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INNER FUNCTIONS IN THE MÖBIUS INVARIANT BESOV-TYPE SPACES

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Abstract An analytic function f in the unit disc $\mathbb{D}$ belongs to $F(p,q,s)$, if

$$\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a)\,\mathrm{d}A(z)$$

is uniformly bounded for all $a \in \mathbb{D}$. Here $g(z,a) = -\log|\varphi_a(z)|$ is the Green function of $\mathbb{D}$, and $\varphi_a(z) = (a-z)/(1-\bar{a}z)$. It is shown that for $0<\gamma<\infty$ and $|w|=1$ the singular inner function $\exp(\gamma(z+w)/(z-w))$ belongs to $F(p,q,s)$, $0<s\leqslant 1$, if and only if $p \leqslant q + \frac{1}{2}(s+3)$. Moreover, it is proved that, if $0<s<1$, then an inner function belongs to the Möbius invariant Besov-type space $B^p_s = F(p,p-2,s)$ for some (equivalently for all) $p > \max\{s, 1-s\}$ if and only if it is a Blaschke product whose zero sequence $\{z_n\}$ satisfies $\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} (1-|\varphi_a(z_n)|^2)^s < \infty$.

Keywords: inner function; Blaschke product; uniformly separated sequence; Bergman space; Q_p-space; Besov-type space

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1. Introduction and main results

An analytic function in the unit disc $\mathbb{D} := \{z : |z| < 1\}$ is called an *inner function* if its modulus equals 1 almost everywhere on the boundary $\mathbb{T} := \{z : |z| = 1\}$. It is well known that every such function can be represented as a product of a Blaschke product and a singular inner function [**16**]. For a given sequence $\{z_n\}$ in $\mathbb{D}$ for which $\sum_{n=1}^{\infty}(1-|z_n|^2)$ converges (with the convention $z_n/|z_n| = 0$ for $z_n = 0$), the *Blaschke product* associated with the sequence $\{z_n\}$ is defined as

$$B(z) := \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1-\bar{z}_n z}.$$

A *singular inner function* is of the form

$$S(z) := \exp\left(\int_{\mathbb{T}} \frac{z+w}{z-w}\,\mathrm{d}\sigma(w)\right),$$

where the measure σ on $\mathbb{T}$ is singular with respect to the Lebesgue measure. If the measure σ is atomic and consists of a point mass concentrated in $w \in \mathbb{T}$, then S is of the form

$$S_{\gamma,w}(z) := \exp\left(\gamma\frac{z+w}{z-w}\right),$$

where $0 < \gamma < \infty$.

The purpose of this paper is to study the classical problem of determining which inner functions (or their derivatives) belong to a given space of analytic functions. In the case of Hardy and related spaces, this problem has been studied, for example, in [**1**,**3**,**4**,**11**,**35**], and in the case of weighted Bergman spaces the reader is referred to [**1**,**24**,**25**,**30**]. Many related results on Blaschke products can be found in [**12**]. For singular inner functions and more recent developments on Blaschke products, see, for example, [**7**,**8**,**14**,**20**,**23**,**31**, **40**,**41**]. The general theory of Hardy and Bergman spaces can be found in [**16**,**18**,**21**,**27**].

The spaces of primary interest in this study are the Möbius invariant Besov-type spaces. For $p > 0$ and $s \geqslant 0$, the *Besov-type space* B_s^p consists of those analytic functions f in $\mathbb{D}$ for which

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-2}g^s(z,a)\,\mathrm{d}A(z) < \infty,$$

where $g(z,a) := -\log|\varphi_a(z)|$ is the Green function of $\mathbb{D}$ and $\varphi_a(z) := (a-z)/(1-\bar{a}z)$. The closure of polynomials in B_s^p is the *small Besov-type space* $B_{s,0}^p$, and it consists of those analytic functions f in $\mathbb{D}$ for which the integral above tends to 0 as a approaches the boundary $\mathbb{T}$. In the study of inner functions the Besov-type spaces form a family which is of special interest, since many classical function spaces can be found among B_s^p by choosing the parameters p and s appropriately. The space B_0^p is the *classical Besov space* B_p that contains no other inner functions than finite Blaschke products [**15**, **30**]. For $0 < s < 1$, the space B_s^2 is the Q_s-space in which the only inner functions are Blaschke products whose zeros $\{z_n\}$ have the density $\sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_a(z_n)|^2)^s < \infty$ [**20**]. Furthermore, it is well known that B_1^2 coincides with BMOA, the space of analytic functions in the Hardy space H^1 whose boundary values have *bounded mean oscillation* on $\mathbb{T}$, and therefore it contains all inner functions. Furthermore, an application of the Garsia norm in BMOA shows that the only inner functions in $B_{1,0}^2 = \mathrm{VMOA}$ (the space of analytic functions of *vanishing mean oscillation*) are finite Blaschke products. In general, a function in B_s^p is always a Bloch function and therefore it cannot exceed logarithmic growth. In particular, B_s^p coincides with the Bloch space $\mathcal{B}$ for all $1 < s < \infty$ [**46**]. See [**5**, **34**, **44**, **45**] for more details on the Q_s-theory and the Bloch space.

The first of the main results in this paper is Theorem 1.1. It determines precisely the values of p, q and s for which the singular inner function $S_{\gamma,w}$ belongs to $F(p,q,s)$ or $F_0(p,q,s)$. For $0 < p < \infty$, $-2 < q < \infty$ and $0 \leqslant s < \infty$ such that $q+s > -1$, the spaces $F(p,q,s)$ and $F_0(p,q,s)$ consist of those analytic functions f in $\mathbb{D}$ for which

$$\|f\|_{F(p,q,s)} := \left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q g^s(z,a)\,\mathrm{d}A(z)\right)^{1/p} < \infty$$

and

$$\lim_{|a|\to 1^-}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q g^s(z,a)\,\mathrm{d}A(z)=0,\quad 0<s<\infty,$$

respectively. It is necessary to assume that $q+s>-1$, since otherwise $F(p,q,s)$ contains constant functions only. Moreover, if $1<s<\infty$, then an analytic function f in $\mathbb{D}$ belongs to $F(p,q,s)$ if and only if $|f'(z)|(1-|z|^2)^{(q+2)/p}$ is uniformly bounded in $\mathbb{D}$. For these and other basic properties of $F(p,q,s)$, see [**36**, **46**].

Theorem 1.1. *Let $0<p<\infty$, $-2<q<\infty$ and $0<s\leqslant 1$ such that $q+s>-1$. Then $S_{\gamma,w}\in F(p,q,s)$ if and only if $p\leqslant q+\frac{1}{2}(s+3)$, and $S_{\gamma,w}\in F_0(p,q,s)$ if and only if $p<q+\frac{1}{2}(s+3)$. Moreover, $S_{\gamma,w}\in F(p,q,0)$ if and only if $p<q+\frac{3}{2}$.*

Recall that, for $0<p<\infty$ and $-1<q<\infty$, the *weighted Bergman space* A^p_q consists of those analytic functions f in $\mathbb{D}$ for which

$$\|f\|_{A^p_q}:=\biggl(\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^q\,\mathrm{d}A(z)\biggr)^{1/p}<\infty.$$

In particular, A^p_0 is the *classical Bergman space* A^p. The last assertion in Theorem 1.1 says that $S'_{\gamma,w}\in A^p_q$ if and only if $p<q+\frac{3}{2}$. This result is essentially known [**29**].

Theorem 1.1 yields the following corollary, which yields precisely the values of p and s for which $S_{\gamma,w}$ belongs to B^p_s or $B^p_{s,0}$. It is worth noting that the Besov-type spaces obey the strict inclusions

$$B^{p_1}_s\subsetneq B^{p_2}_s\quad\text{and}\quad B^{p_1}_{s,0}\subsetneq B^{p_2}_{s,0}\tag{1.1}$$

for all $1-s<p_1<p_2<\infty$ and $0\leqslant s\leqslant 1$ [**46**].

