Problem in Plane Geometry

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Figure 1

In a plane, a point O and a straight line OH drawn through O are given. OH is the bisector of an unknown angle YOX, which it is required to determine by the following conditions:

A point I, given by position in the plane of the figure, is connected with the straight line OH by the given angle $IOH = \theta$, and by the distance $OI = c$, from the point I to the vertex of the angle. This point I is the middle of a chord AB inscribed in the angle YOX, Furthermore the product $OA \times OB$ of the distances to the point O of the extremities of this chord is equal to a given quantity K^2.

The data therefore are:

The bisector OH of the angle YOX or AOB given in direction, and the median OI, of the triangle AOB, given in direction and magnitude.

The polar co-ordinates $\text{IOH} = \theta$, $\text{OI} = c$ which fix the position of the point I with reference to the bisector.

Finally, the product $\text{OA} \times \text{OB} = \text{K}^2$ of the sides OA and OB of the triangle OAB.

The unknowns are:

The lengths $\text{OA} = x$, $\text{OB} = y$, $\text{AB} = 2z$ of the sides of the triangle.

The angle α which the sides OA and OB make with the bisector OH.

The angle ϕ which the median makes with the base AB.

The angles A and B of the triangle.

We shall begin by determining the angle α, which is the key of the solution. Once this angle is determined, the figure may be constructed.

Equations of the Problem

The angles AOI, BOI are expressed in terms of the angle α by the relations

$$\text{AOI} = \alpha + \theta$$
$$\text{IOB} = \alpha - \theta$$

Apply the well-known trigonometrical formula to the triangles IAO, BIO, BAO, as well as to the triangle OII′, which is obtained by drawing through the point I a parallel II′ to the side OB, till it meets the side OA in I′.

To place the chord AB, it will suffice to take on the axis OX a quantity $I'A = OI'$, then to join the point A to the point I We shall have

$$OI' = \frac{x}{2} \quad \text{and } I'I = \frac{y}{2}$$

The four triangles give us the equations

(1) $$z^2 = x^2 + c^2 - 2cx\cos(\alpha + \theta)$$

(2) $$z^2 = y^2 + c^2 - 2cy\cos(\alpha - \theta)$$

(3) $$4z^2 = x^2 + y^2 = 2xy\cos 2\alpha, \text{ and finally}$$

$$c^2 = \frac{x^2}{4} + \frac{y^2}{4} + 2\frac{x}{2} \times \frac{y}{2}\cos 2\alpha$$

changing here the sign of the cosine, because the angle 2α is exterior, and not interior, to the triangle considered.

The last equation may be written

(4) $$4c^2 = x^2 + y^2 + 2xy\cos 2\alpha$$

To these must be added the other condition expressed by

(5) $$xy = K^2$$

A combination of these five equations will give us the unknowns and specially the angle α.

SOLUTION OF EQUATIONS (1)—(5)

In the equations (3) and (4) substitute K^2 for the product xy; then

(3′) $$4z^2 = x^2 + y^2 - 2K^2\cos 2\alpha$$

(4′) $$4c^2 = x^2 + y^2 + 2K^2\cos 2\alpha$$

By addition and subtraction we obtain

(3″) $$z^2 + c^2 = \tfrac{1}{2}(x^2 + y^2)$$

(4″) $$z^2 - c^2 = -K^2\cos 2\alpha$$

The first of these equations expresses the well-known theorem which connects the median with the three sides of a triangle. From the second equation we have

(6) $$z^2 = c^2 - K^2\cos 2\alpha.$$

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Substitute this value of z^2 in equation (3″) and we obtain

$$(7) \qquad x^2+y^2=4c^2-2K^2\cos 2\alpha.$$

If we substitute these values of z^2 and x^2+y^2 in equations (1) and (2), there results

$$\begin{aligned} c^2-K^2\cos 2\alpha &= x^2+c^2-2cx\cos(\alpha+\theta) \\ &= y^2+c^2-2cy\cos(\alpha-\theta). \end{aligned}$$

