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Crossed Products by Semigroups of Endomorphisms and Groups of Partial Automorphisms

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Abstract. We consider a class (A, S, α) of dynamical systems, where S is an Ore semigroup and α is an action such that each α_s is injective and extendible (*i.e.* it extends to a non-unital endomorphism of the multiplier algebra), and has range an ideal of A. We show that there is a partial action on the fixed-point algebra under the canonical coaction of the enveloping group G of S constructed in [15, Proposition 6.1]. It turns out that the full crossed product by this coaction is isomorphic to $A \rtimes_\alpha S$. If the coaction is moreover normal, then the isomorphism can be extended to include the reduced crossed product. We look then at invariant ideals and finally, at examples of systems where our results apply.

Introduction

Crossed products of C^*-algebras by semigroups of endomorphisms have been successfully employed in giving direct, elegant proofs of the uniqueness of C^*-algebras generated by various families of isometries, such as Toeplitz algebras of totally ordered abelian groups [2] and of certain nonabelian groups [15], which include the Toeplitz-Cuntz algebras of [7]. Another variant of crossed product which is different from the classical situation of group actions by automorphisms, involves partial actions by automorphisms, and was introduced by Exel [8] to study circle actions on C^*-algebras. Crossed products by partial actions were treated by McClanahan [21] in the setting of more general groups, and were subsequently characterised from different perspectives by Quigg and Raeburn [26], and Exel, Laca and Quigg [11]. Further, the techniques of partial actions were used by Exel and Laca to construct and study Cuntz-Krieger algebras associated to infinite matrices [10].

Most of the examples of [26] of C^*-algebras which are partial crossed products are also known to be semigroup crossed products; see *e.g.* [5] for the Cuntz algebras $\mathcal{O}_n$ and [15] for the Toeplitz-Cuntz algebras and more general Toeplitz algebras associated by Nica [23] to quasi-lattice ordered groups. The same is true by [15] for the universal C^*-algebra of a quasi-lattice ordered group, which is identified in [11] with a partial crossed product.

We were asked by R. Exel whether the semigroup crossed products of [20] are isomorphic to crossed products by partial actions. It is our goal here to answer this question in the affirmative by realising a certain class of semigroup crossed products as full partial crossed products. The setting of [20] includes many of the important

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examples available at present: the Hecke C^*-algebra of Bost and Connes [4, 16], the generalisations of this in [3, 12] and the two-prime analogue of [19], and certain Hecke C^*-algebras from topological dynamical systems introduced by Brenken [6].

There are two established methods to tackle semigroup crossed products: one is to use direct arguments, based on the universal characterisation of the crossed product as a C^*-algebra generated by covariant representations; the other is to view the semigroup crossed product as a full corner in a unique group crossed product obtained by dilation; *cf.* [14, 24] for the most general case to date, namely that of actions of Ore semigroups. Both methods have their importance. The first one provides, as mentioned in our first paragraph, an effective framework for studying Toeplitz algebras. Further, it was used by Laca and Raeburn [16] and Brenken [6] to identify the Bost-Connes Hecke C^*-algebra as a semigroup crossed product, and afterwards by Laca in his characterisation of phase transitions on certain semigroup crossed products [13], which generalises the remarkable result of Bost and Connes from [4]. Dilation theory was used by Murphy to study nuclearity and simplicity of crossed products by abelian semigroups [22], and by Laca and Raeburn to compute the primitive ideal space of the Bost-Connes Hecke C^*-algebra [17].

To prove Theorem 2.1, which is our main result, we use the universal characterisations of semigroup and partial crossed products, which are also invoked in [11], although there the arguments are of a different nature due to the specific internal structure of the C^*-algebras. Our approach to partial actions and their associated crossed products is based on [26], where the equivalence of the underlying concepts with the similar definitions employed in [21, 11] is established. The features of the systems of [20] together with the ingredients in the main result of [26], which characterises C^*-algebras that are (reduced) partial crossed products, will lie at the base of our choice of semigroup crossed products and their subsequent realisation as partial crossed products.

We recall in Section 1 the basic concepts needed to define semigroup crossed products and partial crossed products, and we introduce our class (A, S, α) of semigroup dynamical systems. We assume that S is an Ore semigroup, for which therefore there is an enveloping group G, such that $G = S^{-1}S$, and G is uniquely determined up to canonical isomorphism. In Section 2 we construct a partial action of G on the fixed point algebra of $A \rtimes S$ under the canonical coaction defined in [15, Proposition 6.1]. For this we use an idea from the proof of [26, Theorem 4.1]. Theorem 2.1 shows that the semigroup crossed product $A \rtimes_\alpha S$ is isomorphic to the full crossed product of the fixed point algebra by the partial action of G. Section 3 specialises to some semigroup systems for which the fixed point algebra will be isomorphic to A. For such systems we compare the ideals of A which are extendibly α-invariant, and the ideals that are invariant under the resulting partial action. The extendibly invariant ideals (called so by Adji, see for instance [1]), preserve short exact sequences under the formation of semigroup crossed products (*cf.* [1] in the case of positive cones of totally ordered abelian groups and [18] for Ore semigroups). Here we show that the slightly intricate condition of extendible invariance is the same as invariance in the sense of partial actions (defined for instance in [11, Section 2]). The last section is devoted to examples.

After this paper was submitted, the author's attention was drawn to the results in [9], from which our Theorem 2.1 can also be deduced.