Corollary 1.2. *Let $0<p<\infty$ and $0\leqslant s<1$ such that $p+s>1$. Then $S_{\gamma,w}\notin B^p_s$ and $S_{\gamma,w}\notin B^p_{1,0}$, but $S_{\gamma,w}\in B^p_1$.*

The second of the main results generalizes in part Corollary 1.2 to the case when the generating measure σ of the singular inner function S is non-atomic.

Theorem 1.3. *Let $0<p<\infty$ and $0\leqslant s<1$ such that $p+s>1$. Then B^p_s does not contain any singular inner functions.*

By Corollary 1.2, Theorem 1.3 and Corollary 2.5, the only possible inner functions in B^p_s, $0\leqslant s<1$, and $B^p_{1,0}$ are Blaschke products. As mentioned earlier, the only inner functions in the classical Besov space $B_p=B^p_0$ are finite Blaschke products. Theorem 1.4 shows that this is no longer true for B^p_s if $0<s\leqslant 1$ and $p>1-s$. This result can also be considered as a refinement of the known fact that an inner function belongs to $Q_s=B^2_s$, $0<s<1$, if and only if it is a Blaschke product whose zeros $\{z_n\}$ satisfy $\sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_a(z_n)|^2)^s<\infty$ [**20**].

Theorem 1.4. *Let $0<s<1$. Then an inner function belongs to the Möbius invariant Besov-type space $B^p_s=F(p,p-2,s)$ for some (equivalently for all) $p>\max\{s,1-s\}$ if and only if it is the Blaschke product associated with a sequence $\{z_n\}$ which satisfies*

$$\sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_a(z_n)|^2)^s<\infty.\tag{1.2}$$

For $0 < s \leqslant \frac{1}{2}$, the assertion in Theorem 1.4 is sharp in the sense that $1 - s \geqslant s$ and the condition $p > 1 - s$ only guarantees that the space B^p_s is not trivial.

The assertion in Theorem 1.4 clearly fails for $s = 1$ since $B^2_1 = \mathrm{BMOA}$ contains all bounded analytic functions, while the condition (1.2) for $s = 1$ is satisfied if and only if $\{z_n\}$ is a finite union of uniformly separated sequences [**17**, **32**]. Recall that a sequence $\{z_n\} \subset \mathbb{D}$ is called *uniformly separated* if there exists a δ such that

$$\inf_j \prod_{k \neq j} |\varphi_{z_k}(z_j)| \geqslant \delta > 0.$$

A famous result by Carleson [**9**] states that $\{z_n\}$ is an interpolating sequence for the space H^∞ of all bounded analytic functions in $\mathbb{D}$ if and only if it is uniformly separated. Therefore, the Blaschke products associated with uniformly separated sequences are often called *interpolating Blaschke products*. It is also worth observing that (1.2) for $s = 2$ is satisfied if and only if $\{z_n\}$ is a finite union of uniformly discrete sequences [**19**]. A sequence $\{z_n\} \subset \mathbb{D}$ is called *uniformly discrete* (or *separated*), if there exists a δ such that

$$\inf_{k \neq j} |\varphi_{z_k}(z_j)| \geqslant \delta > 0.$$

For $\xi \in \mathbb{T}$ and $M \in [1, \infty)$, the domain $\{z \in \mathbb{D} : |1 - \bar{\xi} z| \leqslant M(1 - |z|^2)\}$ is called a *Stolz angle* with vertex at ξ. Blaschke products whose zeros lie in such an angular domain and which belong to Q_s have been studied in [**14**]. Theorem 1.4 and the proofs of [**14**, Theorems 1 and 2] yield Corollary 1.5. A decreasing sequence $\{a_n\}$ of positive real numbers tending to 0 is called *asymptotically concentrated* if, for any $k \in \mathbb{N} := \{1, 2, \dots\}$, there is an infinite sequence $\{n_j\} \subset \mathbb{N}$, depending on k, such that $(a_{n_j}/a_{n_j+k}) \to 1$ as $j \to \infty$.

Corollary 1.5. *Let $0 < s < 1$ and $p > \max\{s, 1 - s\}$, and let B be a Blaschke product with zeros $\{z_n\}$ in a Stolz angle. Then $B \in B^p_s$ if and only if $\{1 - |z_n|\}$ is not asymptotically concentrated.*

The question of when the derivative of a Blaschke product with zeros in a Stolz angle belongs to the weighted Bergman spaces has been studied, for example, in [**24**–**26**].

Another immediate consequence of Theorem 1.4 follows by [**37**, Theorem 2.1]. Corollary 1.6 gives a sufficient condition for the zeros of a Blaschke product such that it belongs to B^p_s. It is known that the converse implication is not true in general [**37**].

Corollary 1.6. *Let $0 < s < 1$, and let B be the Blaschke product associated with a sequence $\{z_n\}$. If there exists a positive constant C such that*

$$\sum_{n=k+1}^{\infty} (1 - |z_n|^2)^s \leqslant C(1 - |z_k|^2)^s$$

for all $k \in \mathbb{N}$, then $B \in B^p_s$ for all $p > \max\{s, 1 - s\}$.

Let us return to the case when $s = 1$, excluded in Theorem 1.4, which does not seem to be so straightforward. Since $B_1^2 = \mathrm{BMOA}$ and $B_{1,0}^2 = \mathrm{VMOA}$, the inclusions in (1.1) ensure that B_1^p contains all inner functions if $p \geqslant 2$, and $B_{1,0}^p$ contains no inner functions other than finite Blaschke products when $p \leqslant 2$. By Corollary 1.2, the singular inner function $S_{\gamma,w}$ belongs to B_1^p for all $p > 0$, and does not belong to $B_{1,0}^p$ for any $p > 0$. Moreover, Theorem 4.2 (b), below, shows that if the zero-sequence $\{z_n\}$ of a Blaschke product B satisfies

$$\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2) \log \frac{1}{1 - |\varphi_a(z_n)|^2} < \infty, \tag{1.3}$$

then $B \in B_1^1$. Conversely, if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B'(z)|(1 - |z|^2)^{-1} \log \frac{1}{1 - |z|} (1 - |\varphi_a(z)|^2) \, \mathrm{d}A(z) < \infty, \tag{1.4}$$

then the zero-sequence $\{z_n\}$ of B satisfies (1.3) (see the discussion before the proof of Theorem 1.4). If $B_{1,\log}^1$ denotes the space of all analytic functions in $\mathbb{D}$ satisfying (1.4), then clearly $B_{1,\log}^1 \subsetneq B_1^1$. By the discussion at the end of § 3, the singular inner function S does not belong to $B_{1,\log}^1$ and, as just pointed out, the zero-sequence $\{z_n\}$ of a Blaschke product in $B_{1,\log}^1$ satisfies (1.3). Complete characterizations of inner functions in B_1^p and $B_{1,0}^p$ remain as open problems.

