}{\lambda} |r_{k+2} - s_1|.$$

Similarly, in general, we obtain

$$|r_{k+\ell+1} - s_\ell| = |f_{k+\ell}(r_{k+\ell}) - f_{k+\ell}(s_{\ell-1})| \geq \frac{k+\ell-1}{\lambda} |r_{k+\ell} - s_{\ell-1}|.$$

Combining these inequalities, it is not difficult to see that

$$|r_{k+\ell+1} - s_\ell| \geq \frac{(k+1)(k+2)\cdots(k+\ell-1)}{\lambda^{\ell-1}} |r_{k+2} - s_1|.$$

Since $r_{k+2} - s_1 > 0$ and $(k+1)(k+2)\cdots(k+\ell-1)/\lambda^{\ell-1} \to \infty$ as $\ell \to \infty$, we have $|r_{k+\ell+1} - s_\ell| \to \infty$ as $\ell \to \infty$, in contradiction to the facts that $0 < r_n < \lambda$ and $0 < s_n < \lambda$. Hence, $x_n < r_n < x_{n-1}$ for all $n \geq 2$.

Now we can use Lemma 5 to derive the threshold t_λ of the optimal stopping rule τ_λ. Recall that $\tilde{t}_\lambda = \min\{n \mid r_n \leq 1\}$ and $t_\lambda = \max\{\tilde{t}_\lambda, 2\}$. If $0 < \lambda \leq 1$ then $r_1 = \lambda \leq 1$, and so $\tilde{t}_\lambda = 1$

and $t_\lambda = 2$. If $\lambda > 1$ then $r_1 = \lambda > 1$, and so $\tilde{t}_\lambda \geq 2$ and $t_\lambda = \tilde{t}_\lambda$. In this case if, for some $k \geq 2$, $x_{k-1} = 1$, then, by Lemma 5, $r_k < x_{k-1} = 1 < r_{k-1}$ and so $t_\lambda = k$. If $x_k < 1 < x_{k-1}$ then $r_{k+1} < x_k < r_k \leq 1 < x_{k-1} < r_{k-1}$ or $r_{k+1} < x_k < 1 < r_k < x_{k-1}$, and so $t_\lambda = k$ or $k + 1$. By the definition of x_n, the statement $x_{k-1} = 1$ is just

$$\tfrac{1}{2}(2\lambda + k - 1 - \sqrt{(k-1)^2 + 4(k-2)\lambda}) = 1,$$

which can be simplified to the form $k = \lambda^2 - \lambda + 2$. Similarly, the statement $x_k < 1 < x_{k-1}$ is just

$$\tfrac{1}{2}(2\lambda + k - \sqrt{k^2 + 4(k-1)\lambda}) < 1 < \tfrac{1}{2}(2\lambda + k - 1 - \sqrt{(k-1)^2 + 4(k-2)\lambda}),$$

which can be simplified to the form $k < \lambda^2 - \lambda + 2 < k + 1$. To summarize,

(i) if $\lambda \leq 1$ then $t_\lambda = 2$;

(ii) if $\lambda > 1$ then $t_\lambda = \lambda^2 - \lambda + 2$ when $\lambda^2 - \lambda \in \mathbb{N}$, and $t_\lambda \in \{\lfloor \lambda^2 - \lambda \rfloor + 2, \lfloor \lambda^2 - \lambda \rfloor + 3\}$ when $\lambda^2 - \lambda \notin \mathbb{N}$.

This completes the proof of Theorem 1.

4. Proofs of Lemmas 2 and 3

In this section we compute ${}_nP_0$ and ${}_nP_1$ and then derive the recurrence relations (11) and (12). We first need a generalized version of the inclusion–exclusion formula.

Theorem 3. (Generalized inclusion–exclusion formula.) *Let $A_1, A_2, \ldots$ be a sequence of events. For each positive integer k, set $S_k = \sum_{i_1<i_2<\cdots<i_k} \mathrm{P}(A_{i_1} A_{i_2} \cdots A_{i_k})$. Let ℓ be a fixed positive integer, and let q_ℓ denote the probability that exactly ℓ of $A_1, A_2, \ldots$ occur. Then*

$$q_\ell = \sum_{k=\ell}^{\infty} (-1)^{k-\ell} C_\ell^k S_k$$

provided that $\sum_{k=\ell}^{\infty} C_\ell^k S_k$ is finite. Here $C_\ell^k = k!/[\ell!\,(k-\ell)!]$.

Similarly, let $\tilde{q}_\ell$ denote the probability that at least ℓ of $A_1, A_2, \ldots$ occur. Then

$$\tilde{q}_\ell = \sum_{i=\ell}^{\infty} (-1)^{i-\ell} C_{\ell-1}^{i-1} S_i$$

provided that $\sum_{i=\ell}^{\infty} C_{\ell-1}^{i-1} S_i$ is finite.

Proof. For any positive integers n and k, set ${}_nS_k = \sum_{i_1<i_2<\cdots<i_k\leq n} \mathrm{P}(A_{i_1} A_{i_2} \cdots A_{i_k})$. Let ${}_nq_\ell$ denote the probability that exactly ℓ of $A_1, A_2, \ldots, A_n$ occur. Then, by the inclusive–exclusive formula we have

$${}_nq_\ell = \sum_{k=\ell}^{n} (-1)^{k-\ell} C_\ell^k \, {}_nS_k.$$

If $\sum_{k=\ell}^{\infty} C_\ell^k S_k$ is finite then each S_k is finite. This implies that ${}_nS_k \to S_k$ as $n \to \infty$ for each k and so $\sum_{k=\ell}^{n} (-1)^{k-\ell} C_\ell^k \, {}_nS_k \to \sum_{k=\ell}^{\infty} (-1)^{k-\ell} C_\ell^k S_k$ as $n \to \infty$, by the dominated convergence theorem. On the other hand, it is clear that ${}_nq_\ell \to q_\ell$ as $n \to \infty$, by the definitions of ${}_nq_\ell$ and q_ℓ. Hence, $q_\ell = \sum_{k=\ell}^{\infty} (-1)^{k-\ell} C_\ell^k S_k$.

The proof of the second part of the theorem is similar and is thus omitted.

