Instability, Chaos and Predictability
in Celestial Mechanics and
Stellar Dynamics

ISBN 1-56072-054-9

On Instability Leading to Chaos in Dynamical Systems

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ABSTRACT

Instability of orbits in dynamical systems leading to chaos has been reviewed briefly. Stability criteria for some unimodal mapping which provide various periodic regimes during the period doubling bifurcations has been discussed in detail. Stability conditions are also reviewed for standard map (or Chirikov-Taylor map), and results obtained for range of values of the non-linear parameter appearing in the map have been studied. Strange attractor has also been discussed.

1. INTRODUCTION

Studies on nonlinear dynamics and emergence of chaos are of growing interest at the present time. Chaotic phenomena have brought new mathematical ideas and analytical technique. The subject is fascinating because of its interplay of science, mathematics and technology. In past two decades scientists of various disciplines, (e.g. Ref.[1]-[11],[13]-[19],[54]), have come to the common conclusion that "a simple system may give rise to a complex behaviour and a complex system may give rise to a simple behaviour". Almost all nonlinear systems exhibit chaotic motion and so, chaotic phenomena are a wide class of natural events found in the physical world. Chaos happens more frequently than order. Chaos is a science of computer age. Modern computational technique and elegant mathematics made chaos to emerge as one of the most exciting and intriguing areas of science. Therefore, the investigations on chaos has brought a great revolution to modern scientists. Chaos has provided tools to explain evolution of dynamical systems to a vast range of real phenomena.

Hadamard (1898), [33], was first to observe sensitive dependence of chaotic trajectories on initial conditions which was also realized, later on, by Duhem (1906), [34], and Poincare (1908), [36],[31]. Poincare observed that "a fully deterministic dynamics does not necessarily imply an explicit prediction on the evolution of a dynamical system". Poincare's observation again came to light, after a long gap and with great excitement, by various researchers, ([1],[2],[6]-[9], [12],[13],[19]-[33],[37]-[53],[56], (Ref.[38],[40],[41] together known as a KAM theory)) and a theory of dynamical chaos was born.

Stability of a nonlinear system is very complicated than in the linear case. Here one has to go through its local as well as global aspects. The chaotic motion in a dynamical system is a result of dynamic instability of orbits in the system. Stability of motion of a nonlinear system which contains some parameters, say λ, may change in the vicinity of a fixed point. The fixed point becomes unstable when λ attains a critical value and a 2-cycle is born. This 2-cycle becomes unstable when λ further changes and attains an another critical value and a 4-cycle is born and so on. Thus we see that whenever a stable motion becomes unstable a bifurcation starts and a period doubling phenomenon occurs, ([1],[2],[6], [11],[20],[22],[24]-[28],[37],[45],[48],[51],[52]). The various critical values of λ obey a general rule

$$\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \to \delta = 4.6692016 \qquad (1.1)$$

where λ_n stands for critical value of λ at the nth bifurcation. δ is known as Feigenbaum's constant. From the above procedure one may accurately determine that critical value for λ after which the motion turns to be chaotic.

In the present work a brief review has been presented on instability leading to chaos. As an illustration, the population model, also known as logistic mapping, has been used as a model for the descrete dynamical system.

2. DYNAMICAL SYSTEMS, CHAOS AND UNPREDICTABILITY

2.1 Dynamical System:

A dynamical system is that whose evolution from some initial state, (prescribed), can be described by some rule(s) in the form of mathematical equations. The evolution of such a system is best described by the so-called phase space. The dynamical systems mostly we encounter are nonlinear in nature whose evolution is controlled by certain parameter which we have already stated.

2.2 What is Chaos?

Until now, there is no generally accepted definition of chaos. However, as it appears through recent literatures, (e.g. Ref.[1],[2],[6]-[9],[13],[16],[19]-[32] etc.), chaos can be well understood from the following ideas:

i) Chaos is an effect of instability of orbits in a dynamical system.

ii) The phenomenon related to the occurance of randomness and unpredictability in a completely deterministic system is called chaos.

iii) Irregular behaviour in both conservative and dissipative systems is termed as chaos.

iv) Chaotic behaviour simply looks markedly more irregular than regular behaviour.

v) Chaotic trajectories show sensitive dependence on initial conditions i.e., chaotic trajectories show an average exponential divergence of initially nearby trajectories.

vi) Chaos describes a situation where typical solutions (or orbits) of a differential equation (or typical evolution of some other model determining deterministic evolution) do not converge to a stationary or periodic function (of time) but continue to exhibit a seemingly unpredictable behaviour.

vii) Chaos can be thought as a new regime of nonlinear oscillations, as overlap of resonances, as accumulations of many instabilities, etc.

viii) Chaos implies that the knowledge of initial data is insufficient for long time prediction.