The remainder of this paper is organized as follows. Section 2 contains general results on inner functions and other auxiliary results. Some of these results may be of independent interest. For example, Lemma 2.1 on radial integrability of inner functions plays an important role in the proofs of some of the main results, and it also generalizes [**4**, Theorem 1], [**20**, Lemma 4.1] and [**42**, Lemma 2.2]. While Theorem 2.3 extends Lemma 2.1, Lemma 2.2 gives a fairly elementary proof for [**28**, Lemma 4.6] and [**36**, Lemma 5.3.1], and Corollary 2.4 might be of interest for those who wish to study inner–outer factorizations in $F(p, q, s)$. Moreover, Corollary 2.5 says that, under certain conditions on p, q and s, a finite product of inner functions belongs to $F(p, q, s)$ if and only if every member in the product does. Section 3 contains the proofs of Theorems 1.1 and 1.3. Section 4 begins with a brief discussion on Blaschke products whose first derivatives belong to a given weighted Bergman space (see Theorems A and 4.1). Theorem 4.2 establishes a sufficient condition for the zeros of a Blaschke product such that it belongs to B_s^p, $0 < p, s < 1$, and a necessary condition in the case when $p \geqslant 1$ is obtained in Theorem 4.3. These results, together with some auxiliary results from § 2, finally yield a proof of Theorem 1.4.

2. Results on inner functions and auxiliary results

The first result of this section indicates the values of p, q and s which are of interest when studying inner functions in the spaces $F(p, q, s)$ and $F_0(p, q, s)$. Here $\mathcal{A}$ denotes the closure of polynomials in the space H^∞, and $\mathcal{B}^\alpha$ stands for the *α-Bloch space* which consists of those analytic functions f in $\mathbb{D}$ such that $|f'(z)|(1 - |z|^2)^\alpha$ is uniformly bounded in $\mathbb{D}$. In particular, $\mathcal{B}^1$ is the Bloch space $\mathcal{B}$.

Proposition A. Let $0 < p, \alpha < \infty$, $-2 < q < \infty$ and $0 \leqslant s \leqslant 1$ such that $q + s > -1$. Then

(1) $\mathcal{A} \not\subset \mathcal{B}^\alpha$, $0 < \alpha < 1$,

(2) $\mathcal{A} \subsetneq H^\infty \subsetneq \mathcal{B}^\alpha$, $1 \leqslant \alpha < \infty$,

(3) $\mathcal{A} \subsetneq H^\infty \subsetneq \mathcal{B} \subsetneq F_0(p, q, s) \subset F(p, q, s)$, $p < q + s + 1$,

(4) $\mathcal{A} \not\subset F(p, q, s)$, $p > q + s + 1$,

(5) $\mathcal{A} \subsetneq \mathrm{VMOA} \subset F_0(p, q, s)$, $p = q + s + 1 \geqslant 2$,

(6) $H^\infty \subsetneq \mathrm{BMOA} \subset F(p, q, s)$, $p = q + s + 1 \geqslant 2$,

(7) $\mathcal{A} \not\subset F(p, q, s)$, $p = q + s + 1 < 2$.

Cases (1)–(5) can be found in [**36**, § 2.3]. In particular, (5) follows by [**36**, Corollary 2.3.5], which is a consequence of [**13**, Theorem 8]. Furthermore, (6) follows by the inclusions $H^\infty \subsetneq$ BMOA and (1.1). Case (7) follows by [**22**, Theorem 2]. Indeed, if $q + s = p - 1$, then

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s \, \mathrm{d}A(z) \geqslant \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1} \, \mathrm{d}A(z),$$

and by [**22**, Theorem 2] there is a function $f \in \mathcal{A}$ for which the last integral diverges if $0 < p < 2$.

It is easy to see that the α-Bloch space $\mathcal{B}^\alpha$ and the classical Besov space $B_p = B_0^p$ are both subsets of VMOA for any $0 < \alpha < 1$ and $1 < p < \infty$. Since the only inner functions in VMOA are finite Blaschke products, Proposition 2 implies that the following four cases are of interest when $q + s > -1$ and $0 < s \leqslant 1$:

(1) $q + 1 \leqslant p < q + 2$ for $F(p, q, 0)$,

(2) $q + s + 1 < p \leqslant q + 2$ for both $F(p, q, s)$ and $F_0(p, q, s)$,

(3) $p = q + s + 1 < 2$ for both $F(p, q, s)$ and $F_0(p, q, s)$,

(4) $2 \leqslant p = q + s + 1$ for $F_0(p, q, s)$ only.

Two quantities a and b are said to be *comparable*, denoted by $a \simeq b$, if the quotient a/b is bounded and bounded away from 0. Moreover, $a \lesssim b$ means that $a \leqslant Cb$ for some positive constant C, and $a \gtrsim b$ is understood in an analogous manner.

The following result on radial integrability of inner functions plays an important role in some of the proofs. It also generalizes [**4**, Theorem 1], [**20**, Lemma 4.1] and [**42**, Lemma 2.2].

Lemma 2.1. *Let S be an inner function and let $1 \leqslant p < \infty$ and $-1 < q < \infty$ such that $p > q+1$. Then, for any $0 \leqslant \delta < 1$, there is a constant C, depending only on p and q, such that*

$$\begin{aligned} C\int_\delta^1 (1-|S(r\mathrm{e}^{\mathrm{i}\theta})|^2)^p(1-r^2)^{q-p}\,\mathrm{d}r &\leqslant \int_\delta^1 |S'(r\mathrm{e}^{\mathrm{i}\theta})|^p(1-r^2)^q\,\mathrm{d}r \\ &\leqslant \int_\delta^1 (1-|S(r\mathrm{e}^{\mathrm{i}\theta})|^2)^p(1-r^2)^{q-p}\,\mathrm{d}r \qquad (2.1) \end{aligned}$$

for almost all θ in $[0, 2\pi)$.

Proof. The second inequality in (2.1) follows by the Schwarz–Pick lemma

$$|S'(z)|(1-|z|^2) \leqslant 1-|S(z)|^2, \quad z \in \mathbb{D},$$

and it holds for all $p > 0$ and $q > -1$.

To prove the first inequality in (2.1), the proof of [**20**, Lemma 4.1] is followed. Let $p \geqslant 1$ and $p > q+1$. Since

$$1-|S(r\mathrm{e}^{\mathrm{i}\theta})| \leqslant \int_r^1 |S'(s\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}s = \int_0^1 |S'((t+(1-t)r)\mathrm{e}^{\mathrm{i}\theta})|(1-r)\,\mathrm{d}t \qquad (2.2)$$

for almost all θ in $[0, 2\pi)$, Minkowski's integral inequality (Fubini's theorem in the case when $p = 1$) yields

$$\begin{aligned} &\biggl(\int_\delta^1 (1-|S(r\mathrm{e}^{\mathrm{i}\theta})|^2)^p(1-r^2)^{q-p}\,\mathrm{d}r\biggr)^{1/p} \\ &\qquad\lesssim \int_0^1 \biggl(\int_\delta^1 |S'((t+(1-t)r)\mathrm{e}^{\mathrm{i}\theta})|^p(1-r)^q\,\mathrm{d}r\biggr)^{1/p}\mathrm{d}t \\ &\qquad= \int_0^1 \biggl(\int_{\delta+t(1-\delta)}^1 |S'(u\mathrm{e}^{\mathrm{i}\theta})|^p(1-u)^q\,\mathrm{d}u\biggr)^{1/p}(1-t)^{-(q+1)/p}\,\mathrm{d}t \\ &\qquad\lesssim \biggl(\int_\delta^1 |S'(u\mathrm{e}^{\mathrm{i}\theta})|^p(1-u)^q\,\mathrm{d}u\biggr)^{1/p}, \end{aligned}$$

and the assertion follows. □

The first inequality in (2.1) is especially easy to prove if $q < 0$. Namely, if $p \geqslant 1$, then the inequality in (2.2), Hölder's inequality and Fubini's theorem yield

$$\begin{aligned} \int_\delta^1 (1-|S(r\mathrm{e}^{\mathrm{i}\theta})|)^p(1-r)^{q-p}\,\mathrm{d}r &\leqslant \int_\delta^1 \int_r^1 |S'(s\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}s(1-r)^{q-1}\,\mathrm{d}r \\ &= \int_\delta^1 |S'(s\mathrm{e}^{\mathrm{i}\theta})|^p \int_\delta^s (1-r)^{q-1}\,\mathrm{d}r\,\mathrm{d}s \\ &\leqslant -\frac{1}{q}\int_\delta^1 |S'(s\mathrm{e}^{\mathrm{i}\theta})|^p(1-s)^q\,\mathrm{d}s. \end{aligned}$$