We can now prove Lemma 2.

Proof of Lemma 2. We first compute ${}_nP_1$. By definition,

$$ {}_nP_1 = \mathrm{P}\bigg(\sum_{k=n}^{\infty} I_k I_{k+1} = 1 \;\bigg|\; I_n = 1\bigg) = \mathrm{P}\bigg(I_{n+1} + \sum_{k=n+1}^{\infty} I_k I_{k+1} = 1\bigg). $$

Define $A_1 = \{I_{n+1} = 1\}$ and $A_j = \{I_{n+j-1} I_{n+j} = 1\}$ for $j \geq 2$. Then

$$ \mathrm{P}\bigg(I_{n+1} + \sum_{k=n+1}^{\infty} I_k I_{k+1} = 1\bigg) = \mathrm{P}(\text{exactly one of } A_1, A_2, \ldots \text{occurs}). $$

For each positive integer k, set

$$ S_k = \sum_{i_1 < i_2 < \cdots < i_k} \mathrm{P}(A_{i_1} A_{i_2} \cdots A_{i_k}). $$

We claim that $S_k = \lambda^k (\lambda + 1)_{[k]} / (k!\,(\lambda + n)_{[k]})$. Then

$$ \begin{aligned} \sum_{k=1}^{\infty} C_1^k S_k &= \sum_{k=1}^{\infty} k \frac{\lambda^k (\lambda + 1)_{[k]}}{k!\,(\lambda + n)_{[k]}} \\ &= \sum_{k=1}^{\infty} \frac{\lambda^k (\lambda + 1)_{[k]}}{(k-1)!\,(\lambda + n)_{[k]}} \\ &\leq \sum_{k=1}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda \mathrm{e}^{\lambda} \\ &< \infty. \end{aligned} $$

Therefore, by Theorem 3 we have

$$ \begin{aligned} \mathrm{P}(\text{exactly one of } A_1, A_2, \ldots \text{ occurs}) &= \sum_{k=1}^{\infty} (-1)^{k-1} C_1^k S_k \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} k \frac{\lambda^k (\lambda + 1)_{[k]}}{k!\,(\lambda + n)_{[k]}} \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{j+1} (\lambda + 1)_{[j+1]}}{j!\,(\lambda + n)_{[j+1]}}, \end{aligned} $$

as required.

As for the computation of ${}_nP_0$, we observe that

$$ \begin{aligned} {}_nP_0 &= \mathrm{P}\bigg(\sum_{k=n}^{\infty} I_k I_{k+1} = 0 \;\bigg|\; I_n = 1\bigg) \\ &= \mathrm{P}\bigg(I_{n+1} + \sum_{k=n+1}^{\infty} I_k I_{k+1} = 0\bigg) \\ &= \mathrm{P}(\text{none of } A_1, A_2, \ldots \text{ occur}) \end{aligned} $$

$$
\begin{aligned}
&= 1 - \mathrm{P}(\text{at least one of } A_1, A_2, \dots \text{ occurs}) \\
&= 1 - \sum_{k=1}^{\infty} (-1)^{k-1} S_k \\
&= 1 - \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\lambda^k (\lambda+1)_{[k]}}{k!\,(\lambda+n)_{[k]}} \\
&= \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^j (\lambda+1)_{[j]}}{j!\,(\lambda+n)_{[j]}},
\end{aligned}
$$

as required. Here we have used the fact that $\sum_{k=1}^{\infty} S_k$ is finite, which follows from

$$\sum_{k=1}^{\infty} C_1^k S_k < \infty.$$

It remains to prove that $S_k = \lambda^k(\lambda+1)_{[k]}/(k!\,(\lambda+n)_{[k]})$. This result can be proved by mathematical induction on both n and k. We write it down as the following lemma. Note that the S_k here is just the $S_k^{(n)}$ in the following lemma.

Lemma 6. *Let $B_i = \{I_i I_{i+1} = 1\}$ and $\tilde{B}_i = \{I_{i+1} = 1\}$ for $i \ge 1$. Set, for any positive integers n and k,*

$$S_k^{(n)} = \sum_{n<i_2<i_3<\cdots<i_k} \mathrm{P}(\tilde{B}_n B_{i_2} B_{i_3} \cdots B_{i_k}) + \sum_{n<i_1<i_2<\cdots<i_k} \mathrm{P}(B_{i_1} B_{i_2} \cdots B_{i_k}).$$

Then $S_k^{(n)} = \lambda^k(\lambda+1)_{[k]}/(k!\,(\lambda+n)_{[k]})$.

Proof. For $k = 1$ and each n,

$$S_1^{(n)} = \mathrm{P}(\tilde{B}_n) + \sum_{i>n} \mathrm{P}(B_i) = \frac{\lambda}{\lambda+n} + \sum_{i>n} \frac{\lambda}{\lambda+i-1}\frac{\lambda}{\lambda+i} = \frac{\lambda(\lambda+1)_{[1]}}{1!\,(\lambda+n)_{[1]}}.$$

Suppose that the assertion is true for $k \le m$ and each n. Then, for $k = m+1$ and each n,

$$
\begin{aligned}
S_{m+1}^{(n)} &= \sum_{n<i_2<i_3<\cdots<i_{m+1}} \mathrm{P}(\tilde{B}_n B_{i_2} B_{i_3} \cdots B_{i_{m+1}}) + \sum_{n<i_1<i_2<\cdots<i_{m+1}} \mathrm{P}(B_{i_1} B_{i_2} \cdots B_{i_{m+1}}) \\
&= \mathrm{P}(\tilde{B}_n) \sum_{n<i_2<i_3<\cdots<i_{m+1}} \mathrm{P}(B_{i_2} B_{i_3} \cdots B_{i_{m+1}} \mid \tilde{B}_n) \\
&\quad + \sum_{j>n} \mathrm{P}(B_j) \sum_{j<i_2<i_3<\cdots<i_{m+1}} \mathrm{P}(B_{i_2} B_{i_3} \cdots B_{i_{m+1}} \mid B_j).
\end{aligned}
$$