Solving these two equations with respect to $\cos(\alpha+\theta)$ and $\cos(\alpha-\theta)$, we have

$$(8) \qquad \begin{cases} \cos(\alpha+\theta)=\dfrac{x^2+K^2\cos 2\alpha}{2cx} \\ \cos(\alpha-\theta)=\dfrac{y^2+K^2\cos 2\alpha}{2cy} \end{cases}$$

The final equation which will give us $\cos\alpha$ is obtained by multiplication, from equations (8). It is

$$(9) \quad \cos(\alpha+\theta)\cos(\alpha-\theta)=\frac{(x^2+K^2\cos 2\alpha)(y^2+K^2\cos 2\alpha)}{2cx\times 2cy}$$

$$=\frac{x^2y^2+K^2\cos 2\alpha\,(x^2+y^2)+K^4\cos^2 2\alpha}{4c^2\,xy}$$

and this lends itself to important simplifications.

1°. The first member simplifies to $\cos^2\alpha-\sin^2\theta$.

2°. The second member contains only the product xy, its square x^2y^2, and the sum x^2+y^2, all of them functions which may be expressed by means of $\cos\alpha$, from the relations (5) and (7).

We have the identity

$$\begin{aligned} &x^2y^2+K^2\cos 2\alpha\,(x^2+y^2)+K^4\cos^2 2\alpha \\ &=K^4+K^2\cos 2\alpha\,(4c^2-2K^2\cos 2\alpha)+K^4\cos^2 2\alpha \\ &=K^2(K^2+4c^2\cos 2\alpha-2K^2\cos^2 2\alpha+K^2\cos^2 2\alpha) \\ &=K^2(K^2+4c^2\cos 2\alpha-K^2\cos^2 2\alpha). \end{aligned}$$

Hence by these transformations, and by suppressing, in numerator and denominator, the factor K^2, equation (9) becomes

$$(10) \qquad \cos^2\alpha-\sin^2\theta=\frac{K^2+4c^2\cos 2\alpha-K^2\cos^2 2\alpha}{4c}$$

Equation (10) is a biquadratic in $\cos\alpha$, that is, an equation of the second degree in $\cos^2\alpha$.

Put therefore $\cos^2 a = u$, calling u a new unknown. We have then

$$\cos 2a = 2u - 1, \ \cos^2 2a = 4u^2 - 4u + 1$$

and equation (10) takes the form

$$u - \sin^2\theta = \frac{K^2}{4c^2} + 2u - 1 - \frac{K^2}{4c^2}(4u^2 - 4u + 1)$$

Arrange with respect to u, and divide by $\frac{K^2}{c^2}$, then

$$u^2 - \left(1 + \frac{c^2}{K^2}\right)u + \frac{c^2}{K^2}\cos^2\theta = 0. \tag{11}$$

The two values of u are

$$u = \frac{1}{2} + \frac{c^2}{2K^2} \pm \sqrt{\frac{1}{4} + \frac{c^2}{2K^2} + \frac{c^4}{4K^4} - \frac{c^2}{K^2}\cos^2\theta} \tag{12}$$

In order that the roots may be real, we must have the inequality

$$\frac{c^4}{4K^4} + \frac{c^2}{2K^2} + \frac{1}{4} - \frac{c^2}{K^2}\cos^2\theta > 0$$

Now, identically

$$\frac{c^2}{2K^2} - \frac{c^2}{K^2}\cos^2\theta = \frac{c^2}{2K^2}(1 - 2\cos^2\theta)$$

$$= -\frac{c^2}{2K^2}\cos 2\theta$$

The inequality, therefore, after clearing of fractions, becomes

$$c^4 + K^4 - 2c^2K^2\cos 2\theta > 0$$

and we see that this condition will always be fulfilled.

Imagine a triangle A′O′B′ whose sides are proportional respectively to the squares c^2 and K^2 and which contain between them the angle 2θ.

The first member will be the square of the third side A′B′ homogeneous to the square of a length, and consequently always positive.