In a similar manner one can show that

$$\int_\delta^1 (1-|S(re^{i\theta})|^2)^p(1-r^2)^{-p}\,dr \leqslant \int_\delta^1 |S'(re^{i\theta})|^p \log\frac{1}{1-r}\,dr \tag{2.3}$$

for almost all θ in $[0,2\pi)$. Defining $\log_{n+1} x := \log_n(\log x)$, $\log_1 x := \log x$, $\exp_{n+1} x := \exp_n(\exp(x))$ and $\exp_1 x := \exp x$, the general form of (2.3) can be written as

$$\int_\delta^1 (1-|S(re^{i\theta})|)^p \frac{(1-r)^{-p}}{\prod_{j=1}^n \log_n 1/(1-r)}\,dr \leqslant \int_\delta^1 |S'(re^{i\theta})|^p \log_{n+1}\frac{1}{1-r}\,dr,$$

valid for almost all θ in $[0,2\pi)$, provided that $\delta > 1 - 1/(\exp_n 1)$.

Next, the reasoning in the proof of Lemma 2.1 is used to prove an area integral version involving an arbitrary analytic function (see Theorem 2.3). To do this, the following lemma is needed. It is worth observing that the proof presented here gives a fairly elementary proof for [**28**, Lemma 4.6] and [**36**, Lemma 5.3.1].

Lemma 2.2. *Let f be an analytic function in $\mathbb{D}$, and let $0 < p < \infty$, $-1 < q < \infty$, $0 \leqslant t < \infty$ and $0 \leqslant \delta < 1$. Then there is a positive constant C, depending only on q and t, such that*

$$\int_\delta^1\int_0^{2\pi} |f(re^{i\theta})|^p(1-r^2)^q\,d\theta\,dr \leqslant C\int_{\mathbb{D}\setminus\Delta(0,\delta)} |f(z)|^p(1-|z|^2)^q|z|^t\,dA(z), \tag{2.4}$$

where $\Delta(0,\delta) := \{z : |z| < \delta\}$.

Proof. If $\frac{1}{2} \leqslant \delta < 1$, then (2.4) holds with $C = 2^{t+1}$. Now let $0 \leqslant \delta < \frac{1}{2}$. Since

$$\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta$$

is a non-decreasing function of r by Hardy's convexity theorem [**16**, Theorem 1.5], it follows that

$$\begin{aligned}
\int_0^1\int_0^{2\pi} &|f(re^{i\theta})|^p(1-r^2)^q\,d\theta\,dr \\
&\leqslant \int_0^{2\pi} |f(\tfrac{1}{2}e^{i\theta})|^p\,d\theta \int_0^{1/2}(1-r^2)^q\,dr \\
&\quad + 2^{t+1}\int_{1/2}^1\int_0^{2\pi} |f(re^{i\theta})|^p(1-r^2)^q r^t r\,d\theta\,dr \\
&= C(q,t)\int_0^{2\pi} |f(\tfrac{1}{2}e^{i\theta})|^p\,d\theta \int_{1/2}^1 (1-r^2)^q r^t r\,dr \\
&\quad + 2^{t+1}\int_{\mathbb{D}\setminus\Delta(0,1/2)} |f(z)|^p(1-|z|^2)^q|z|^t\,dA(z) \\
&\leqslant (C(q,t) + 2^{t+1})\int_{\mathbb{D}\setminus\Delta(0,1/2)} |f(z)|^p(1-|z|^2)^q|z|^t\,dA(z),
\end{aligned}$$

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where

$$C(q,t) := \frac{\int_0^{1/2}(1-r^2)^q \, \mathrm{d}r}{\int_{1/2}^1 (1-r^2)^q r^{t+1} \, \mathrm{d}r}.$$

The inequality (2.4) with $C = C(q,t) + 2^{t+1}$ for $0 \leqslant \delta < \frac{1}{2}$ now clearly follows. □

Theorem 2.3. *Let S be an inner function and let $1 \leqslant p < \infty$, $-2 < q < \infty$, $0 < p^* < \infty$ such that $p > q+1 > 0$. Then, for any analytic function f in $\mathbb{D}$ and $0 \leqslant \delta < 1$, the following quantities are comparable:*

(1) $\displaystyle\int_{\mathbb{D}\setminus\Delta(0,\delta)} |f(z)|^{p^*}(1-|S(z)|^2)^p(1-|z|^2)^{q-p} \, \mathrm{d}A(z);$

(2) $\displaystyle\int_{\mathbb{D}\setminus\Delta(0,\delta)} |f(z)|^{p^*}|S'(z)|^p(1-|z|^2)^q \, \mathrm{d}A(z).$

Proof. By the Schwarz–Pick lemma it suffices to show that there exists a positive constant, depending only on p, q and p^*, such that the integral in (1) is less than or equal to this constant times the integral in (2). To do this, define

$$F(r\mathrm{e}^{\mathrm{i}\theta}) := \sup_{0<t\leqslant r} |(f(t\mathrm{e}^{\mathrm{i}\theta}))|^{p^*/p}.$$

Reasoning similar to that in the proof of Lemma 2.1 yields

$$\int_{\mathbb{D}\setminus\Delta(0,\delta)} |f(z)|^{p^*}(1-|S(z)|^2)^p(1-|z|^2)^{q-p} \, \mathrm{d}A(z) \lesssim \int_0^{2\pi}\int_\delta^1 |F(u\mathrm{e}^{\mathrm{i}\theta})|^p|S'(u\mathrm{e}^{\mathrm{i}\theta})|^p(1-u)^q \, \mathrm{d}u \, \mathrm{d}\theta.$$

By the Hardy–Littlewood maximal theorem [**16**, Theorem 1.9],

$$\int_0^{2\pi} |F(u\mathrm{e}^{\mathrm{i}\theta})|^p|S'(u\mathrm{e}^{\mathrm{i}\theta})|^p \, \mathrm{d}\theta \lesssim \int_0^{2\pi} |f(u\mathrm{e}^{\mathrm{i}\theta})|^{p^*}|S'(u\mathrm{e}^{\mathrm{i}\theta})|^p \, \mathrm{d}\theta,$$

and therefore

$$\int_{\mathbb{D}\setminus\Delta(0,\delta)} |f(z)|^{p^*}(1-|S(z)|^2)^p(1-|z|^2)^{q-p} \, \mathrm{d}A(z) \lesssim \int_0^{2\pi}\int_\delta^1 |f(u\mathrm{e}^{\mathrm{i}\theta})|^{p^*}|S'(u\mathrm{e}^{\mathrm{i}\theta})|^p(1-u^2)^q \, \mathrm{d}u \, \mathrm{d}\theta,$$

from which the assertion follows by Lemma 2.2. □

Another immediate consequence of Theorem 2.3 is Corollary 2.4, which may be of interest for those who wish to study inner–outer factorizations in $F(p,q,s)$.