It is not difficult to verify that

$$\sum_{n<i_2<i_3<\cdots<i_{m+1}} \mathrm{P}(B_{i_2} B_{i_3} \cdots B_{i_{m+1}} \mid \tilde{B}_n) = S_m^{(n+1)}$$

and

$$\sum_{j<i_2<i_3<\cdots<i_{m+1}} \mathrm{P}(B_{i_2} B_{i_3} \cdots B_{i_{m+1}} \mid B_j) = S_m^{(j+1)}.$$

https://doi.org/10.1239/aap/1282924061 Published online by Cambridge University Press

By induction we have

$$
\begin{aligned}
S_{m+1}^{(n)} &= \mathrm{P}(\tilde{B}_n) S_m^{(n+1)} + \sum_{j>n} \mathrm{P}(B_j) S_m^{(j+1)} \\
&= \frac{\lambda}{\lambda+n} \frac{\lambda^m (\lambda+1)_{[m]}}{m!\,(\lambda+n+1)_{[m]}} + \sum_{j>n} \frac{\lambda^2}{(\lambda+j-1)(\lambda+j)} \frac{\lambda^m (\lambda+1)_{[m]}}{m!\,(\lambda+j+1)_{[m]}} \\
&= \frac{\lambda^{m+1}(\lambda+1)_{[m]}}{m!\,(\lambda+n)_{[m+1]}} + \sum_{j>n} \frac{\lambda^{m+2}(\lambda+1)_{[m]}}{m!\,(\lambda+j-1)_{[m+2]}} \\
&= \frac{\lambda^{m+1}(\lambda+1)_{[m]}}{m!\,(\lambda+n)_{[m+1]}} \\
&\quad + \frac{\lambda^{m+2}(\lambda+1)_{[m]}}{m!} \frac{1}{m+1} \sum_{j>n} \left(\frac{1}{(\lambda+j-1)_{[m+1]}} - \frac{1}{(\lambda+j)_{[m+1]}} \right) \\
&= \frac{\lambda^{m+1}(\lambda+1)_{[m]}}{m!\,(\lambda+n)_{[m+1]}} + \frac{\lambda^{m+2}(\lambda+1)_{[m]}}{m!} \frac{1}{(m+1)(\lambda+n)_{[m+1]}} \\
&= \frac{\lambda^{m+1}(\lambda+1)_{[m+1]}}{(m+1)!\,(\lambda+n)_{[m+1]}}.
\end{aligned}
$$

Hence, the assertion is true for $k = m+1$ and each n. By the induction principle, the assertion is true for all k and n.

Finally, we prove Lemma 3.

Proof of Lemma 3. To verify (11), we use (9) and (10) to deduce that

$$
\begin{aligned}
&(\lambda+n)_{n+1} P_0 - {}_{n+1}P_1 \\
&\quad = (\lambda+n) \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^j (\lambda+1)_{[j]}}{j!\,(\lambda+n+1)_{[j]}} - \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{j+1} (\lambda+1)_{[j+1]}}{j!\,(\lambda+n+1)_{[j+1]}} \\
&\quad = (\lambda+n) \left\{ 1 + \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\lambda^{j+1} (\lambda+1)_{[j+1]}}{(j+1)!\,(\lambda+n+1)_{[j+1]}} \right\} \\
&\qquad - \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{j+1} (\lambda+1)_{[j+1]}}{j!\,(\lambda+n+1)_{[j+1]}} \\
&\quad = (\lambda+n) + \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\lambda^{j+1} (\lambda+1)_{[j+1]}}{(j+1)!\,(\lambda+n+1)_{[j+1]}} (\lambda+n+j+1) \\
&\quad = (\lambda+n) + \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\lambda^{j+1} (\lambda+1)_{[j+1]}}{(j+1)!\,(\lambda+n+1)_{[j]}} \\
&\quad = (\lambda+n) \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^j (\lambda+1)_{[j]}}{j!\,(\lambda+n)_{[j]}} \\
&\quad = (\lambda+n)_n P_0.
\end{aligned}
$$

For (12), using (9) and (10), we have

$$
\begin{aligned}
&\lambda(\lambda+1)_{n+1}P_0 - \lambda_{n+1}P_1 \\
&\quad = \lambda(\lambda+1)\sum_{j=0}^{\infty}(-1)^j\frac{\lambda^j(\lambda+1)_{[j]}}{j!\,(\lambda+n+1)_{[j]}} - \lambda\sum_{j=0}^{\infty}(-1)^j\frac{\lambda^{j+1}(\lambda+1)_{[j+1]}}{j!\,(\lambda+n+1)_{[j+1]}} \\
&\quad = \lambda(\lambda+1)\left\{1+\sum_{j=0}^{\infty}(-1)^{j+1}\frac{\lambda^{j+1}(\lambda+1)_{[j+1]}}{(j+1)!\,(\lambda+n+1)_{[j+1]}}\right\} \\
&\qquad - \lambda\sum_{j=0}^{\infty}(-1)^j\frac{\lambda^{j+1}(\lambda+1)_{[j+1]}}{j!\,(\lambda+n+1)_{[j+1]}} \\
&\quad = \lambda(\lambda+1)+\sum_{j=0}^{\infty}(-1)^{j+1}\frac{\lambda^{j+2}(\lambda+1)_{[j+1]}}{(j+1)!\,(\lambda+n+1)_{[j+1]}}(\lambda+j+2) \\
&\quad = \lambda(\lambda+1)+\sum_{j=0}^{\infty}(-1)^{j+1}\frac{\lambda^{j+2}(\lambda+1)_{[j+2]}}{(j+1)!\,(\lambda+n+1)_{[j+1]}} \\
&\quad = \sum_{j=0}^{\infty}(-1)^j\frac{\lambda^{j+1}(\lambda+1)_{[j+1]}}{j!\,(\lambda+n+1)_{[j]}} \\
&\quad = (\lambda+n)\sum_{j=0}^{\infty}(-1)^j\frac{\lambda^{j+1}(\lambda+1)_{[j+1]}}{j!\,(\lambda+n)_{[j+1]}} \\
&\quad = (\lambda+n)_n P_1.
\end{aligned}
$$

It remains to verify (13). From (11) and (12), we have

$$
r_n = \frac{(\lambda+n)_n P_1}{(\lambda+n)_n P_0} = \frac{\lambda(\lambda+1)_{n+1}P_0 - \lambda_{n+1}P_1}{(\lambda+n)_{n+1}P_0 - {}_{n+1}P_1} = \frac{\lambda(\lambda+1)-\lambda r_{n+1}}{\lambda+n-r_{n+1}},
$$

form which (13) follows.