2.3 Unpredictability

The instability, non-deterministic behaviour and chaotic motion are consequences of the results of non-accurate predictability of the evolution of a dynamical system. Accurate prediction of the evolution of a dynamical system is extremely difficult as it depends on the facts, [65], that how accurately (a) the principles of dynamics describing such evolution be formulate i.e. the laws governing such evolution, (b) the mathematical model be established ?i.e., the differential equations established how accurately representing the system? (c) the approximation of the solution be made ? (d) the initial conditions and other numerical parameters be used?

The above conditions are very important because a minute error may cause a perfectly deterministic periodic motion to a chaotic one and vice-versa, Szebehely ([61]-[63]).

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2.4 Sensitive Dependence on Initial Conditions

Accuracy in initial conditions is a very important factor because the chaotic trajectories show sensitive dependence on the initial conditions,[66]. This means that for a small change $\delta \underline{x}(o)$ in the initial condition $\underline{x}(o)$: $\underline{x}(o) \rightarrow \underline{x}(o) + \delta \underline{x}(o)$, the point $\underline{x}(t)$ at time t may change as $\underline{x}(t) \rightarrow \underline{x}(t) + \delta \underline{x}(t)$, such that $\delta \underline{x}(t) \sim \delta \underline{x}(o) e^{\Lambda t}$, where Λ is called the Lyapunov characteristic exponent (L.C.E.). Lyapunov exponents are tools to determine whether or not the system is chaotic. The exponential convergence or divergence of initially nearby trajectories provide a conclusive way to distinguish between torus and chaos by estimating Lyapunov exponents Λ_i: $\Lambda < 0$ means stability whereas $\Lambda > 0$ stands for unstable behaviour and if $\Lambda_{max} > 0$ the system is defined to be chaotic, [52], and in this case a deterministic motion turns to be a chaotic one (Fig. 1).

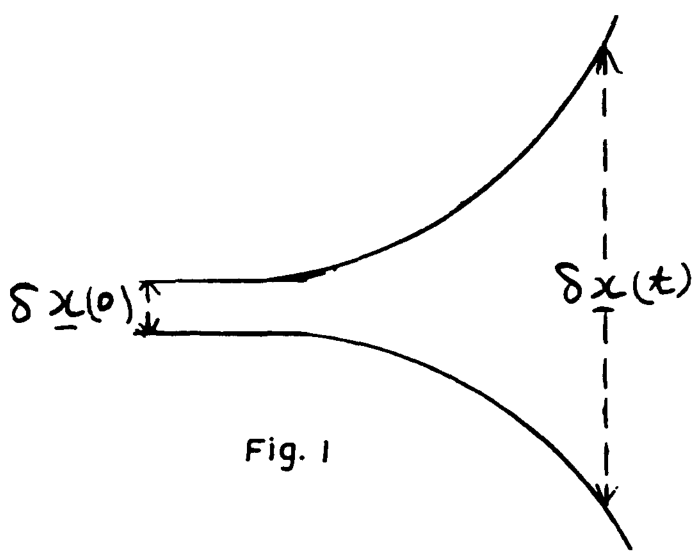

Fig. 1

3. BIFURCATIONS: STABILITY OF FIXED POINTS

The study of bifurcations has developed into a major area of mathematical and applied research. Jacobi ([56]) has used the term "branching off" for bifurcation which is due to Poincaré,[57]. The term bifurcation mostly now applied to situations involving (i) the study of the change in the number of fixed points as the control parameters λ are varying and (ii) the dynamic studies concerned with the change in the topology, (the phase portraits), as λ are changing. By changing λ a bifurcation may occur at any time and the phase portrait may change to a topologically non-equivalent portrait.

Let us consider a descrete dynamical system whose evolution is represented by the deterministic map

$$x_{n+1} = F(x_n) \tag{3.1}$$

The equilibrium value x^*, for which $x^* = F(x^*)$, is said to be the fixed point of F. Then, x^* is said to be stable if the sequence of iterates $x_1, x_2, \ldots, x_n, \ldots$ of F converges to x^*,

$$\text{i.e.} \qquad \lim_{k\to\infty} x_k = x^*, \tag{3.2}$$

for all initial values $x(o)$ of x, or alternatively, if

$$|F'(\lambda, x^*)| = \left| \frac{x_{n+1} - x^*}{x_n - x^*} \right| = \left| \frac{dF}{dx^*} \right| < 1 , \tag{3.3}$$

$$\text{and } x^* \text{ is unstable if } \left| \frac{dF}{dx^*} \right| > . \tag{3.4}$$

Figures 2(a) and 2(b) respectively representing stable and unstable fixed points.