1 Semigroup Systems and Partial Systems

We recall (see *e.g.* [15]) that a semigroup dynamical system (A, S, α) consists of a C^*-algebra A, a semigroup S, and an action α of S by endomorphisms of A. A pair (π, V), where π is a non-degenerate representation of A on a Hilbert space and V is a representation of S by isometries on the same space, is called a *covariant representation* if

$$\pi\big(\alpha_s(a)\big) = V_s\pi(a)V_s^*, \quad \text{for } a \text{ in } A \text{ and } s \text{ in } S.$$

The crossed product, denoted $A \rtimes_\alpha S$, is generated as a C^*-algebra by a covariant representation (i_A, i_S), where $i_A\colon A \to A \rtimes_\alpha S$ preserves approximate units and $i_S\colon S \to M(A \rtimes_\alpha S)$ is a semigroup homomorphism. This pair is universal, in the sense that to any covariant representation (π, V) of (A, S, α) corresponds a non-degenerate representation $\pi \times V$ of the crossed product, such that $(\pi \times V) \circ i_A = \pi$, and $\overline{\pi \times V} \circ i_S = V$.

Let S be a cancellative semigroup which is right-reversible, in the sense that for any s, t in S we have $Ss \cap St \neq \varnothing$. Such S is called an *Ore semigroup* (see [14] for examples relevant to C^*-algebras). There is an enveloping group G such that $G = S^{-1}S$, which is unique up to canonical isomorphism. We shall assume that $S \cap S^{-1} = \{e\}$, where e is the unit element of G.

The semigroup systems (A, S, α) that are of interest to us have the properties:

(i) S is an Ore semigroup with enveloping group G.
(ii) The action $\alpha\colon S \to \mathrm{End}(A)$ is by injective endomorphisms;
(iii) For each s in S, α_s is extendible, *i.e.* it extends uniquely to a strictly continuous homomorphism $\overline{\alpha_s} : M(A) \to M(A)$, which takes the identity $1_{M(A)}$ into a proper projection.
(iv) $\alpha_s(A)$ is a non-zero closed ideal of A, for every $s \in S$, and any two such ideals have non-zero intersection.

The assumption (iii) guarantees that the pair $(\bar{\pi}, V)$ is covariant for $\big(M(A), S, \bar{\alpha}\big)$ whenever (π, V) is covariant (*cf.* [1] or [18, Remark 1.2]). By [24, Theorem 4.5], (i)—(iii) imply that there is a crossed product $(A \rtimes_\alpha S, i_A, i_S)$, such that i_A is injective and

$$A \rtimes_\alpha S = \overline{\mathrm{span}}\{i_S(s)^* i_A(a) i_S(t) \mid s, t \in S, a \in A\}. \tag{1.1}$$

If G is a discrete group, let i_G denote its embedding as canonical unitaries in $C^*(G)$. Recall from [27, 25] that a coaction δ of G on a C^*-algebra B is an injective non-degenerate homomorphism $\delta\colon B \to M\big(B \otimes_{\max} C^*(G)\big)$ satisfying the coaction identity

$$(\delta \otimes \mathrm{id}_{C^*(G)}) \circ \delta = (\mathrm{id}_B \otimes \delta_G) \circ \delta,$$

where $\delta_G\colon C^*(G) \to M\big(C^*(G) \otimes_{\max} C^*(G)\big)$ is induced by $g \to i_G(g) \otimes i_G(g)$. The existence of a canonical coaction on a crossed product $A \rtimes S$ is proved in [15] when A is unital, but holds in general (by a similar proof), as follows.

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Proposition 1.1 ([15, Proposition 6.1]) *Given an arbitrary semigroup system (A, S, α) such that the enveloping group G of S is discrete, there is an injective coaction*

$$\delta\colon A \rtimes_\alpha S \to M\big((A \rtimes_\alpha S) \otimes_{\max} C^*(G)\big),$$

such that $\delta \circ i_A(a) = i_A(a) \otimes 1$, and $\delta \circ i_S(s) = i_S(s) \otimes i_G(s)$.

Let $B := A \rtimes_\alpha S$, and set like in [26] $B_g = \{b \in B \mid \delta(b) = b \otimes g\}$ for $g \in G$ (we drop the i_G from the notation); these are the spectral subspaces. Observe that

$$B_e = \{b \in B \mid \delta(b) = b \otimes e\}.$$

is the fixed point algebra of the coaction.

We recall from [26, Definition 1.1] that a partial action of a discrete group G on a C^*-algebra C consists of a pair $(\{D_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ of closed ideals of C and isomorphisms $\beta_g\colon D_{g^{-1}} \to D_g$, such that $D_e = C$ and β_{gh} extends $\beta_g\beta_h$ on the domain $\beta_{h^{-1}}(D_{g^{-1}})$ of the product, for all $g, h \in G$.

Definition 1.2 ([26, Definition 1.7 and Definition 1.10]) A partial representation of G on a Hilbert space $\mathcal{H}$ is a map $u\colon G \to \mathbb{B}(\mathcal{H})$ such that the u_g are partial isometries with commuting range projections, $u_e u_e^* = I$, $u_g^* u_g = u_{g^{-1}} u_{g^{-1}}^*$ and $u_g u_h \preceq u_{gh}$ for all $g, h \in G$, where two partial isometries u, v on $\mathcal{H}$ satisfy $u \preceq v$ precisely when $uu^* = uv^*$

A covariant representation of a partial dynamical system (C, G, β) is a pair (π, u) consisting of a non-degenerate representation π of C and a partial representation u of G on the same Hilbert space such that for all g in G and c in $D_{g^{-1}}$,

$$u_g u_g^* = \pi(p_g) \quad \text{and} \tag{1.2}$$

$$\pi\big(\beta_g(c)\big) = u_g \pi(c) u_g^*, \tag{1.3}$$

where p_g denotes the projection in C^{**} which is the identity of D_g^{**}.