5. Monotonicity of t_λ

We are also interested in the property of t_λ. In fact, we can prove that t_λ is increasing in λ.

Theorem 4. *The threshold t_λ is increasing in λ.*

Intuitively, this result is quite natural. Because $\mathrm{E}(I_n) = \lambda/(\lambda+n-1)$ is increasing in λ, for larger λ, it is more likely that the last consecutive record occurs after time n. To prove Theorem 4, we need to analyze r_n, viewed as a function of λ. From now on, ${}_nP_0(\lambda) = {}_nP_0$, ${}_nP_1(\lambda) = {}_nP_1$, and $r_n(\lambda) = {}_nP_1/{}_nP_0$.

Lemma 7. *For each positive integer n and each $\lambda > 0$, $r_n'(\lambda)$ exists and $r_n'(\lambda) \to 0$ as $n \to \infty$.*

Lemma 8. *For each positive integer n, $r_n'(\lambda) > 0$ for all $\lambda > 0$. Hence, $r_n(\lambda)$ is increasing in λ. Moreover, for each n, $r_n(\lambda) \to \infty$ as $\lambda \to \infty$.*

We will prove Lemmas 7 and 8 later. Now we use Lemma 8 to prove Theorem 4.

Proof of Theorem 4. First note that, by Theorem 1, $t_\lambda = 2$ when $0 < \lambda \le 1$. Therefore, it suffices to argue only for $\lambda > 1$. Suppose that $\lambda > 1$. Then $t_\lambda = \tilde{t}_\lambda$ and so

$$t_\lambda = \min\{n \mid r_n(\lambda) \le 1\}.$$

Let $\lambda_1 > \lambda_2 > 1$. If $r_n(\lambda_1) \le 1$ for some n then $r_n(\lambda_2) \le 1$ since, by Lemma 8, $r_n(\lambda)$ is increasing in λ. This argument implies that $t_{\lambda_1} \ge t_{\lambda_2}$. Hence, t_λ is increasing in λ.

Proof of Lemma 7. To prove that $r_n'(\lambda)$ exists, it suffices to prove that ${}_nP_0$ and ${}_nP_1$ are differentiable with respect to λ. From (9), we have

$${}_nP_0(\lambda) = \sum_{j=0}^{\infty}(-1)^j H_j(\lambda), \quad \text{where} \quad H_j(\lambda) = \frac{\lambda^j (\lambda+1)_{[j]}}{j!\,(\lambda+n)_{[j]}}.$$

For $H_j(\lambda)$, we see that $H_0'(\lambda) = 0$ and, for $j \ge 1$,

$$\begin{aligned} H_j'(\lambda) &= H_j(\lambda)\left\{\frac{j}{\lambda} + \sum_{k=0}^{j-1}\left(\frac{1}{\lambda+1+k} - \frac{1}{\lambda+n+k}\right)\right\} \\ &= H_j(\lambda)\left\{\frac{j}{\lambda} + \sum_{k=0}^{j-1}\frac{n-1}{(\lambda+1+k)(\lambda+n+k)}\right\}. \end{aligned}$$

This equation yields $H_j'(\lambda) > 0$ and

$$\begin{aligned} H_j'(\lambda) &\le H_j(\lambda)\left\{\frac{j}{\lambda} + \frac{(n-1)j}{n\lambda}\right\} \\ &\le \frac{2j}{\lambda} H_j(\lambda) \\ &= \frac{2j}{\lambda}\frac{\lambda^j(\lambda+1)_{[j]}}{j!\,(\lambda+n)_{[j]}} \\ &\le \frac{2(\lambda+1)}{\lambda+n}\frac{\lambda^{j-1}}{(j-1)!}. \end{aligned}$$

It follows that

$$\sum_{j=0}^{k}|(-1)^j H_j'(\lambda)| \le \sum_{j=0}^{k} H_j'(\lambda) \le \sum_{j=1}^{k}\frac{2(\lambda+1)}{\lambda+n}\frac{\lambda^{j-1}}{(j-1)!} \le \frac{2(\lambda+1)}{\lambda+n}\mathrm{e}^{\lambda}. \tag{14}$$

This just says that $\sum_{j=0}^{\infty}(-1)^j H_j'(\lambda)$ converges uniformly on any bounded interval $(0, a)$. Therefore, we have

$${}_nP_0'(\lambda) = \sum_{j=0}^{\infty}(-1)^j H_j'(\lambda).$$

Furthermore, by (14),

$$|{}_nP_0'(\lambda)| \le \sum_{j=0}^{\infty}|(-1)^j H_j'(\lambda)| \le \frac{2(\lambda+1)}{\lambda+n}\mathrm{e}^{\lambda},$$

which implies that ${}_nP_0' \to 0$ as $n \to \infty$.

https://doi.org/10.1239/aap/1282924061 Published online by Cambridge University Press

In a similar way, we can use (10) or (11) to prove that ${}_nP_1'(\lambda)$ exists for all $\lambda > 0$. Hence, $r_n'(\lambda)$ exists for each positive integer n and each $\lambda > 0$.