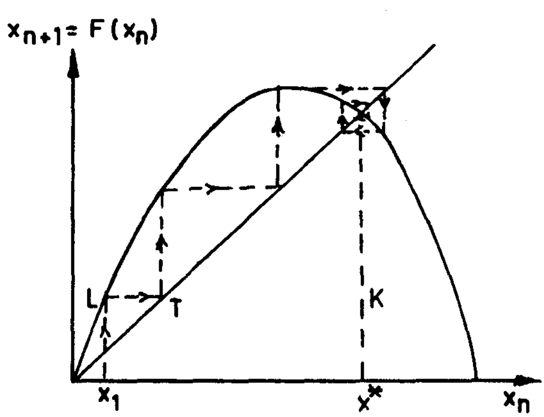

Figure 2(a): Stable fixed point (Attractor)

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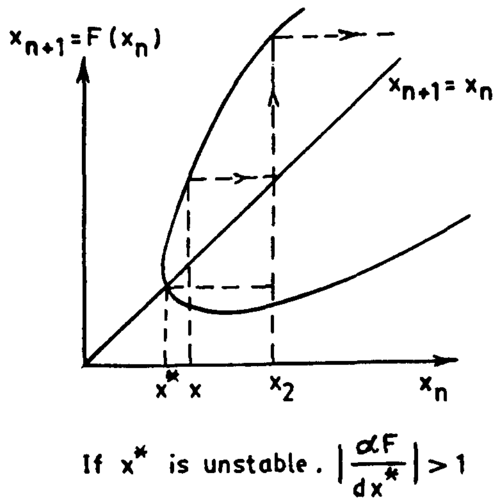

Figure 2(b): Unstable fixed point

A stable fixed point is termed as an attractor because the points in its neighbourhood approach to it when iterated.

As stated earlier, when the controlling parameters λ of F vary and attain some critical value, say λ_c, stability of x^* is disturbed and bifurcation starts and we observe a period doubling scenario, ([6],[11],[12],[24]-[30],[37],[51],[52]), of which a periodic pattern of period p or a p-cycle is defined by

$$x_{i+p} = x_i, \quad x_{i+k} \neq x_i, \text{ for all } k < p, \tag{3.5}$$

and i greater than certain N.

Thus, p = 1 corresponds to the fixed point x^*. An attractor x^* is not only the isolated point, there might be p-point in p-cycle, $x_1^*, x_2^*, \ldots, x_p^*$ such that

$$\begin{aligned} x_{i+1}^* &= F(x_i^*), \quad i = 1,2,\ldots,p-1 \\ x_1^* &= F(x_p^*) \end{aligned} \tag{3.6}$$

The set $\{x_1^*, x_2^*, \ldots, x_p^*\}$ is called a p-point limit cycle. The p-cycle is stable if

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$$\left| \prod_{i=1}^{p} F'(\lambda, x_i^*) \right| < 1 \quad , \tag{3.7}$$

where the chain rule of differentiation has been used. That is, the p-point limit cycle is stable if each x_i^* in $\{x_1^*, x_2^*, \ldots, x_p^*\}$ is a stable fixed point of $F^{(p)}$.

If the set $\{x_i^*\}$, $i = 1,2,\ldots,p$ is a global attractor then for almost every initial point x_0, the sequence $x_n = F^{(n)}(x_0)$ approaches the sequence $x_1^*, x_2^*, \ldots, x_n^*, x_1^*, x_2^*, \ldots, x_n^*, \ldots$

4. INSTABILITY LEADING TO CHAOS

The sequence of critical values of λ where bifurcations occur obey the general rule (1.1). It has been observed that the limiting value δ in (1.1) approaches by third or fourth bifurcation. So by changing λ one may accurately determine that critical value of λ after which the motion turns to be chaotic.

To illustrate chaotic behaviour, Feigenbaum and many others ([24]-[28],[45], etc.) have used population model,[67], which is also known as logistic mapping and is written as

$$f(x) = \lambda x(1-x), \tag{4.1}$$

$x = 0$ and $x = \frac{\lambda-1}{\lambda}$ are fixed points of f. The period two points of f, including the fixed points of f, are given by the roots of the equation

$$f^2(x) - x = 0$$

or
$$[f(x)-x]\,[\lambda^2 x^2 - \lambda(\lambda+1)x + (\lambda+1)] = 0. \tag{4.2}$$

Thus, the period two points of f (not fixed points of f) are given by

$$\frac{\lambda+1 + \sqrt{(\lambda-3)(\lambda+1)}}{2\lambda} \quad \text{and} \quad \frac{(\lambda+1) - \sqrt{(\lambda-3)(\lambda+1)}}{2\lambda}$$

Now if (i) $\lambda < 3$, there are no such real points,

(ii) $\lambda = 3$, there is exactly one point, the fixed point of f

and (iii) $\lambda > 3$, there are two different points.