Corollary 2.4. *Let S be an inner function and let $1 \leqslant p < \infty$, $-2 < q < \infty$ and $0 \leqslant s, p^* < \infty$ such that $p > q + s + 1 > 0$. Then, for any analytic function f in $\mathbb{D}$ and $a \in \mathbb{D}$, the following quantities are comparable:*

(1) $\displaystyle\int_{\mathbb{D}} |f(z)|^{p^*}(1 - |S(z)|^2)^p(1 - |z|^2)^{q-p}(1 - |\varphi_a(z)|^2)^s \,\mathrm{d}A(z)$;

(2) $\displaystyle\int_{\mathbb{D}} |f(z)|^{p^*}|S'(z)|^p(1 - |z|^2)^q(1 - |\varphi_a(z)|^2)^s \,\mathrm{d}A(z)$.

The last result of this section is a consequence of Lemma 2.1 and [**2**, Theorem 6]. The reasoning involves Carleson measures. For $0 < s < \infty$, a positive measure μ on $\mathbb{D}$ is a *bounded s-Carleson measure*, if $\mu(Q(I)) = O(|I|^s)$, where $|I|$ denotes the arc length of a subarc I of $\mathbb{T}$ and $Q(I) := \{z \in \mathbb{D} : z/|z| \in I,\ 1 - |I| \leqslant |z|\}$ is the Carleson box based on I. Moreover, if $\mu(Q(I)) = o(|I|^s)$ as $|I| \to 0$, then μ is a *compact* (or *vanishing*) *s-Carleson measure*. These measures (for $s = 1$) were introduced by Carleson [**9, 10**] (see [**33**] for a list of relevant references). It is known that an analytic function f in $\mathbb{D}$ belongs to $F(p, q, s)$ (respectively, $F_0(p, q, s)$), $0 < s < \infty$, if and only if the measure μ such that $\mathrm{d}\mu(z) = |f'(z)|^p(1 - |z|^2)^{q+s}\,\mathrm{d}A(z)$ is a bounded (respectively, compact) s-Carleson measure [**6, 46**].

Corollary 2.5. *Let $1 \leqslant p < \infty$, $-2 < q < \infty$ and $0 \leqslant s < \infty$ such that $p > q + s + 1 > 0$. Let $S := \prod_{j=1}^n S_j$, where S_j is an inner function for all $j = 1, 2, \dots, n$. Then $S \in F(p, q, s)$ if and only if $S_j \in F(p, q, s)$ for all $j = 1, 2, \dots, n$. Similarly, $S \in F_0(p, q, s)$ if and only if $S_j \in F_0(p, q, s)$ for all $j = 1, 2, \dots, n$. Moreover, if $q = p - 2$, then these assertions hold for all $p > 1 - s$.*

Proof. Let first $1 \leqslant p < \infty$ and $S_j \in F(p, q, s)$ for all $j = 1, 2, \dots, n$. Define $S := \prod_{j=1}^n S_j$. Then

$$S' = \sum_{j=1}^{n} S_j' \prod_{k \neq j} S_k,$$

from which Hölder's inequality yields $|S'(z)|^p \leqslant n^{p-1} \sum_{j=1}^n |S_j'(z)|^p$, and it follows that $S \in F(p, q, s)$.

Now let $S \in F(p, q, s)$. Since $|S_j(z)| \geqslant |S(z)|$ for all $z \in \mathbb{D}$ and $j = 1, 2, \dots, n$, Lemma 2.1 implies that

$$\begin{aligned}\int_{Q(I)} |S_j'(z)|^p(1 - |z|^2)^{q+s}\,\mathrm{d}A(z) &\leqslant \int_{Q(I)} (1 - |S(z)|^2)^p(1 - |z|^2)^{q+s-p}\,\mathrm{d}A(z)\\ &\lesssim \int_{Q(I)} |S'(z)|^p(1 - |z|^2)^{q+s}\,\mathrm{d}A(z),\end{aligned}$$

from which [**46**, Theorem 2.4] yields $S_j \in F(p, q, s)$ for all $j = 1, \dots, n$. The assertion for small spaces follows readily by the proof above.

If $q = p - 2$ and $0 < p < 1$, then the assertions follow by using the inequality $|S'(z)|^p \leqslant \sum_{j=1}^n |S_j'(z)|^p$, an equivalent seminorm in $F(p, q, s)$, established in [**46**] and defined by

$$\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s\,\mathrm{d}A(z)\right)^{1/p}, \tag{2.5}$$

the change of variable $z=\varphi_a(w)$ and [**2**, Theorem 6]. □

Note that the assertions of Corollary 2.5 in the case when $p\geqslant 1$ can also be proved by using (2.5) and Corollary 2.4 with $f\equiv 1$.

3. Singular inner functions

This section contains the proofs of Theorems 1.1 and 1.3 on singular inner functions. Recall that if the generating measure σ of the singular inner function S is atomic and consists of a point mass concentrated in $w\in\mathbb{T}$, then S is of the form

$$S_{\gamma,w}(z)=\exp\left(\gamma\frac{z+w}{z-w}\right),$$

where $0<\gamma<\infty$, and then

$$|S'_{\gamma,w}(z)|=\frac{2\gamma}{|z-w|^2}\exp\left(-\gamma\frac{1-|z|^2}{|z-w|^2}\right). \tag{3.1}$$

Proof of Theorem 1.1. Let $q+s+1\leqslant p<q+s+\frac{3}{2}$ and let $w=\mathrm{e}^{\mathrm{i}\theta}$. Then, by (3.1),

$$\begin{aligned}J(t)&:=\int_0^1|S'_{\gamma,\mathrm{e}^{\mathrm{i}\theta}}(r\mathrm{e}^{\mathrm{i}t})|^p(1-r^2)^{q+s}r\,\mathrm{d}r\\&\lesssim\int_0^1\exp\left(-p\gamma\frac{1-r}{|1-r\mathrm{e}^{\mathrm{i}(t-\theta)}|^2}\right)\frac{\gamma^p(1-r)^{q+s}}{|1-r\mathrm{e}^{\mathrm{i}(t-\theta)}|^{2p}}\,\mathrm{d}r,\end{aligned}$$

and since $|1-r\mathrm{e}^{\mathrm{i}(t-\theta)}|^2=(1-r)^2+rc^2$, where $c:=|1-\mathrm{e}^{\mathrm{i}(t-\theta)}|$, it follows that

$$J(t)\lesssim\int_0^1\exp\left(-p\gamma\frac{u}{u^2+c^2(1-u)}\right)\left(\frac{\gamma u}{u^2+c^2(1-u)}\right)^p\frac{\mathrm{d}u}{u^{p-q-s}}.$$

Without loss of generality, assume $c\leqslant 1$. Then

$$\begin{aligned}J(t)&\lesssim\int_0^c\exp\left(-\frac{p}{2}\frac{\gamma u}{c^2}\right)\left(\frac{\gamma u}{c^2}\right)^p\frac{\mathrm{d}u}{u^{p-q-s}}+\int_c^1\exp\left(-\frac{p}{2}\frac{\gamma}{u}\right)\left(\frac{\gamma}{u}\right)^p\frac{\mathrm{d}u}{u^{p-q-s}}\\&=\left(\frac{\gamma}{c^2}\right)^{p-q-s-1}\int_0^{\gamma/c}\exp(-\tfrac{1}{2}pv)v^{q+s}\,\mathrm{d}v+\gamma^{q+s+1-p}\int_\gamma^{\gamma/c}\exp(-\tfrac{1}{2}pv)v^{2p-q-s-2}\,\mathrm{d}v\\&=:J_1(t)+J_2(t).\end{aligned}$$

The quantity $J_2(t)$ is bounded by a positive constant, depending only on p, q, s and γ, for all $t\in[0,2\pi]$. Moreover, since $q+s>-1$, $J_1(t)\lesssim c^{2(q+s+1-p)}$, and thus $J(t)\lesssim$