Next we want to prove that $r_n'(\lambda) \to 0$ as $n \to \infty$. For this, we first note that

$$r_n'(\lambda) = \frac{{}_nP_1'\,{}_nP_0 - {}_nP_0'\,{}_nP_1}{{}_nP_0^2},$$

where we abbreviate the variable λ for brevity. In view of (11), ${}_nP_1$ and ${}_nP_1'$ can be expressed in terms of ${}_nP_0$, ${}_nP_0'$, ${}_{n-1}P_0$, and ${}_{n-1}P_0'$:

$$\begin{aligned} {}_nP_1 &= (\lambda + n - 1)({}_nP_0 - {}_{n-1}P_0), \\ {}_nP_1' &= ({}_nP_0 - {}_{n-1}P_0) + (\lambda + n - 1)({}_nP_0' - {}_{n-1}P_0'). \end{aligned}$$

Therefore, the above $r_n'(\lambda)$ can be expressed in terms of ${}_nP_0$, ${}_nP_0'$, ${}_{n-1}P_0$, and ${}_{n-1}P_0'$, that is,

$$\begin{aligned} r_n'(\lambda) &= \frac{\{({}_nP_0 - {}_{n-1}P_0) + (\lambda + n - 1)({}_nP_0' - {}_{n-1}P_0')\}\,{}_nP_0}{{}_nP_0^2} \\ &\quad - \frac{{}_nP_0'(\lambda + n - 1)({}_nP_0 - {}_{n-1}P_0)}{{}_nP_0^2} \\ &= \frac{(\lambda + n - 1)({}_nP_0'\,{}_{n-1}P_0 - {}_nP_0\,{}_{n-1}P_0')}{{}_nP_0^2} + \frac{{}_nP_0 - {}_{n-1}P_0}{{}_nP_0} \\ &= \frac{(\lambda + n - 1)\{{}_nP_0'({}_{n-1}P_0 - 1) - {}_{n-1}P_0'({}_nP_0 - 1) + ({}_nP_0' - {}_{n-1}P_0')\}}{{}_nP_0^2} \\ &\quad + \frac{{}_nP_0 - {}_{n-1}P_0}{{}_nP_0}. \end{aligned}$$

Now it is very easy to verify that $r_n'(\lambda) \to 0$ as $n \to \infty$, using the fact that ${}_nP_0 \to 1$ as $n \to \infty$ (Lemma 1(i)) and the following claims:

(i) $(\lambda + n - 1)\,{}_nP_0'({}_{n-1}P_0 - 1) \to 0$ as $n \to \infty$;

(ii) $(\lambda + n - 1)\,{}_{n-1}P_0'({}_nP_0 - 1) \to 0$ as $n \to \infty$;

(iii) $(\lambda + n - 1)({}_nP_0' - {}_{n-1}P_0') \to 0$ as $n \to \infty$.

Since $|{}_nP_0'| \le 2(\lambda + 1)\mathrm{e}^{\lambda}/(\lambda + n)$ and ${}_nP_0 \to 1$ as $n \to \infty$, (i) and (ii) follow. To prove (iii), using (9) yields

$$\begin{aligned} {}_nP_0 - {}_{n-1}P_0 &= \sum_{j=0}^{\infty}(-1)^j \frac{\lambda^j(\lambda+1)_{[j]}}{j!\,(\lambda+n)_{[j]}} - \sum_{j=0}^{\infty}(-1)^j \frac{\lambda^j(\lambda+1)_{[j]}}{j!\,(\lambda+n-1)_{[j]}} \\ &= \sum_{j=1}^{\infty}(-1)^{j+1} \frac{\lambda^j(\lambda+1)_{[j]}}{(j-1)!\,(\lambda+n-1)_{[j+1]}}, \end{aligned}$$

and so

$${}_nP_0' - {}_{n-1}P_0' = \sum_{j=1}^{\infty}(-1)^{j+1} \frac{\lambda^j(\lambda+1)_{[j]}}{(j-1)!\,(\lambda+n-1)_{[j+1]}} \left\{\frac{j}{\lambda} + \sum_{k=1}^{j}\frac{1}{\lambda+k} - \sum_{k=0}^{j}\frac{1}{\lambda+n-1+k}\right\}.$$

Because it is clear that, for $j \ge 1$,

$$0 < \frac{j}{\lambda} + \sum_{k=1}^{j} \frac{1}{\lambda + k} - \sum_{k=0}^{j} \frac{1}{\lambda + n - 1 + k} < \frac{2j}{\lambda},$$

we have

$$\begin{aligned} |{}_nP_0' - {}_{n-1}P_0'| &\le \sum_{j=1}^{\infty} \frac{\lambda^j (\lambda+1)_{[j]}}{(j-1)!\,(\lambda+n-1)_{[j+1]}} \frac{2j}{\lambda} \\ &\le \frac{\lambda+1}{(\lambda+n-1)(\lambda+n)} \sum_{j=1}^{\infty} \frac{2j\lambda^{j-1}}{(j-1)!} \\ &= \frac{2(\lambda+1)^2 \mathrm{e}^{\lambda}}{(\lambda+n-1)(\lambda+n)}. \end{aligned}$$

It follows that

$$|(\lambda+n-1)({}_nP_0' - {}_{n-1}P_0')| \le (\lambda+n-1)\frac{2(\lambda+1)^2\mathrm{e}^{\lambda}}{(\lambda+n-1)(\lambda+n)} = \frac{2(\lambda+1)^2\mathrm{e}^{\lambda}}{\lambda+n}.$$

Since $2(\lambda+1)\mathrm{e}^{\lambda}/(\lambda+n) \to 0$ as $n \to \infty$, we see that $(\lambda+n-1)({}_nP_0' - {}_{n-1}P_0') \to 0$ as $n \to \infty$ and (iii) follows.

Finally, we proceed to prove Lemma 8.

Proof of Lemma 8. From Lemma 7 we know that $r_n'(\lambda)$ exists and $r_n'(\lambda) \to 0$ as $n \to \infty$. In view of (13), we have

$$r_{n+1}' = 1 + \frac{(n-1)(r_n - \lambda r_n')}{(r_n-\lambda)^2},$$

where $r_{n+1}' = r_{n+1}'(\lambda)$ and $r_n' = r_n'(\lambda)$. For each fixed λ, consider the sequence of functions

$$F_n(x) = 1 + \frac{(n-1)(r_n - \lambda x)}{(r_n-\lambda)^2}, \qquad x \in \mathbb{R},$$

for $n = 2, 3, \dots$. Note that the sequence $\{r_n'\}_{n\ge 2}$ satisfies the relation $r_{n+1}' = F_n(r_n')$.