Given a partial action β of a discrete group G on a C^*-algebra C, Quigg and Raeburn [26, Section 3] construct a full crossed product, denoted by $C \rtimes_\beta G$, as the enveloping C^*-algebra of the $*$-algebra spanned by the functions $F(c, g) \in l^1(G, C)$ defined by $F(c, g)(h) = c$ if $g = h$ and $F(c, g)(h) = 0$ when $h \neq g$. Consequently (but non-trivially), $C \rtimes_\beta G$ is generated by a universal covariant representation (ι, m) in the double dual $(C \rtimes_\beta G)^{**}$, in the sense that $C \rtimes_\beta G$ is the closure of span $\{\iota(c)m_g\}$, and for an arbitrary covariant representation (π, u), there is a representation $\pi \times u$ of $C \rtimes_\beta G$ satisfying

$$(\pi \times u)\big(\iota(c)m_g\big) = \pi(c)u_g, \quad \text{for all } g \text{ in } G \text{ and } c \text{ in } D_g.$$

2 The Main Results

Our main theorem identifies the semigroup crossed product from (1.1) with a full partial crossed product of the fixed-point algebra B_e.

Theorem 2.1 *If B is the crossed product of a system (A, S, α) which satisfies (i)—(iv), then there exists a partial action β of G on B_e such that*

$$B_e \rtimes_\beta G \cong B. \tag{2.1}$$

Remark 2.2 Note that when $B_e \cong A$, we obtain an isomorphism $A \rtimes_\alpha S \cong A \rtimes_\beta G$.

The proof of the theorem will proceed in several steps: first, we construct a partial representation m of G in B^{**} with the properties described in [26, Theorem 4.1(ii)]. Second, we produce out of m a partial action β on B_e, by following the idea of the proof of the implication (ii) $\Rightarrow$ (i) from [26, Theorem 4.1]. Finally, we derive (2.1) via an application of [26, Proposition 3.1].

Recall from [26] that $D_g = B_g B_g^* = \overline{\operatorname{span}}\{bc^* \mid b, c \in B_g\}$ are ideals of B_e such that, if p_g denotes the identity of D_g^{**} regarded as a projection in B^{**}, then the multiplier bimodule of B_g is

$$M(B_g) = \{b \in p_g B^{**} p_{g^{-1}} \mid D_g b \cup b D_{g^{-1}} \subset B_g\}, \quad \text{for } g \text{ in } G. \tag{2.2}$$

The assumptions (i)—(iv) on (A, S, α) imply the following result, which will be used repeatedly in the sequel.

Lemma 2.3 *The unit of $\alpha_s(A)^{**}$ is the projection $\overline{\alpha_s}(1_{M(A)})$ in $M(A)$, and belongs to the centre of A^{**}. In particular,*

$$\overline{\alpha_s}(1_{M(A)})\overline{\alpha_t}(1_{M(A)}) = \overline{\alpha_t}(1_{M(A)})\overline{\alpha_s}(1_{M(A)}) \quad \textit{for } s, t \textit{ in } S. \tag{2.3}$$

Proof Let (u_λ) be an approximate unit for A. Then an easy calculation shows that $\big(\alpha_s(u_\lambda)\big)$ is an approximate unit for $\alpha_s(A)$. Since α_s is extendible, the net $\alpha_s(u_\lambda)$ converges in the strict topology, thus in the strong*-topology, to the projection $\overline{\alpha_s}(1_{M(A)})$ in $M(A)$. Hence this projection is the weak*-limit of an approximate unit for $\alpha_s(A)$, which means that it is the unit of $\alpha_s(A)^{**}$ regarded in A^{**}. It is central in A^{**} because $\alpha_s(A)$ is an ideal. The equality (2.3) follows then immediately. ■

Lemma 2.4 *For g in G we have:*

(i) $B_g = \overline{\operatorname{span}}\{i_S(s)^* i_A(a) i_S(t) \mid s^{-1}t = g, s, t \in S, a \in A\}$;
(ii) $D_g = \overline{\operatorname{span}}\big\{ i_S(w)^* i_A\big(\overline{\alpha_z}(1)a\big)\, i_S(w) \mid w^{-1}z = g, w, z \in S, a \in A\big\}$;
(iii) *If $g = w^{-1}z$, the projection p_g has the form*

$$i_S(w)^* \overline{i_A}\big(\overline{\alpha_z}(1)\big)\, i_S(w).$$

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Proof The assertion (i) is immediate from (1.1) and the defining properties of δ. Take elements $i_S(s)^* i_A(a) i_S(t)$ in B_g and $i_S(p)^* i_A(b) i_S(r)$ in B_g^*. Thus $g = s^{-1}t = r^{-1}p$, and we can find $u, v \in S$ such that $ut = vp =: z \in S$. Then $us = vr =: w \in S$, and hence by Lemma 2.3

$$\begin{aligned} i_S(s)^* i_A(a) i_S(t) i_S(p)^* i_A(b) i_S(r) \\ &= i_S(s)^* i_A(a) i_S(u)^* i_S(ut) i_S(vp)^* i_S(v) i_A(b) i_S(r) \\ &= i_S(w)^* i_A\big(\overline{\alpha_z}(1)\alpha_u(a)\alpha_v(b)\big)\, i_S(w). \end{aligned}$$

The assertion (ii) follows.

Towards proving (iii), we show first that two writings of g, as for example $g = s^{-1}t = w^{-1}z$, give rise to the same element of $M(B)$. Indeed, let $e, f \in S$ such that $ew = fs$. Then $ez = ft$, and we have by covariance of $(\overline{i_A}, i_S)$ that

$$\begin{aligned} i_S(s)^* \overline{i_A}\big(\overline{\alpha_t}(1)\big)\, i_S(s) &= i_S(s)^* i_S(f)^* i_S(f) i_S(t) i_S(t)^* i_S(f)^* i_S(f) i_S(s) \\ &= i_S(fs)^* i_S(ft) i_S(ft)^* i_S(fs) \\ &= i_S(ew)^* i_S(ez) i_S(ez)^* i_S(ew) \\ &= i_S(w)^* \overline{i_A}\big(\overline{\alpha_z}(1)\big)\, i_S(w), \end{aligned}$$

as claimed. Let now $d = i_S(w)^* i_A(\overline{\alpha_z}(1)a) i_S(w)$ be a typical element of the spanning set of D_g. Then (2.3) and covariance of $(\overline{i_A}, i_S)$ imply that

$$\begin{aligned} p_g d &= i_S(w)^* \overline{i_A}\big(\overline{\alpha_z}(1)\overline{\alpha_w}(1)\overline{\alpha_z}(1)a\big)\, i_S(w) \\ &= i_S(w)^* i_A\big(\overline{\alpha_z}(1)a\big)\, i_S(w) = d. \end{aligned} \tag{2.4}$$