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$c^{2(q+s+1-p)}+1$. It follows that

$$\begin{aligned}\int_{Q(I)}|S'_{\gamma,w}(z)|^p(1-|z|^2)^{q+s}\,\mathrm{d}A(z) &\lesssim \int_\theta^{\theta+|I|}\frac{1}{|1-\mathrm{e}^{\mathrm{i}(t-\theta)}|^{2(p-q-s-1)}}\,\mathrm{d}t+|I|\\ &\simeq \int_0^{|I|}\frac{\mathrm{d}t}{(1-\cos t)^{p-q-s-1}}\,\mathrm{d}t+|I|\\ &\simeq \int_0^{|I|}\frac{\mathrm{d}t}{t^{2(p-q-s-1)}}+|I|\\ &\simeq |I|^{2(q+s+1-p)+1}+|I| \qquad (3.2)\end{aligned}$$

for all $I\subset\mathbb{T}$ such that $|I|\leqslant 1$. If $q=p-\frac{1}{2}(s+3)$, then (3.2) implies that $S_{\gamma,w}\in F(p,p-\frac{1}{2}(s+3),s)$, and the nesting property of $F(p,q,s)$ with respect to p yields $S_{\gamma,w}\in F(p,q,s)$ for all $p\leqslant q+\frac{1}{2}(s+3)$. Moreover, if $q+s+1<p<q+\frac{1}{2}(s+3)$ and $0<s<1$, then $S_{\gamma,w}\in F_0(p,q,s)$ by (3.2), since $2(q+s+1-p)+1-s>0$ and $1-s>0$. It follows that $S_{\gamma,w}\in F_0(p,q,s)$ for all $p<q+\frac{1}{2}(s+3)$, provided that $0<s<1$. The case when $s=1$ follows by Proposition 2 (3).

To see that the conditions $p\leqslant q+\frac{1}{2}(s+3)$ and $p<q+\frac{1}{2}(s+3)$ are also necessary, assume first that $s>0$ and choose $I:=I(w):=\{\mathrm{e}^{\mathrm{i}t}:\theta\leqslant t\leqslant\theta+|I|\}$. Then reasoning similar to that above yields

$$\begin{aligned}\int_{1-|I|}^1|S'_{\gamma,\mathrm{e}^{\mathrm{i}\theta}}(r\mathrm{e}^{\mathrm{i}t})|^p(1-r^2)^{q+s}r\,\mathrm{d}r &\gtrsim \int_0^c\exp\left(-\frac{8p}{3}\frac{\gamma u}{c^2}\right)\left(\frac{\gamma u}{c^2}\right)^p\frac{\mathrm{d}u}{u^{p-q-s}}\\ &\geqslant \left(\frac{\gamma}{c^2}\right)^{p-q-s-1}\int_0^\gamma\exp\left(-\frac{8p}{3}v\right)v^{q+s}\,\mathrm{d}v\\ &=: J_1(t),\end{aligned}$$

where $J_1(t)\gtrsim c^{2(q+s+1-p)}$. Further, the reasoning in (3.2) gives

$$\int_{Q(I)}|S'_{\gamma,w}(z)|^p(1-|z|^2)^{q+s}\,\mathrm{d}A(z)\gtrsim|I|^{2(q+s+1-p)+1}.$$

It follows that if $q+\frac{1}{2}(s+3)<p<q+s+\frac{3}{2}$, then $S_{\gamma,w}\notin F(p,q,s)$, and similarly if $q+\frac{1}{2}(s+3)\leqslant p<q+s+\frac{3}{2}$, then $S_{\gamma,w}\notin F_0(p,q,s)$. The case when $s=0$ follows by observing the proof above. $\square$

Theorem 1.1, with an additional restriction $p\geqslant 1$, can also be proved by using Lemma 2.1 and following the proof of [**20**, Corollary 4.2]. The proof presented here uses some ideas from the proof of [**43**, Lemma 2.6].

Proof of Theorem 1.3. Let $1<p<\infty$, $0<s<1$ and $I\subset\mathbb{T}$ with $|I|\leqslant\delta<\frac{1}{2}$, and let $z=r\mathrm{e}^{\mathrm{i}\theta}\in Q(I)$. If

$$S(z)=\exp\left(\int_{\mathbb{T}}\frac{z+w}{z-w}\,\mathrm{d}\sigma(w)\right),$$

where σ is a non-atomic singular measure on $\mathbb{T}$, then

$$|S(z)| = \exp\biggl(-\int_{\mathbb{T}} \frac{1-|z|^2}{|w-z|^2}\,\mathrm{d}\sigma(w)\biggr) = \exp\biggl(-\int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{w}z|^2}\,\mathrm{d}\sigma(w)\biggr). \tag{3.3}$$

By Lemma 2.1 and (3.3),

$$\begin{aligned} J_I(\theta) &:= \int_{1-|I|}^{1} |S'(r\mathrm{e}^{\mathrm{i}\theta})|^p(1-r^2)^{p-2+s} r\,\mathrm{d}r \\ &\simeq \int_{1-|I|}^{1} (1-|S(r\mathrm{e}^{\mathrm{i}\theta})|^2)^p(1-r^2)^{s-2}\,\mathrm{d}r \\ &\geqslant \int_{1-|I|}^{1} \biggl(1-\exp\biggl(-\int_{\theta-|I|}^{\theta+|I|} \frac{1-r}{|\mathrm{e}^{\mathrm{i}t}-r\mathrm{e}^{\mathrm{i}\theta}|^2}\mathrm{d}\sigma(\mathrm{e}^{\mathrm{i}t})\biggr)\biggr)^p(1-r)^{s-2}\,\mathrm{d}r \\ &\geqslant \int_{1-|I|}^{1} \biggl(1-\exp\biggl(-\frac{\sigma(I)(1-r)}{2|I|^2}\biggr)\biggr)^p(1-r)^{s-2}\,\mathrm{d}r \end{aligned}$$

for almost all θ in $[0,2\pi)$, and therefore

$$\begin{aligned} J_I &:= \int_{Q(I)} |S'(z)|^p(1-|z|^2)^{p-2+s}\,\mathrm{d}A(z) \\ &\gtrsim |I|\int_{1-|I|}^{1} \biggl(1-\exp\biggl(-\frac{\sigma(I)(1-r)}{2|I|^2}\biggr)\biggr)^p(1-r)^{s-2}\,\mathrm{d}r \\ &= |I|^s\biggl(\frac{\sigma(I)}{|I|}\biggr)^{1-s}\int_0^{\sigma(I)/|I|}(1-\mathrm{e}^{-t/2})^p t^{s-2}\,\mathrm{d}t. \end{aligned} \tag{3.4}$$

If $S \in B_s^p$, then $J_I = O(|I|^s)$ by [**46**, Theorem 2.4], and (3.4) implies that

$$\biggl(\frac{\sigma(I)}{|I|}\biggr)^{1-s}\int_0^{\sigma(I)/|I|}(1-\mathrm{e}^{-t/2})^p t^{s-2}\,\mathrm{d}t = O(1)$$

as $|I| \to 0$. But this is a contradiction since $\limsup_{|I|\to 0}\sigma(I)/|I| = \infty$ by [**39**, Theorem 8.11]. Thus, the assumption $S \in B_s^p$ is wrong, and since the space B_s^p gets larger with p by (1.1) it follows that $S \notin B_s^p$ for $p > 1-s$. $\square$

It will be shown next that the singular inner function S does not belong to $B^1_{1,\log}$. By [**6**, Lemma 2.1], an analytic function f belongs to $B^1_{1,\log}$ if and only if

$$\sup_I \frac{1}{|I|}\int_{Q(I)} |f'(z)|\log\frac{1}{1-|z|}\,\mathrm{d}A(z) < \infty.$$

Assuming now that $S \in B^1_{1,\log}$ and following the proof of Theorem 1.3, the inequality in (2.3) with $p = 1$ yields

$$\int_0^{\sigma(I)/|I|}(1-\mathrm{e}^{-t/2})t^{-1}\,\mathrm{d}t = O(1).$$

This is again clearly a contradiction since $\limsup_{|I|\to 0}\sigma(I)/|I| = \infty$. Thus, the inner function S does not belong to $B^1_{1,\log}$.