Let $x \le 0$. Then we can prove that $F_n(x) > 1$ and $F_{n+1}(F_n(x)) < 0$. For this, we observe that, since $x \le 0$, it follows that

$$F_n(x) = 1 + \frac{(n-1)(r_n - \lambda x)}{(r_n-\lambda)^2} \ge 1 + \frac{(n-1)r_n}{(r_n-\lambda)^2},$$

which implies that $F_n(x) > 1$. Furthermore, the above inequality implies that

$$\begin{aligned} F_{n+1}(F_n(x)) &= 1 + \frac{n(r_{n+1} - \lambda F_n(x))}{(r_{n+1}-\lambda)^2} \\ &\le 1 + \frac{n\{r_{n+1} - \lambda(1 + (n-1)r_n/(r_n-\lambda)^2)\}}{(r_{n+1}-\lambda)^2} \\ &= \frac{(r_{n+1}-\lambda)^2(r_n-\lambda)^2 - n\lambda(r_n-\lambda)^2 + nr_{n+1}(r_n-\lambda)^2 - n(n-1)\lambda r_n}{(r_{n+1}-\lambda)^2(r_n-\lambda)^2} \\ &< 0, \end{aligned}$$

https://doi.org/10.1239/aap/1282924061 Published online by Cambridge University Press

using the inequalities

$$(r_{n+1}-\lambda)^2(r_n-\lambda)^2 - n\lambda(r_n-\lambda)^2 < 0$$

and

$$nr_{n+1}(r_n-\lambda)^2 - n(n-1)\lambda r_n < 0.$$

Note that the above two inequalities can be verified easily from the observation that, for $n \geq 2$, $r_{n+1} < x_n < r_n < \lambda$, by Lemma 5, and so

$$(r_n-\lambda)^2 < (x_n-\lambda)^2 = \left(\frac{2(n-1)\lambda}{\sqrt{n^2+4n(n-1)\lambda}+n}\right)^2 < (n-1)\lambda,$$

and similarly $(r_{n+1}-\lambda)^2 < n\lambda$.

Now suppose that $r'_n \leq 0$ for some n. Then $r'_{n+1} = F_n(r'_n) > 1$ and $r'_{n+2} = F_{n+1}(r'_{n+1}) = F_{n+1}(F_n(r'_n)) < 0$. Arguing in the same way, we have $r'_{n+3} > 1$ and $r'_{n+4} < 0$, and, in general, $r'_{n+2k+1} > 1$ and $r'_{n+2k+2} < 0$ for $k \geq 1$. This contradicts the fact that $r'_n \to 0$ as $n \to \infty$, Thus, $r'_n > 0$ for all n.

6. Probability of selecting the last consecutive record

We have proved that the optimal stopping rule is of threshold type, i.e.

$$\tau_\lambda = \min\{n \mid n \geq t_\lambda,\ I_{n-1}I_n = 1\}.$$

It is natural to ask about the probability of selecting the last consecutive record using the optimal stopping rule τ_λ. Fortunately, this probability is not difficult to figure out and has the following neat form.

Theorem 5. *The probability of selecting the last consecutive record using the optimal stopping rule τ_λ is*

$$Q_\lambda = \frac{\lambda^2}{t_\lambda+\lambda-2}\,{}_{t_\lambda-1}P_0.$$

In particular, if $\lambda^2-\lambda \in \mathbb{N}$ then $Q_\lambda = {}_{t_\lambda-1}P_0$.

Proof. For each positive integer n, let p_n denote the probability of selecting the last consecutive record using the stopping rule with threshold n: stop at the first $k \geq n$ with $I_{k-1}I_k = 1$. It is not difficult to see that

$$p_n = \mathrm{P}\left(\sum_{k=n}^{\infty} I_{k-1}I_k = 1\right).$$

In the following we want to prove that

$$p_n = \frac{\lambda^2}{n+\lambda-2}\,{}_{n-1}P_0,$$

from which the first assertion follows and then the last assertion follows from Theorem 1(ii).

https://doi.org/10.1239/aap/1282924061 Published online by Cambridge University Press

Recalling the definitions of ${}_nP_0$ and ${}_nP_1$, we have, for $n \geq 2$,

$$
\begin{aligned}
{}_{n-1}P_1 &= \mathrm{P}\biggl(I_n + \sum_{k\geq n} I_k I_{k+1} = 1\biggr) \\
&= \sum_{s=0,1} \mathrm{P}\biggl(I_n + \sum_{k\geq n} I_k I_{k+1} = 1 \biggm| I_n = s\biggr)\mathrm{P}(I_n = s) \\
&= \mathrm{P}\biggl(\sum_{k\geq n+1} I_k I_{k+1} = 1\biggr)\mathrm{P}(I_n = 0) + \mathrm{P}\biggl(I_{n+1} + \sum_{k\geq n+1} I_k I_{k+1} = 0\biggr)\mathrm{P}(I_n = 1) \\
&= p_{n+2}\frac{n-1}{\lambda+n-1} + {}_nP_0\frac{\lambda}{\lambda+n-1}.
\end{aligned}
$$

It follows that

$$
p_{n+2}\frac{n-1}{\lambda+n-1} = {}_{n-1}P_1 - {}_nP_0\frac{\lambda}{\lambda+n-1}. \tag{15}
$$

On the other hand, for $n \geq 2$,

$$
\begin{aligned}
p_n &= \mathrm{P}\biggl(\sum_{k\geq n} I_{k-1}I_k = 1\biggr) \\
&= \sum_{s=0,1} \mathrm{P}\biggl(\sum_{k\geq n} I_{k-1}I_k = 1 \biggm| I_{n-1} = s,\ I_n = 1\biggr)\mathrm{P}(I_{n-1} = s,\ I_n = 1) \\
&\quad + \mathrm{P}\biggl(\sum_{k\geq n} I_{k-1}I_k = 1 \biggm| I_n = 0\biggr)\mathrm{P}(I_n = 0) \\
&= \sum_{s=0,1} \mathrm{P}\biggl(I_{n+1} + \sum_{k\geq n+2} I_{k-1}I_k = 1 - s\biggr)\mathrm{P}(I_{n-1} = s,\ I_n = 1) \\
&\quad + \mathrm{P}\biggl(\sum_{k\geq n+2} I_{k-1}I_k = 1\biggr)\mathrm{P}(I_n = 0) \\
&= {}_nP_0\frac{\lambda^2}{(\lambda+n-2)(\lambda+n-1)} + {}_nP_1\frac{(n-2)\lambda}{(\lambda+n-2)(\lambda+n-1)} + p_{n+2}\frac{n-1}{\lambda+n-1}.
\end{aligned}
$$