A similar computation works when d is a finite combination of elements from the spanning set of D_g, because we can change the expression of p_g accordingly, by the first part of the proof of (iii). Hence p_g acts as a unit on D_g, and we conclude that $p_g = 1_{M(D_g)}$. ∎

Proposition 2.5 *The formula*

$$m_{s^{-1}t} = i_S(s)^* i_S(t) \quad \text{for } s, t \in S, \tag{2.5}$$

defines a partial representation $m\colon G \to B^{**}$, *such that*

$$m_g \in M(B_g) \text{ and } m_g m_g^* = p_g \text{ for all } g \text{ in } G.$$

Proof We begin by proving that (2.5) defines a partial representation. First, to see that it is independent of the expression for $g \in S^{-1}S$, suppose that $g = s^{-1}t = r^{-1}p$, and take u, v in S such that $ur = vs$. Then $vt = up$, and

$$i_S(s)^* i_S(t) = i_S(vs)^* i_S(vt) = i_S(ur)^* i_S(up) = i_S(r)^* i_S(p),$$

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showing that m_g is independent of the decomposition of g in $S^{-1}S$. Covariance of $(\overline{i_A}, i_S)$ and (2.3) show that $m_g^* m_g$ is a projection, so m_g is a partial isometry. To show that the ranges of two such partial isometries commute, suppose $g = s^{-1}t$, $h = r^{-1}p$, and take u, v in S such that $ur = vs := w \in S$. Then

$$\begin{aligned} m_g m_g^* m_h m_h^* &= i_S(s)^* i_S(t) i_S(t)^* i_S(s) i_S(r)^* i_S(p) i_S(p)^* i_S(r) \\ &= i_S(vs)^* \overline{i_A}\big(\overline{\alpha_{vt}}(1)\overline{\alpha_w}(1)\overline{\alpha_{up}}(1)\big)\, i_S(ur) \\ &= i_S(w)^* \overline{i_A}\big(\overline{\alpha_{up}}(1)\overline{\alpha_w}(1)\overline{\alpha_{vt}}(1)\big)\, i_S(w) \\ &= m_h m_h^* m_g m_g^*. \end{aligned}$$

That $m_e m_e^* = 1$ is immediate, and so is $m_{g^{-1}} m_{g^{-1}} = m_g^* m_g$. It remains to prove that

$$m_g m_h m_h^* m_g^* = m_g m_h m_{gh} \quad \text{for } g, h \text{ in } G. \tag{2.6}$$

Again with $g = s^{-1}t$, $h = r^{-1}p$, take y, z in S such that $yt = zr =: w \in S$. By covariance of $(\overline{i_A}, i_S)$, the left hand side of (2.6) becomes

$$i_S(ys)^* \overline{i_A}\big(\overline{\alpha_w}(1)\overline{\alpha_{zp}}(1)\overline{\alpha_w}(1)\big)\, i_S(ys),$$

which is the same as the right hand side $i_S(ys)^* \overline{i_A}\big(\overline{\alpha_w}(1)\overline{\alpha_{zp}}(1)\big)\, i_S(ys)$, by Lemma 2.3.

To establish the second half of the proposition, note that

$$m_g m_g^* = i_S(s)^* i_S(t) i_S(t)^* i_S(s) = p_g,$$

for any $g = s^{-1}t$ in G. By (2.2), it remains to prove that

$$m_g \in p_g B^{**} p_{g^{-1}} \quad \text{and} \quad D_g m_g \cup m_g D_{g^{-1}} \subset B_g,$$

for every g in G. For $g = s^{-1}t$, the first inclusion follows from

$$\begin{aligned} m_g &= i_S(s)^* i_S(t) = \overline{i_A}(1_{M(A)}) i_S(s)^* i_S(t) \overline{i_A}(1_{M(A)}) \\ &= i_S(s)^* \overline{i_A}\big(\overline{\alpha_s}(1)\overline{\alpha_t}(1)\big)\, i_S(t) \\ &= p_g i_S(s)^* i_S(t) p_{g^{-1}} \in p_g M(B) p_{g^{-1}}. \end{aligned}$$

Let $d = i_S(w)^* i_A(\overline{\alpha_z}(1)a) i_S(w) \in D_g$, where $w^{-1}z = g$, $a \in A$. Then (2.3) shows that

$$\begin{aligned} d m_g &= i_S(w)^* i_A\big(\overline{\alpha_z}(1)a\big)\, i_S(w) i_S(w)^* i_S(z) \\ &= i_S(w)^* i_A\big(\overline{\alpha_z}(1)a\overline{\alpha_w}(1)\big)\, i_S(z) \\ &= i_S(w)^* i_A(a) i_S(z) \in B_g. \end{aligned}$$

The same line of argument applies when d is a finite combination of elements from the spanning set of D_g, thus yielding the inclusion $D_g m_g \subset B_g$. The other inclusion is proved in a similar manner. ■

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Proposition 2.6 *There is a partial action β on B_e, such that $\beta_g\colon D_{g^{-1}} \to D_g$ is an isomorphism, for each g in G.*

Proof Since the partial representation m satisfies the conditions of [26, Theorem 4.1(ii)], we know from the proof of that theorem that $\operatorname{Ad} m_g\colon D_{g^{-1}} \to D_g$ is a partial action of G on B_e. ∎

Proposition 2.7 *If (π, V) is a covariant representation of (A, S, α), then $(\pi \times V, u)$, where $u_{s^{-1}t} := V_s^* V_t$, is a covariant representation of (B_e, G, β).*

Conversely, the restriction $(\rho \circ i_A, v|_S)$ of a covariant representation (ρ, v) of (B_e, G, β) is a covariant representation of (A, S, α).