Thus a point two or 2-cycle is created as λ increased through 3. Also, note that the fixed point is attracting for $\lambda < 3$ and

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repelling for $\lambda > 3$ and the attracting nature is passed on to the period two cycle which exists only for $\lambda > 3$ and is attracting. We can illustrate this by plotting f and f^2 against x for various values of λ, (above, below and equal to 3) Figure 3 (a) and 3 (b) are such plottings.

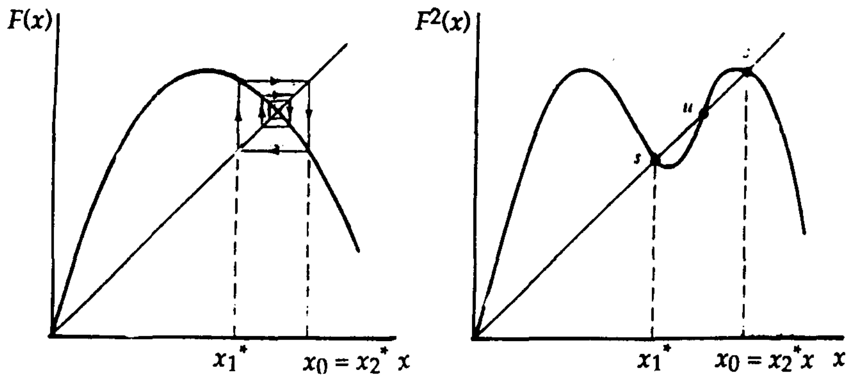

Figure 3 (a) and 3 (b)

As λ increases further, the 2-cycle becomes repelling and an attractive 4-cycle is born; then a 8-cycle and so on and we observe the period doubling scenario. For example

λ = 2.9 gives an attractive 1-cycle

λ = 3.1 gives an attractive 2-cycle

λ = 3.5 gives an attractive 4-cycle

λ = 3.56 gives an attractive 8-cycle

λ = 3.566 gives an attractive 16-cycle

i.e., there is a set $\{\lambda_{c_n}\}$ of critical values of λ such that if $\lambda_{c_n} < \lambda < \lambda_{c_{n+1}}$ there is a stable period 2^n-cycle and

$\lim_{n\to\infty} \lambda_{c_n} = \lambda_\infty$, (finite). In the control phase space the above period doubling bifurcations look like Fig. 4. The $\pm$ sign for branches indicate that each bifurcation point $\bar{x}_{(n)}$ can be associated with its branch by a $\pm$ subscript, (e.g. (n) = +, +, - , + , - , -).

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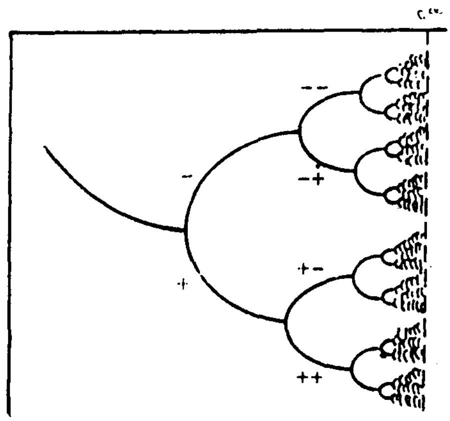

Figure 4:

It has been observed that beyond the critical values of for which motion show period doubling, chaotic motion exist in a band of parameter values. When these bands are of finite width, periodic windows be developed for the parameter within which the motion may again undergo period doubling bifurcation and again leading to chaotic motion, ([6],[11],[12],[51],[52]). Thus in the above example as λ increases to 4, at various values of λ, an attractive q-cycle is born followed immediately by period doubling sequence of attractive 2q, 4q, 8q,... cycles.

When $\lambda > 4$, then there is a cantor set J in [0,1] such that (i) $x \in J$ iff $f(x) \in J$, (ii) $f^n(x) \to -\infty$ if $x \notin J$ and (iii) f has a chaotic action on J.

Therefore, we can conclude that the period doubling phenomenon is the most celebrated scenario for chaotic motion.