One could see the same thing otherwise by noticing that $c^4 + K^4 - 2c^2K^2$ is the square, necessarily positive, of $c^2 - K^2$; and that the introduction of the factor $\cos 2\theta$, numerically inferior to unity, into the negative term cannot modify this result.

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The equation (12) therefore assigns to the unknown quantity u real and consequently positive values. To each one of these values of u correspond two values of $\cos\alpha$,

$$\cos\alpha = \pm \sqrt{u}$$

But the negative root is inadmissible, for $\cos\alpha$ negative defines an angle α comprised between $\frac{\pi}{2}$ and π, and would make of the angle 2α, which is that of the triangle BOA, an angle greater than π. The value π, attributed to the angle 2α, is a limiting value which it is impossible to attain; the chord AB drawn through a given point I could not be inserted in an angle equal to two right angles, whose sides would be the prolongation of each other.

There are therefore two solutions to the problem, since two values of u, real and positive, satisfy equation (12); but there are not four, since the negative determinations of $\cos\alpha$ are wanting.

Sequel to the Solution

When the values of $\cos\alpha$ have been found, Figure 1 can be constructed by drawing the corresponding lines OX, OY, and this completes the solution graphically.

The calculation of x, y, z conducts likewise to the complete solution of the problem.

The value of z is obtained by the equation (6) $z^2 = c^2 - K^2\cos 2\alpha$; then those of x and y by the solution of equations (8)

$$(8) \qquad \begin{cases} x^2 - 2cx\cos(\alpha+\theta) + K^2\cos 2\alpha = 0 \\ y^2 - 2cy\cos(\alpha-\theta) + K^2\cos 2\alpha = 0 \end{cases}$$

Hence

$$z = \sqrt{c^2 - K^2\cos 2\alpha}$$
$$x = c\cos(\alpha+\theta) \pm \sqrt{c^2\cos^2(\alpha+\theta) - K^2\cos 2\alpha}$$
$$y = c\cos(\alpha-\theta) \pm \sqrt{c^2\cos^2(\alpha-\theta) - K^2\cos 2\alpha}$$

The positive value of z alone is kept. As to x and y the two values which the formulae give are all positive if they are real, and that takes for granted the inequality

$$\frac{c^2}{K^2} > \frac{\cos 2\alpha}{\cos^2(\alpha+\theta)}$$

But these two values are not admissible without examination; the unknowns x and y are bound to verify the equation

(5) $$xy = K^2$$

One can therefore admit only those simultaneous values of x and y which satisfy this condition. Any combination taken arbitrarily among the values of x and y will not verify equation (5).

Let x', x'' be the two values of x
y', y'' ,, ,, ,, ,, ,, y

We shall have, by equations (8)

(a) $$\begin{cases} x'\ x'' = K^2 \cos 2a \\ y'\ y'' = K^2 \cos 2a \end{cases}$$

Suppose that the combination $(x'\ y')$ agrees with equation (5) so that

(b) $$x'\ y' = K^2$$

From the two equations (a) by multiplication we have

$$x'\ y'\ x''\ y'' = K^4 \cos^2 2a\ ;$$

and since, by hypothesis, $x'\ y' = K^2$

it follows that $x''\ y'' = K^2 \cos^2 2a$

This shows that the combination (x'', y'') does not satisfy the imposed condition (5), except in the case when $2a = 0$, and this case may be put aside to begin with, since it annuls the angle AOB.

Only those roots of equations (8) should be associated together which verify equation (5).

Knowing the sides of the triangles BOI, IOA, the angles of these triangles may be deduced either by the proportion of the sines, or by any other geometrical means.