Proof Suppose that (π, V) is a covariant representation of (A, S, α). Then $\pi \times V$ is a representation of B such that $(\pi \times V) \circ i_A = \pi$ and $\overline{\pi \times V} \circ i_S = V$. The same proof as that of (2.5) yields that u is a partial representation of G on B_e. That $(\pi \times V, u)$ is covariant requires $(\pi \times V)(p_g) = u_g u_g^*$ and $(\pi \times V)\big(\beta_g(b)\big) = u_g(\pi \times V)(b)u_g^*$. The first identity is immediate from covariance of (π, V) and Lemma 2.4. The second identity is straightforward for elements b of the form $i_S(w)^* i_A(a) i_S(w)$, where $a \in A$, $w \in S$, and can be easily extended to finite linear combinations.

Suppose now that a covariant representation (ρ, v) of (B_e, G, β) is given. To see that $v|_S$ is an isometric representation of S, note that when $g \in S$, $D_{g^{-1}} = B_e$. Thus for s in S, we have

$$v_s^* v_s = \bar{\rho}(p_{g^{-1}}) = \bar{\rho}(1_{M(B_e)}) = 1_{\mathbb{B}(H_\rho)},$$

which shows that v_s is an isometry. Since $v_s v_t \preceq v_{st}$ for any s, t in S, $v_s v_t = v_{st}$ on $(v_s v_t)^* v_s v_t(\mathcal{H}_\rho)$, which equals $\mathcal{H}_\rho$ because v_s and v_t are isometries. Thus $v|_S$ is an isometric representation of S on $\mathcal{H}_\rho$. Covariance follows since

$$\begin{aligned}(\rho \circ i_A)\big(\alpha_s(a)\big) &= \rho\big(i_S(s) i_A(a) i_S(s)^*\big) = \rho\Big(\beta_s\big(i_A(a)\big)\Big) \\ &= v_s \rho\big(i_A(a)\big) v_s^*.\end{aligned}$$ ∎

Letting ι denote the embedding $B_e \hookrightarrow B$, we notice that (ι, m) is a covariant representation of (B_e, G, β). We have $C^*(\iota, m) = B$ because, as pointed out in the proof of (ii) ⇒ (i) of [26, Theorem 4.1], $B = \overline{\sum_g B_g}$. Thus there is a surjective homomorphism

$$\iota \times m\colon B_e \rtimes_\beta G \to B. \tag{2.7}$$

Proof of Theorem 2.1 We aim to prove that $\iota \times m$ from (2.7) is faithful.

Towards this end, we wish to apply [26, Proposition 3.1], which says that $\iota \times m$ is faithful if for any covariant representation (ρ, v) of (B_e, G, β) there is a homomorphism $\Psi\colon C^*(\iota, m) \to C^*(\rho, v)$, such that

$$\Psi\big(\iota(b) m_g\big) = \rho(b) v_g, \quad \text{for } b \in D_g. \tag{2.8}$$

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Pick such (ρ, v). Thus we know from Proposition 2.7 that $(\rho \circ i_A, v|_S)$ is covariant for (A, S, α). Then the representation $\Psi := (\rho \circ i_A) \times v|_S$ of B restricted to A and S equals $\rho \circ i_A$ and v, and maps $B = C^*(\iota, m)$ into $C^*(\rho, v)$.

To prove (2.8), suppose to begin with that $b = i_S(w)^* i_A(\overline{\alpha_z}(1)a) i_S(w)$, where $g = w^{-1}z$. Then the left hand side becomes

$$\begin{aligned}\Psi\big(\iota(b)m_g\big) &= (\rho \circ i_A) \times v|_S\Big(i_S(w)^* i_A\big(\overline{\alpha_z}(1)a\big)\, i_S(w) i_S(w)^* i_S(z)\Big)\\ &= v_w^* \rho \circ i_A\big(\overline{\alpha_w}(1) a \overline{\alpha_z}(1)\big)\, v_z\\ &= v_w^* \rho\big(i_A(a)\big)\, v_z.\end{aligned}$$

Since v is a partial representation of G and an isometric representation of S, we have for $w, z \in S$ that

$$\begin{aligned}v_{w^{-1}z} &= v_w^* v_w v_{w^{-1}z} = v_{w^{-1}} v_w v_{w^{-1}z}\\ &= v_{w^{-1}} v_z = v_w^* v_z.\end{aligned}$$

Therefore the right hand side of (2.8) becomes

$$\begin{aligned}\rho(b) v_g &= \rho\bigg(\beta_{w^{-1}}\Big(i_A\big(\overline{\alpha_z}(1)a\big)\Big)\bigg)\, v_g\\ &= v_w^* \rho\Big(i_A\big(\overline{\alpha_z}(1)a\big)\Big)\, v_w v_g\\ &= v_w^* \rho\Big(i_A\big(\overline{\alpha_z}(1)a\big)\Big)\, v_w v_{w^{-1}z}\\ &= v_w^* \rho\Big(i_A\big(\overline{\alpha_z}(1)a\big)\Big)\, v_w v_w^* v_z\\ &= v_w^* \rho\Big(i_A\big(\overline{\alpha_w}(1) a \overline{\alpha_z}(1)\big)\Big)\, v_z\\ &= v_w^* \rho\big(i_A(a)\big)\, v_z,\end{aligned}$$

which is the same as the left hand side. A similar manipulation works for an arbitrary element b of span $\big\{ i_S(w)^* i_A\big(\overline{\alpha_z}(1)a\big)\, i_S(w) \mid w^{-1}z = g, a \in A \big\}$.