4. Blaschke products

We begin with a brief discussion on Blaschke products whose first derivatives belong to a given weighted Bergman space A^p_q. The proof of Theorem A can be found in [**30**] (see also [**4**, **38**]).

Theorem A. *Let $0 < p < \infty$ and $-1 < q < \infty$. Let B be the Blaschke product associated with a sequence $\{z_n\}$.*

(1) *If*
$$\sum_{n=1}^{\infty}(1-|z_n|^2)^{2+q-p} < \infty, \quad \max\{\tfrac{1}{2}q+1, q+1\} < p < q+2,$$
then $B' \in A^p_q$.

(2) *Furthermore, if*
$$\sum_{n=1}^{\infty}(1-|z_n|^2)^p \log\frac{1}{1-|z_n|^2} < \infty, \quad p = \tfrac{1}{2}q+1 \geqslant q+1,$$
then $B' \in A^p_q$.

(3) *Furthermore, if*
$$\sum_{n=1}^{\infty}(1-|z_n|^2)^{\delta} < \infty$$
for some $\delta < \frac{1}{2}q+1$, then $B' \in A^p_q$ for any $p < \frac{1}{2}q+1$.

If B is the Blaschke product associated with a uniformly separated sequence $\{z_n\}$ (i.e. B is an interpolating Blaschke product) such that $B' \in A^p_q$, then reasoning similar to that in the proof of [**25**, Theorem 5] shows that
$$\sum_{n=1}^{\infty}(1-|z_n|^2)^{q+2-p} < \infty. \tag{4.1}$$

Therefore, assertion (1) in Theorem A is sharp. We next prove a slightly strengthened version of this result in the case when $p \geqslant 1$ without appealing to [**25**, Lemma 6]. Some details of the proof of Theorem 4.1 will be needed while proving Theorem 1.4. Recall that $\{z_n\}$ is a finite union of uniformly separated sequences if and only if
$$\sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_a(z_n)|^2) < \infty. \tag{4.2}$$

Theorem 4.1. *Let $1 \leqslant p < \infty$ and $-1 < q < \infty$ such that $q+1 < p < q+2$. Let B be the Blaschke product associated with a sequence $\{z_n\}$ which satisfies (4.2). If $B' \in A^p_q$, then (4.1) is satisfied.*

Proof. Since $1-r^2 \leqslant -2\log r$ for all $0<r\leqslant 1$, it follows that

$$2\log|B(z)| = \sum_{n=1}^{\infty}\log(|\varphi_{z_n}(z)|^2) \leqslant -\sum_{n=1}^{\infty}(1-|\varphi_{z_n}(z)|^2),$$

and therefore $|B(z)|^2 \leqslant \exp(-\sum_{n=1}^{\infty}(1-|\varphi_{z_n}(z)|^2))$. Hence,

$$1-|B(z)|^2 \geqslant 1-\exp\biggl(-\sum_{n=1}^{\infty}(1-|\varphi_{z_n}(z)|^2)\biggr), \tag{4.3}$$

and since the function $(1-\mathrm{e}^{-x})/x$ is decreasing in $(0,\infty)$, the assumption (4.2) yields

$$\begin{aligned}1-|B(z)|^2 &\geqslant \sum_{n=1}^{\infty}(1-|\varphi_{z_n}(z)|^2)\frac{1-\exp(-\sup_{z\in\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_{z_n}(z)|^2))}{\sup_{z\in\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_{z_n}(z)|^2)}\\ &\simeq \sum_{n=1}^{\infty}(1-|\varphi_{z_n}(z)|^2). \qquad (4.4)\end{aligned}$$

By Lemma 2.1, the inequality $(\sum|b_k|)^p \geqslant (\sum|b_k|^p)$ and (4.4), we obtain

$$\begin{aligned}\|B'\|_{A_q^p}^p &\gtrsim \int_{\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_{z_n}(z)|^2)^p(1-|z|^2)^{q-p}\,\mathrm{d}A(z)\\ &\geqslant \sum_{n=1}^{\infty}(1-|z_n|^2)^p\int_{\mathbb{D}}\frac{(1-|z|^2)^q}{|1-\bar{z}_n z|^{2p}}\,\mathrm{d}A(z). \qquad (4.5)\end{aligned}$$

Let $I_n := \{\mathrm{e}^{\mathrm{i}\zeta} : |\zeta-\arg z_n| \leqslant \frac{1}{2}(1-|z_n|)\}$, so that $|I_n| = 1-|z_n|$. It is easy to see that

$$|1-\bar{z}_n z| \leqslant (2+\tfrac{\sqrt{5}}{2})(1-|z_n|)$$

for $z\in Q(I_n)$. This combined with (4.5) yields

$$\begin{aligned}\|B'\|_{A_q^p}^p &\gtrsim \sum_{n=1}^{\infty}(1-|z_n|^2)^p\int_{Q(I_n)}\frac{(1-|z|^2)^q}{|1-\bar{z}_n z|^{2p}}\,\mathrm{d}A(z)\\ &\gtrsim \sum_{n=1}^{\infty}(1-|z_n|^2)^{1-p}\int_{|z_n|^2}^{1}(1-r)^q\,\mathrm{d}r\\ &\simeq \sum_{n=1}^{\infty}(1-|z_n|^2)^{q+2-p},\end{aligned}$$

and the desired assertion follows. $\square$

It is worth noting that if B is the Blaschke product associated with a sequence $\{z_n\}$ which satisfies (4.2) and

$$\int_{\mathbb{D}}|B'(z)|\log\frac{1}{1-|z|}\,\mathrm{d}A(z) < \infty,$$

then the inequality (2.3) and the proof of Theorem 4.1 show that the sum in Theorem A (2) with $p = 1$ is finite.

We now turn to consider the Besov-type spaces. The following theorem is a partial improvement on [**43**, Theorem 2.12].

Theorem 4.2. *Let B be the Blaschke product associated with a sequence $\{z_n\}$.*

(a) *If $0 < p < 1$ and*

$$\sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_a(z_n)|^2)^p < \infty,$$

then $B \in \bigcap_{s>\max\{p,1-p\}} B^p_s$.

(b) *If $\frac{1}{2} < p \leqslant 1$ and*

$$\sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_a(z_n)|^2)^p \log\frac{1}{1-|\varphi_a(z_n)|^2} < \infty,$$

then $B \in B^p_p$.

(c) *If $0 < s < 1$ and*

$$\sup_{a\in\mathbb{D}}\sum_{n=1}^{\infty}(1-|\varphi_a(z_n)|^2)^s < \infty, \tag{4.6}$$

then $B \in \bigcap_{p>\max\{s,1-s\}} B^p_s$.