Figure 2

If, for example, A′ and B′ be the projections of A and B on OI, then

$$A'I = B'I = c - x\cos(a+\theta)$$
$$= y\cos(a-\theta) - c$$

whence $$\cos\phi = \frac{A'I}{AI} = \frac{c - x\cos(a+\theta)}{z}$$

$$\text{angle IAO} = \pi - (a+\theta+\phi)$$
$$\text{angle IBO} = \phi - a + \theta$$

The area of triangle AOB has for measure

$$\frac{1}{2}xy\sin 2\alpha = \frac{K^2\sin 2\alpha}{2}$$

The altitude OS drawn from O to the base AB of triangle AOB is given by the relation

$$OS = \frac{K^2\sin 2\alpha}{4z}$$

It is also equal to $\quad x\sin A = x\sin(\alpha+\theta+\phi)$

or to $\quad y\sin B = y\sin(\phi-\alpha+\theta)$

whence angle ϕ may be determined if x, y, α, and θ are known.

Particular Case of the Right-Angled Triangle

Figure 3

If angle 2α is right we have

$$\alpha = \frac{\pi}{4}, \qquad u = \cos^2\alpha = \frac{1}{2}$$

The results obtained are verified directly on the figure.

For $\quad x = OA = 2c\cos(\alpha+\theta) = 2c\cos A$

$$y = OB = 2c\cos(\alpha-\theta) = 2c\sin A\,;$$

angles $\alpha+\theta$, $\alpha-\theta$ are complementary

$$z = IB = IA = OI = c$$

Furthermore
$$\begin{aligned} K^2 = xy = 4c^2\sin A\cos A &= 2c^2\sin 2A \\ &= 2c^2\sin 2(\alpha+\theta) \\ &= 2c^2\sin\left(\frac{\pi}{2}+2\theta\right) \\ &= 2c^2\cos 2\theta \end{aligned}$$

The area of triangle AOB is equal to $\frac{1}{2}K^2$

Particular Case of the Isosceles Triangle

In the isosceles triangle defined by the condition $x=y$ we have $\theta=0$, for the mid point I of the base AB is situated on the bisector of angle AOB. We have then

$$x = y = K$$

and equation (11) can be decomposed into two factors

$$u^2 - \left(1+\frac{c^2}{K^2}\right)u + \frac{c^2}{K^2} = (u-1)\left(u-\frac{c^2}{K^2}\right) = 0$$

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Of the two roots $u=1$ and $u=\frac{c^2}{K^2}$, the first may be set aside, since it involves $\alpha=0$ or $\alpha=\pi$; the second gives $c=K\cos\alpha$, a relation readily verified from a figure, as well as the relation $z=c\tan\alpha$.

General Remark

As soon as the angle 2α is determined, if we make the angle θ vary from the value $\theta=0$ to the values $-\alpha$ and $+\alpha$, the straight line AB varies, and envelops a hyperbola which has O for centre and OX, OY for asymptotes; the moveable straight line touches its envelope at its mid point I. The angle ϕ is then the angle formed by the tangent to the curve and the radius vector OI.

If we call r the radius vector expressed in function of the angle θ taken as polar angle, we shall have

$$\tan\phi=\frac{r}{r'}=\frac{rd\theta}{dr}$$

The equation of the curve is

$$r=\frac{a\sin\alpha}{\sqrt{\cos^2\theta-\cos^2\alpha}}$$

Hence, by applying the formula,

$$\tan\phi=\frac{\cos^2\theta-\cos^2\alpha}{\cos\theta\sin\theta}$$

The factor a is the semi-axis of the curve, measured on the bisector OH of the angle of the asymptotes. At this point $\phi=\frac{\pi}{2}$. The asymptotes are given by the equality $\theta=\pm\alpha$.

The quantity a will be expressed in function of the data c and θ_0 by calling θ_0 the particular value of the polar angle furnished by the data of the problem.

Hence $$c=\frac{a\sin\alpha}{\sqrt{\cos^2\theta_0-\cos^2\alpha}}$$

which involves for a the value

$$a=\frac{c\sqrt{\cos^2\theta_0-\cos^2\alpha}}{\sin\alpha}$$

It can be seen that the problem is impossible if $\sin\alpha=0$, for then the semi-axis a of the hyperbola becomes infinite.