Now substituting (15) into the above equation, we have a further expression for p_n:

$$
\begin{aligned}
p_n &= \frac{\lambda^2}{(\lambda+n-2)(\lambda+n-1)}{}_nP_0 + \frac{(n-2)\lambda}{(\lambda+n-2)(\lambda+n-1)}{}_nP_1 \\
&\quad + {}_{n-1}P_1 - \frac{\lambda}{\lambda+n-1}{}_nP_0 \\
&= \frac{(n-2)\lambda}{(\lambda+n-2)(\lambda+n-1)}({}_nP_1 - {}_nP_0) + {}_{n-1}P_1 \\
&= \frac{(n-2)\lambda}{(\lambda+n-2)(\lambda+n-1)}({}_nP_1 - {}_nP_0) + \frac{1}{\lambda+n-1}\{\lambda(\lambda+1)_nP_0 - \lambda_nP_1\} \\
&= \frac{\lambda^2}{(\lambda+n-2)(\lambda+n-1)}\{(\lambda+n-1)_nP_0 - {}_nP_1\}
\end{aligned}
$$

https://doi.org/10.1239/aap/1282924061 Published online by Cambridge University Press

$$
\begin{aligned}
&= \frac{\lambda^2}{(\lambda+n-2)(\lambda+n-1)}(\lambda+n-1)\,{}_{n-1}P_0 \\
&= \frac{\lambda^2}{\lambda+n-2}\,{}_{n-1}P_0,
\end{aligned}
$$

where the third and fifth equations follow from (12) and (11), respectively. This completes the proof.

Because the optimal stopping rule τ_λ is of threshold type, the probability Q_λ can also be expressed in terms of p_n:

$$Q_\lambda = \max_{n \geq 2} p_n(\lambda),$$

where $p_n(\lambda) = p_n = \lambda^2\,{}_{n-1}P_0/(\lambda+n-2)$. Moreover, for each λ, we have, by Theorem 4,

$$\max_{n \geq 2} p_n(\lambda) = p_{t_\lambda}(\lambda).$$

Since t_λ is increasing in λ, by Theorem 5, it follows that, for any $a > 0$ and any $\lambda \in (0, a)$,

$$Q_\lambda = \max_{t_a \geq n \geq 2} p_n(\lambda). \tag{16}$$

Because $p_n(\lambda)$ is a continuous function of λ (we have proved, in Section 5, that ${}_nP_0'(\lambda)$ exists), (16) implies that Q_λ is a continuous function of λ in $(0, a)$. Let $a \to \infty$. Then Q_λ is continuous at every positive value of λ.

Plots of $p_2(\lambda)$, $p_3(\lambda)$, ... are shown in Figure 1. Applying (16) to the data of Figure 1 yields the plot of Q_λ given in Figure 2. Figure 2 suggests two conjectures for Q_λ.

Conjecture 1. *The probability Q_λ attains a maximum at $\lambda = 1$.*

Conjecture 2. *The probability Q_λ approaches some value c as λ goes to ∞.*

While we are unable to prove Conjecture 1, it can be shown that Q_λ has a local maximum at $\lambda = 1$, as argued below. If $0 < \lambda \leq 1$ then, by Theorem 1, $t_\lambda = 2$ and so $Q_\lambda = \lambda\,{}_1P_0 = \lambda \mathrm{e}^{-\lambda}$, which states that Q_λ is increasing in λ when $\lambda \in (0, 1]$. Furthermore, because $r_1(\lambda) = \lambda$ and

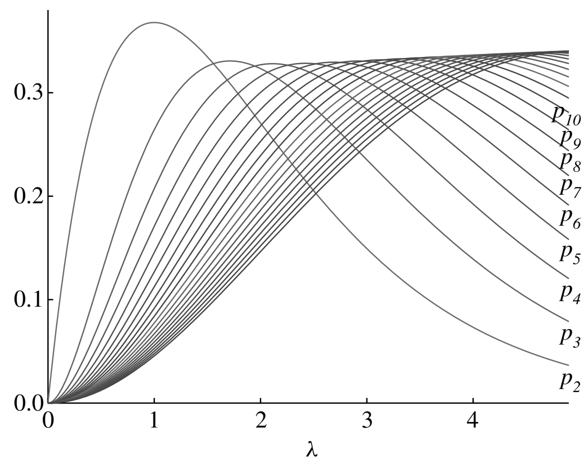

Figure 1.

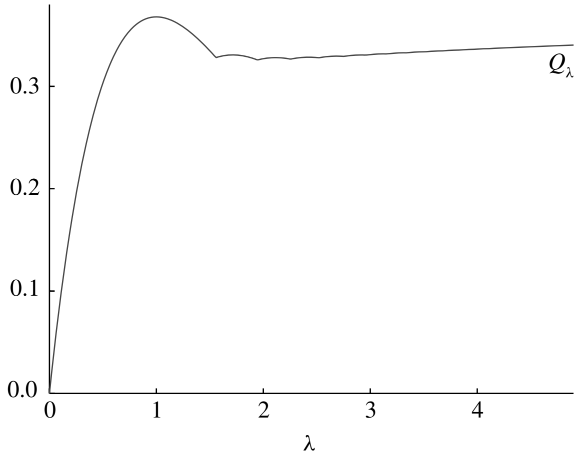

FIGURE 2.

$r_n(\lambda)$ is strictly decreasing in n and strictly increasing in λ, we see that $r_1(1) = 1$, $r_2(1) < 1$, and then $r_2(a) = 1$, $r_1(a) > 1$ for some $a > 1$, by Lemma 8. Now it follows that $t_\lambda = 2$ and $Q_\lambda = \lambda e^{-\lambda}$ for $\lambda \in [1, a]$, which implies that Q_λ is decreasing in λ when $\lambda \in [1, a]$. Hence, Q_λ has a local maximum at $\lambda = 1$ and $Q_1 = e^{-1}$.

For Conjecture 2, we have an affirmative answer as follows.

Theorem 6. *As* $\lambda \to \infty$, $Q_\lambda \to e^{-1}$.