Proof. Since

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty}\frac{|z_n|^2-1}{(1-\bar{z}_n z)(z_n-z)},$$

it follows that

$$\begin{aligned}|B'(z)| &= \left|\sum_{n=1}^{\infty}\frac{1-|z_n|^2}{(1-\bar{z}_n z)(z_n-z)}\right|\left|\prod_{k=1}^{\infty}\frac{|z_k|}{z_k}\frac{z_k-z}{1-\bar{z}_k z}\right| \\ &\leqslant \sum_{n=1}^{\infty}\frac{1-|z_n|^2}{|z_n-z|\,|1-\bar{z}_n z|}\frac{|z_n-z|}{|1-\bar{z}_n z|}|B_n(z)| \\ &\leqslant \sum_{n=1}^{\infty}|\varphi'_{z_n}(z)|, \end{aligned}\tag{4.7}$$

where

$$B_n(z) := \prod_{k\neq n}\frac{|z_k|}{z_k}\frac{z_k-z}{1-\bar{z}_k z}.$$

The inequality (4.7) yields

$$|(B\circ\varphi_a)'(z)| \leqslant \sum_{n=1}^{\infty}|\varphi'_{z_n}(\varphi_a(z))|\,|\varphi'_a(z)| = \sum_{n=1}^{\infty}|\varphi'_{\varphi_a(z_n)}(z)|,$$

and it follows that

$$|(B \circ \varphi_a)'(z)|^p \leqslant \sum_{n=1}^{\infty} |\varphi'_{\varphi_a(z_n)}(z)|^p$$

for $0 < p \leqslant 1$. The change of variable $w = \varphi_a(z)$ gives

$$\begin{aligned}\int_{\mathbb{D}} |B'(w)|^p (1-|w|^2)^{p-2}(1-|\varphi_a(w)|^2)^s \,\mathrm{d}A(w) \\ &= \int_{\mathbb{D}} |(B\circ\varphi_a)'(z)|^p (1-|z|^2)^{s+p-2}\,\mathrm{d}A(z) \\ &\leqslant \int_{\mathbb{D}} \sum_{n=1}^{\infty} \frac{(1-|\varphi_a(z_n)|^2)^p}{|1-\overline{\varphi_a(z_n)}z|^{2p}} (1-|z|^2)^{s+p-2}\,\mathrm{d}A(z) \\ &= \sum_{n=1}^{\infty} (1-|\varphi_a(z_n)|^2)^p \int_{\mathbb{D}} \frac{(1-|z|^2)^{p-2+s}}{|1-\overline{\varphi_a(z_n)}z|^{2p}}\mathrm{d}A(z),\end{aligned}$$

from which Forelli–Rudin estimates [**27**, Theorem 1.7] with the assumptions yield the assertions. □

The following result is a partial converse of Theorem 4.2 (c). The proof uses ideas from [**20**, **42**].

Theorem 4.3. *Let $0 < p < \infty$ and $0 < s < 1$ such that $p + s > 1$, and let B be the Blaschke product associated with a sequence $\{z_n\}$. If $B \in B_s^p$, then (4.6) is satisfied.*

Proof. Since B_s^p gets larger with p by (1.1), we may assume that $B \in B_s^p$ for some $p \geqslant 1$. Let $I \subset \mathbb{T}$ such that $|I| \leqslant \frac{1}{2}$, and define $R := R(I) := \sum_{z_n \in Q(I)} (1 - |z_n|^2)$. By Lemma 2.1 and the inequality (4.3), we have

$$\begin{aligned} J_I &:= \int_{Q(I)} |B'(z)|^p (1-|z|^2)^{p-2+s}\,\mathrm{d}A(z) \\ &\gtrsim \int_{Q(I)} \biggl(1-\exp\biggl(-\sum_{n=1}^{\infty}(1-|\varphi_{z_n}(z)|^2)\biggr)\biggr)^p (1-|z|^2)^{s-2}\,\mathrm{d}A(z) \\ &\geqslant \int_{Q(I)} \biggl(1-\exp\biggl(-(1-|z|^2)\sum_{z_n\in Q(I)} \frac{1-|z_n|^2}{|1-z\bar{z}_n|^2}\biggr)\biggr)^p (1-|z|^2)^{s-2}\,\mathrm{d}A(z) \\ &\geqslant \int_{Q(I)} \biggl(1-\exp\biggl(-\frac{1-|z|}{2|I|^2}R(I)\biggr)\biggr)^p (1-|z|^2)^{s-2}\,\mathrm{d}A(z) \\ &= |I| \int_{1-|I|}^{1} \biggl(1-\exp\biggl(-\frac{1-r}{2|I|^2}R\biggr)\biggr)^p (1-r^2)^{s-2} r\,\mathrm{d}r \\ &\simeq |I|^s \biggl(\frac{R}{|I|}\biggr)^{1-s} \int_0^{R/2|I|} (1-\mathrm{e}^{-u})^p u^{s-2}\,\mathrm{d}u. \end{aligned}$$

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Since $B \in B^p_s$, [**46**, Theorem 2.4] implies that $J_I = O(|I|^s)$, and therefore

$$\left(\frac{R}{|I|}\right)^{1-s} \int_0^{R/2|I|} (1 - \mathrm{e}^{-u})^p u^{s-2} \, \mathrm{d}u \leqslant C$$

for a positive constant C. It follows that $R = R(I) = O(|I|)$, and hence (4.2) is satisfied by [**21**, Chapter VI, Lemma 3.3]. If B_{φ_a} denotes the Blaschke product associated with the sequence $\{\varphi_a(z_n)\}$, then $|B(\varphi_a(z))| = |B_{\varphi_a}(z)|$, and (4.4) yields

$$1 - |B(\varphi_a(z))|^2 = 1 - |B_{\varphi_a}(z)|^2 \gtrsim \sum_{n=1}^{\infty} (1 - |\varphi_{\varphi_a(z_n)}(z)|^2). \tag{4.8}$$

By the inequality $-2\log r \geqslant 1 - r^2$, a change of variables, Lemma 2.1 and (4.8), we finally obtain

$$\begin{aligned}
\|B\|^p_{B^p_s} &\gtrsim \int_{\mathbb{D}} |(B \circ \varphi_a)'(z)|^p (1 - |z|^2)^{p-2+s} \, \mathrm{d}A(z) \\
&\simeq \int_{\mathbb{D}} (1 - |(B \circ \varphi_a)(z)|^2)^p (1 - |z|^2)^{s-2} \, \mathrm{d}A(z) \\
&\gtrsim \int_{\mathbb{D}} \left(\sum_{n=1}^{\infty} (1 - |\varphi_{\varphi_a(z_n)}(z)|^2) \right)^p (1 - |z|^2)^{s-2} \, \mathrm{d}A(z) \\
&\geqslant \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^p \int_{\mathbb{D}} \frac{(1 - |z|^2)^{p-2+s}}{|1 - \overline{\varphi_a(z_n)} z|^{2p}} \, \mathrm{d}A(z),
\end{aligned}$$

from which reasoning similar to that at the end of the proof of Theorem 4.1 yields the assertion. □

The reasoning in the proof of Theorem 4.3 can also be used to obtain a necessary condition for the zeros of a Blaschke product in $B^1_{1,\log}$. Namely, if the Blaschke product B associated with a sequence $\{z_n\}$ belongs to $B^1_{1,\log}$, then the inequality in (2.3) (with the same steps as in the proof of Theorem 4.3) yields

$$\int_0^{R/2|I|} (1 - \mathrm{e}^{-u}) u^{-1} \, \mathrm{d}u \leqslant C$$

for a positive constant C. It again follows that $R = O(|I|)$, and thus (4.2) is satisfied. Following the proof further, we obtain

$$\sup_{a \in \mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2) \log \frac{1}{1 - |\varphi_a(z_n)|^2} < \infty.$$

Proof of Theorem 1.4. Let S be an inner function and let $0 < s < 1$. If $S \in B^p_s$ for some $p > 0$, then by Corollary 1.2, Theorem 1.3 and Corollary 2.5, S must be a Blaschke product. Furthermore, its zero sequence $\{z_n\}$ satisfies (4.6) by Theorem 4.3. Conversely, if the zero sequence $\{z_n\}$ of a Blaschke product B satisfies (4.6), then $B \in B^p_s$ for all $p > \max\{s, 1 - s\}$ by Theorem 4.2. □

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