We conclude that $\iota \times m$ induces an isomorphism of $B_e \rtimes_\beta G$ onto $B = A \rtimes_\alpha S$. ∎

Given a partial system (C, G, β), Quigg and Raeburn [26, Section 3] give an alternative description of a regular representation (π^r, u^r) of $C \rtimes_\beta G$ on $\mathcal{H} \otimes l^2(G)$ constructed in [21], and subsequently prove that $C^*(\pi^r, u^r)$ is independent of the choice of faithful representation of C, for which reason they define the reduced crossed product $C \rtimes_{\beta,r} G$ as $C^*(\pi^r, u^r)$. A coaction δ on a C^*-algebra B is normal, [25, 26], if $(\mathrm{id}_B \otimes \lambda) \circ \delta$ is faithful, where λ is the left regular representation of $C^*(G)$. Using this notion, we are able to tell when our Theorem 2.1 can be obtained in terms of reduced partial crossed products.

Corollary 2.8 *In the setting of Theorem 2.1, if δ is moreover normal, then*

$$B_e \rtimes_\beta G \cong B_e \rtimes_{\beta,r} G.$$

Proof It is part of [26, Theorem 4.1] that the partial action $\beta_g = \operatorname{Ad} m_g$ coming from the partial representation m with the properties described in Proposition 2.5 gives rise to an isomorphism $B \cong B_e \rtimes_{\beta,r} G$. Applying Theorem 2.1 gives the claim. ■

3 Applications

In this section we assume that the semigroup system (A, S, α) with the properties (i)—(iv) moreover satisfies that A has an identity 1, and there is an action $\alpha'\colon S \to \operatorname{End}(A)$, such that $\alpha'_s \circ \alpha_s = \mathrm{id}$ and $\alpha_s \circ \alpha'_s$ is multiplication by the projection $\alpha_s(1)$.

In this situation we show that $B_e \cong A$, and we compare the invariant ideals of (A, S, α) and (A, G, β). In the final section we will present a class of semigroup crossed products for which the present considerations apply.

Proposition 3.1

1) *Suppose that (A, S, α) is a system as described in the beginning of this section. Then i_A induces an isomorphism*
$$A \cong B_e.$$
2) *For $g = s^{-1}t$, denote by $\tilde{D}_g$ the (unique) ideal of A such that $\alpha_s(\tilde{D}_g) = \alpha_s(A) \cap \alpha_t(A)$. Then*
$$D_g \cong i_A(\tilde{D}_g), \quad \textit{for } g \textit{ in } G,$$
and the partial action β of Theorem 2.1 satisfies the identity

$$\beta_g \circ i_A = i_A \circ \alpha_s^{-1} \circ \alpha_t \text{ on } \tilde{D}_g, \quad \text{when } g = s^{-1}t. \tag{3.1}$$

Proof Take a typical element $i_S(s)^* i_A(a) i_S(s)$ of the spanning set of B_e, where $s \in S$ and $a \in A$. Then

$$\begin{aligned} i_S(s)^* i_A(a) i_S(s) &= i_S(s)^* i_S(s) i_S(s)^* i_A(a) i_S(s) \\ &= i_S(s)^* i_A(\alpha_s(1)a) i_S(s) \\ &= i_S(s)^* i_A\big(\alpha_s \circ \alpha'_s(a)\big)\, i_S(s) \\ &= i_A\big(\alpha'_s(a)\big), \end{aligned}$$

which is a typical element of $i_A(A)$, because α'_s is surjective, for each s. Since i_A is faithful (as noted just before (1.1)), it induces an isomorphism $B_e \cong A$, as claimed.

A straightforward computation shows that a spanning element of D_g of the form $i_S(s)^* i_A(\alpha_t(1)a) i_S(s)$, with $a \in A$, $s^{-1}t = g$, becomes $i_A(b)$, where b is in $\tilde{D}_g$ such that

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$\alpha_s(b) = \alpha_t(1)a\alpha_s(1)$. Hence $D_g \cong i_A(\tilde{D}_g)$. Pick thus an element $i_A(a)$ of $D_{g^{-1}}$, where $a \in \tilde{D}_{g^{-1}}$. With $g = s^{-1}t$, it follows that $\alpha_t(a) \in \alpha_s(A) \cap \alpha_t(A) = \alpha_s(\tilde{D}_g)$, so there is a b in $\tilde{D}_g$ such that $\alpha_t(a) = \alpha_s(b)$. Hence

$$\begin{aligned}\beta_g\big(i_A(a)\big) &= m_g i_A(a) m_g^* = i_S(s)^* i_S(t) i_A(a) i_S(t)^* i_S(s)\\ &= i_S(s)^* i_A\big(\alpha_t(a)\big)\, i_S(s)\\ &= i_A(b) = i_A\big(\alpha_s^{-1} \circ \alpha_t(a)\big),\end{aligned}$$

proving (3.1). ∎

We showed in [18] that an ideal J of a system (A, S, α) where S is Ore and the endomorphisms are extendible induces an ideal of $A \rtimes_\alpha S$ if a special form of invariance holds. This condition, called extendibly α-invariance of J (*cf.* [1, 18]), says that for each s in S, $\alpha_s(J) \subset J$ and $\alpha_s(u_\lambda)$ converges strictly in $M(J)$ to $\bar{\psi}\big(\bar{\alpha}_s(1_{M(A)})\big)$, where $(u_\lambda)_\lambda$ is an approximate unit for J, and $\psi\colon A \to M(J)$ is the canonical homomorphism. Recall also that an ideal I of a partial system (C, G, β) is invariant under β if $\beta_g(I \cap D_{g^{-1}}) \subset I$ for all g in G (*cf.* [11]). For a system (A, S, α) like in Proposition 3.1, we clarify the relation between the somewhat involved notion of an extendibly α-invariant ideal and the rather more familiar looking condition of β-invariance of $i_A(J)$.