Proof. By Theorem 5,

$$Q_\lambda = \frac{\lambda^2}{t_\lambda + \lambda - 2} {}_{t_{\lambda-1}}P_0,$$

where $t_\lambda = \lfloor \lambda^2 - \lambda \rfloor + 2$ or $\lfloor \lambda^2 - \lambda \rfloor + 3$, by Theorem 1. Therefore,

$$\frac{\lambda^2}{\lfloor \lambda^2 - \lambda \rfloor + \lambda + 1} \leq \frac{\lambda^2}{t_\lambda + \lambda - 2} \leq \frac{\lambda^2}{\lfloor \lambda^2 - \lambda \rfloor + \lambda},$$

and so $\lambda^2/(t_\lambda + \lambda - 2) \to 1$ as $\lambda \to \infty$.

In the following, we prove that ${}_{t_{\lambda-1}}P_0 \to e^{-1}$ as $\lambda \to \infty$. By (9),

$${}_{t_{\lambda-1}}P_0 = \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^j (\lambda + 1)_{[j]}}{j!\,(\lambda + \lfloor \lambda^2 - \lambda \rfloor + 1)_{[j]}}$$

or

$${}_{t_{\lambda-1}}P_0 = \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^j (\lambda + 1)_{[j]}}{j!\,(\lambda + \lfloor \lambda^2 - \lambda \rfloor + 2)_{[j]}}.$$

In the first case (the second case is similar), we have, for large λ and each $j \geq 0$,

$$\left| (-1)^j \frac{\lambda^j (\lambda + 1)_{[j]}}{j!\,(\lambda + \lfloor \lambda^2 - \lambda \rfloor + 1)_{[j]}} \right| \leq \frac{\lambda^j (\lambda + 1)_{[j]}}{j!\,(\lambda^2)_{[j]}} \leq \frac{2^j}{j!}$$

and

$$\frac{\lambda^j(\lambda+1)_{[j]}}{j!\,(\lambda+\lfloor\lambda^2-\lambda\rfloor+1)_{[j]}} \to \frac{1}{j!} \quad \text{as } \lambda\to\infty.$$

But $\sum_{j=0}^{\infty} 2^j/j! = \mathrm{e}^2$; hence, we see that

$$\sum_{j=0}^{\infty}(-1)^j \frac{\lambda^j(\lambda+1)_{[j]}}{j!\,(\lambda+\lfloor\lambda^2-\lambda\rfloor+1)_{[j]}} \to \sum_{j=0}^{\infty}(-1)^j\frac{1}{j!} = \mathrm{e}^{-1} \quad \text{as } \lambda\to\infty.$$

Similarly, we can prove that

$$\sum_{j=0}^{\infty}(-1)^j \frac{\lambda^j(\lambda+1)_{[j]}}{j!\,(\lambda+\lfloor\lambda^2-\lambda\rfloor+2)_{[j]}} \to \mathrm{e}^{-1} \quad \text{as } \lambda\to\infty.$$

Hence, $Q_\lambda \to \mathrm{e}^{-1}$ as $\lambda\to\infty$.

Acknowledgements

This paper is dedicated to the memory of my advisor, Dr. Ching-Zong Wei. I gratefully acknowledge support from the National Science Council of Taiwan under grant NSC 97-2118-M-018-002. I also wish to thank Dr. Yi-Ching Yao and the anonymous referee for helpful suggestions that improved the paper and made it more readable.

References

Arratia, R., Barbour, A. D. and Tavaré, S. (1992). Poisson process approximations for the Ewens sampling formula. *Ann. Appl. Prob.* **2,** 519–535.

Bruss, F. T. (2000). Sum the odds to one and stop. *Ann. Prob.* **28,** 1384–1391.

Bruss, F. T. and Louchard, G. (2003). Optimal stopping on patterns in strings generated by independent random variables. *J. Appl. Prob.* **40,** 49–72.

Bruss, F. T. and Paindaveine, D. (2000). Selecting a sequence of last successes in independent trials. *J. Appl. Prob.* **37,** 389–399.

Chern, H.-H., Hwang, H.-K. and Yeh, Y.-N. (2000). Distribution of the number of consecutive records. *Random Structures Algorithms* **17,** 169–196.

Chow, Y. S. and Robbins, H. (1961). A martingale system theorem and applications. In Proc. 4th Berkeley Symp. Math. Statist. Prob., Vol. 1, University of California Press, Berkeley, CA, pp. 93–104.

Chow, Y.-S., Robbins, H. and Siegmund, D. (1971). *Great Expectations: the Theory of Optimal Stopping.* Houghton Mifflin, Boston, MA.

Csörgö, S. and Wu, W. B. (2000). On sums of overlapping products of independent Bernoulli random variables. *Ukraïn. Mat. Zh.* **52,** 1304–1309. English translation: *Ukrainian Math. J.* **52,** 1496–1503.

Dynkin, E. B. (1963). Optimal choice of the stopping moment of a Markov process. *Dokl. Akad. Nauk SSSR* **150,** 238–240 (in Russian).

Hahlin, L. O. (1995). Double records. Res. Rep. 1995:12, Department of Mathematics, Uppsala University.

Holst, L. (2007). Counts of failure strings in certain Bernoulli sequences. *J. Appl. Prob.* **44,** 824–830.

Hsiau, S.-R. (2007). Stopping at the last renewal up to a terminal time. *Statist. Sinica* **17,** 725–733.

Hsiau, S.-R. and Yang, J.-R. (2002). Selecting the last success in Markov-dependent trials. *J. Appl. Prob.* **39,** 271–281.

Joffe, A., Marchand, É., Perron, F. and Popadiuk, P. (2004). On sums of products of Bernoulli variables and random permutations. *J. Theoret. Prob.* **17,** 285–292.

Mori, T. F. (2001). On the distribution of sums of overlapping products. *Acta Sci. Math.* **67,** 833–841.

Sethuraman, J. and Sethuraman, S. (2004). On counts of Bernoulli strings and connections to rank orders and random permutations. In *A festschrift for Herman Rubin* (IMS Lecture Notes Monogr. Ser. **45**), Institute of Mathematical Statistics, Beachwood, OH, pp. 140–152.