Lemma 3.2 *Given a system (A, S, α) like in Proposition 3.1, an ideal J of A is extendibly α-invariant if and only if $\alpha_s(J) = \alpha_s(A) \cap J$, for all s in S.*

Proof Suppose to begin with that J is extendibly α-invariant. Thus $\alpha_s(J) \subset J$ and $1_{M(\alpha_s(J))} = \psi\big(\alpha_s(1)\big)$ for each s, and therefore

$$\alpha_s(J) \subset \psi\big(\alpha_s(A)\big) \cdot J = \alpha_s(A) \cap J.$$

To prove the reverse inclusion, we represent A on Hilbert space, and show that any state ω on A annihilating $\alpha_s(J)$ also annihilates $\psi\big(\alpha_s(A)\big)\, J$. Pick such ω. We may assume that there is ξ in H such that $\omega(a) = (a\xi|\xi)$, for all a in A. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for J. Then $\big(\alpha_s(u_\lambda)\big)_{\lambda\in\Lambda}$ is an approximate unit for $\alpha_s(J)$ and the assumption on ω implies that $\omega\big(\alpha_s(u_\lambda)\big) = 0$, from which by passing to weak*-limit we obtain that $1_{M(\alpha_s(J))}\xi = 0$. Hence $\psi\big(\alpha_s(1)\big)\,\xi = 0$, and for $a \in A$, $j \in J$, we have

$$\begin{aligned}\omega\Big(\psi\big(\alpha_s(a)\big)\, j\Big) &= \Big(\psi\big(\alpha_s(a)\big)\, j\xi|\xi\Big) = \big(\alpha_s(a) j\xi|\xi\big)\\ &= \big(\alpha_s(1)\alpha_s(a) j\xi|\xi\big) = \Big(\psi\big(\alpha_s(1)\big)\,\big(\alpha_s(a) j\big)\,\xi|\xi\Big)\\ &= \Big(\alpha_s(a) j\xi|\psi\big(\alpha_s(1)\big)\,\xi\Big) = 0.\end{aligned}$$

Thus $\omega\Big(\psi\big(\alpha_a(A)\big)\, J\Big) = 0$, as claimed.

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Conversely, the hypothesis immediately implies that $\alpha_s(J) \subset J$, which means that J is α-invariant, and $\alpha_s(J) = \alpha_s(A)J$. We deduce from this last identity that the unit of $M\big(\alpha_s(J)\big)$ coincides with the unit of $M\Big(\psi\big(\alpha_s(A)\big)\,J\Big)$, which an easy computation proves to be $\psi\big(\alpha_s(1)\big)$. Since the unit of $M\big(\alpha_s(J)\big)$ is the strict limit of any approximate unit for $\alpha_s(J)$, both conditions required of J in order for it to be extendibly invariant are fulfilled. The proof of the lemma is complete. ■

Proposition 3.3

1) *An ideal J of A is extendibly α-invariant if and only if $i_A(J)$ is β-invariant.*
2) *Moreover, if any of these forms for invariance holds, then*

$$J \rtimes_\alpha S \cong i_A(J) \rtimes_\beta G. \tag{3.2}$$

Proof Towards proving 1), assume first that $i_A(J)$ is β-invariant. Thus for any g in G,

$$\beta_g(i_A(J) \cap D_{g^{-1}}) \subset i_A(J).$$

By Proposition 3.1 and injectivity of i_A, this is the same as

$$\alpha_s^{-1} \circ \alpha_t(J \cap \tilde{D}_{t^{-1}s}) \subset J, \quad \text{for arbitrary } s, t \text{ in } S. \tag{3.3}$$

Notice that $\tilde{D}_{t^{-1}} = A$ and $\tilde{D}_s = \alpha_s(A)$ when $s, t \in S$. Hence inserting $s = e$ and $t = e$ in (3.3) will imply that $\alpha_t(J) \subset J$ and $\alpha_s^{-1}\big(J \cap \alpha_s(A)\big) \subset A$, for any $s, t \in S$. Therefore

$$\alpha_s(J) = J \cap \alpha_s(A) \quad \text{for any } s \text{ in } S, \tag{3.4}$$

which by Lemma 3.2 shows precisely that J is extendibly α-invariant.

Conversely, assume that J is extendibly α-invariant. We must show that (3.3) holds for any $g = s^{-1}t \in G$. It suffices to show that

$$\alpha_s^{-1}\big(\alpha_t(J) \cap \alpha_s(A) \cap \alpha_t(A)\big) \subset J,$$

or, equivalently, that

$$\alpha_t(J) \cap \alpha_s(A) \cap \alpha_t(A) \subset \alpha_s(J)$$

for arbitrary s, t in S. But this follows from the identity $\alpha_r(J) = \alpha_r(A) \cap J$ for $r \in S$, which comes from Lemma 3.2.

To prove 2), recall from the proof of Theorem 2.1 that

$$\iota \times m\colon i_A(J) \rtimes_\beta G \to B$$

is an injective homomorphism. Hence it will induce an isomorphism onto its image. It suffices to prove that this image is $J \rtimes_\alpha S$. Since $i_A(J)$ is β-invariant, $i_A(J) \rtimes_\beta G$ is the ideal

$$\overline{\operatorname{span}}\{\iota(b)m_g \mid g \in G, b \in D_g \cap i_A(J)\}$$

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of $B_e \rtimes_\beta G$ (see [11, Proposition 3.1]). Recalling the definition (2.5) of m and the isomorphism $B_e \cong i_A(A)$ provided by Proposition 3.1, we deduce that $\iota \times m\big(i_A(J) \rtimes_\beta G\big)$ equals

$$\begin{aligned}\overline{\operatorname{span}}\{i_A(a) i_S(s)^* i_S(t) \mid s,t \in S, a \in J \cap \tilde{D}_{s^{-1}t}\}\\ &= \overline{\operatorname{span}}\big\{ i_S(s)^* i_A\big(\alpha_s(a)\big)\, i_S(t) \mid s,t \in S, a \in J \cap \tilde{D}_{s^{-1}t}\big\}\\ &= \overline{\operatorname{span}}\{i_S(s)^* i_A(f) i_S(t) \mid s,t \in S, f \in \alpha_s(J \cap \tilde{D}_{s^{-1}t})\}.\end{aligned}$$

Note that extendibly invariance yields

$$\alpha_s(J \cap \tilde{D}_{s^{-1}t}) = J \cap \alpha_s(\tilde{D}_{s^{-1}t}) = \alpha_s(A) \cap \alpha_t(A) \cap J,$$

for arbitrary $s,t \in S$. On the other hand, extendibly invariance of J implies that $J \rtimes_\alpha S$ is the ideal $\overline{\operatorname{span}}\{i_S(s)^* i_A(a) i_S(t) \mid s,t \in S, a \in J\}$ of $A \rtimes_\alpha S$ (*cf.* [18, Theorem 1.7]). A small rearranging shows that

$$\begin{aligned}J \rtimes_\alpha S &= \overline{\operatorname{span}}\big\{ i_S(s)^* i_A\big(\alpha_s(1) a \alpha_t(1)\big)\, i_S(t) \mid s,t \in S, a \in J\big\}\\ &= \overline{\operatorname{span}}\{i_S(s)^* i_A(e) i_S(t) \mid s,t \in S, e \in \alpha_s(A) \cap \alpha_t(A) \cap J\}.\end{aligned}$$

Comparing the generating sets for $J \rtimes_\alpha S$ and $\iota \times m\big(i_A(J) \rtimes_\beta G\big)$ shows that the two ideals are the same, as required. ∎

4 Examples

Example 4.1 Assume that X is a compact Hausdorff space and θ is a homomorphism of an Ore semigroup S into the set

$$\{f \colon X \to X \mid f \text{ is continuous, injective, and } f(X) \text{ is open}\},$$

such that $\theta_s(e) = \mathrm{id}_X$, and $\theta_s(X) \cap \theta_t(X) \neq \varnothing$, for any $s,t \in S$. By [18, Proposition 4.1], θ induces an action $\alpha \colon S \to \operatorname{End}\big(C(X)\big)$, which is defined by

$$\alpha_s(f)(x) = \begin{cases} f \circ \theta_s^{-1}(x) & \text{if } x \in \theta_s(X),\\ 0 & \text{if } x \in X \setminus \theta_s(X).\end{cases} \tag{4.1}$$

Then $\alpha_s\big(C(X)\big)$ is the ideal $C_0\big(\theta_s(X)\big)$, for $s \in S$. The endomorphisms α_s are unital, and therefore trivially extendible. It follows that the system $\big(C(X), S, \alpha\big)$ is of the form studied in Section 1, and Theorem 2.1 says that $C(X) \rtimes_\alpha S$ is a partial crossed product.

Example 4.2 It was shown in [18, Section 4] that the dynamical systems studied in [20] are of the form (4.1): the semigroup S is the direct sum $\mathbb{N}^k$ of copies of the semigroup of non-negative integers, for k in $\mathbb{N} \cup \infty$, and the space X is the Pontryagin dual of a discrete abelian group Γ arising from a direct system over $\mathbb{N}^k$. The injective

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map θ_m is $\hat{\beta}_m\colon \hat{\Gamma} \to \hat{\Gamma}$ defined by $\hat{\beta}_m(\gamma) = \gamma \circ \beta_m$, for every m in $\mathbb{N}^k$, where $\beta\colon \mathbb{N}^k \to \mathrm{End}(\Gamma)$ is an action by surjective endomorphisms naturally determined in the course of building the direct limit. This $\hat{\beta}$ satisfies $\hat{\beta}_n(\hat{\Gamma}) \subset \hat{\beta}_m(\hat{\Gamma})$, when $m \leq n$ in $\mathbb{N}^k$. Examples of crossed products that fit the framework of [20], and for which accordingly we have a general characterisation of faithful representations, are the Hecke C^*-algebra of Bost and Connes [4, 16], the generalisations of this in [12] and two-prime analogue in [19], and certain Hecke C^*-algebras from topological dynamical systems [6].

The action $\mathbb{N}^k \to \mathrm{End}(\Gamma)$ induces an action β of $\mathbb{N}^k$ by endomorphisms of $C^*(\Gamma)$, hence of $C(\hat{\Gamma})$ via the Fourier transform, which satisfies

$$\beta_m \circ \alpha_m = \mathrm{id} \quad \text{and} \quad \alpha_m \circ \beta_m(f) = \alpha_m(1) f,$$

for $m \in \mathbb{N}^k$, $f \in C(\hat{\Gamma})$ [20]. Hence the system $(C(\hat{\Gamma}), \mathbb{N}^k, \alpha)$ fits the framework of Section 3. Notice that the coaction δ of Proposition 1.1 is simply the dual action of $\hat{\mathbb{Z}}^k$ induced by α [27, Remark 3.5], because $\mathbb{Z}^k$ is abelian. Since the full and the reduced partial crossed products coincide for amenable groups, *cf.* [21, Proposition 4.2] and [26, Remark 3.7], we have by Proposition 3.1 the following corollary of Theorem 2.1.

Corollary 4.3 *For the system* $\big(C(\hat{\Gamma}), \mathbb{N}^k, \alpha\big)$*, there are isomorphisms*

$$C(\hat{\Gamma}) \rtimes_\alpha \mathbb{N}^k \cong C(\hat{\Gamma}) \rtimes_\beta \mathbb{Z}^k \cong C(\hat{\Gamma}) \rtimes_{\beta,r} \mathbb{Z}^k. \tag{4.2}$$

Moreover, the extendibly α-invariant ideals of $\big(C(\hat{\Gamma}), \mathbb{N}^k, \alpha\big)$ are determined by the β-invariant ideals of $\big(C(\hat{\Gamma}), \mathbb{Z}^k, \beta\big)$, and viceversa.

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