In case of the logistic mapping [45], written as $x_n = 1 - \lambda x^2_{n-1}$, the period doubling scenario appears at critical values of λ given by

$$p = 1, \quad 0 < \lambda < \lambda_{c_1} = 0.75$$

$$p = 2, \quad \lambda_{c_1} < \lambda < \lambda_{c_2} = 1.25$$

$$p = 4, \quad \lambda_{c_2} < \lambda < \lambda_{c_3} = 1.3680989$$

$$p = 8, \quad \lambda_{c_3} < \lambda < \lambda_{c_3} = 1.3940461$$

which quickly converges to an aperiodic orbit at $n \to \infty$, the value $\lambda_\infty = 1.401155...$ In the range $(\lambda_\infty, 2)$ there exists an infinite number of periodic windows immersed in the background of aperiodic regime. Figure 5 represent bifurcation diagram for the mapping $x_{n+1} = \lambda - x_n^2$, Ref. [37],[68].

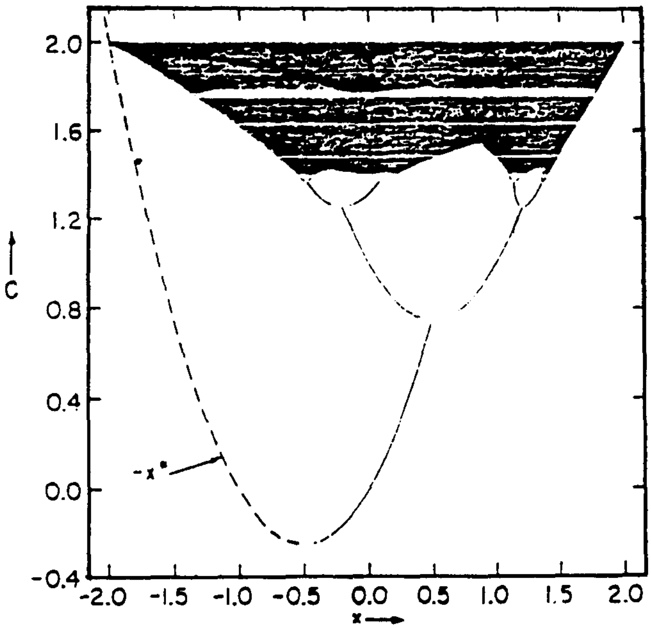

Figure 5: Bifurcation diagram for equation: $x_{n+1} = C - x_n^2$.
From Grebog, Ott and Yorke (1982)

The problem of turbulence is a long standing phenomenon of physics and as it appears in recent literature, ([22],[25] [27],[50],[51],[52],[55]), the chaotic phenomena are relevant to the onset mechanism of turbulence. When the Reynold's number R is small enough, the fluid flow is laminar and statioary corresponding to a suitable fixed point in its phase space. However, when R is increased to a certain critical value R_c, the fixed point is no more stable, (or of attracting nature) and a stable limit cycle closed around the fixed point be established. When R again be increased to another critical value R_{c_2}, the limit cycle loses its stability and a stable 2-Torus, (an attracting closed tube), around the unstable limit cycle be established and so on. Landau and Hopf, ([58]-[60]), identified the final state of this infinite process with an infinite number of incommensurable frequencies as fully turbulent. This is known as Landau-Hopf route to turbulence. Later on, ([50],[51]), it has been showed that

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three consecutive bifurcations are enough to have erratic motion by interweaving trajectories attracted to a low dimensional manifold in the same phase space called strange attractor. The motion on strange attractors be identified with turbulence. Thus, in the scheme we have,

Fixed Point → Limit Cycle → 2-Torus → Strange Attractor
Chaos

This concludes that the quasiperiodic motion on a 2-Torus may have loose stability and may give birth to chaos directly.

5. STRANGE ATTRACTOR: ([23],[2],[7],[24]-[28],[56])

As stated earlier a stable fixed point is an attractor. An attracting 2-cycle is the stable two-period cycle and, in general, an attracting p-cycle means stable p-period cycle.

We say that the descrete dynamical system posses chaos if it has some strange attractor. Strange attractors are mathematical objects but computers have given them life and draw pictures of them. A strange attractor is first an attractor which consists of an infinity of points in the plane of m-dimensional space. These points correspond to the state of a chaotic motion. We call it "strange" as it has the structure of a fractal set. The intersection of this object with a straight line results in a "Cantor Set". It has an infinitely nested structure which can be seen in each repeated minor magnification.

A strange attractor is defined for a map f as an infinite point set Ω such that (i) $f(\Omega) = \Omega$, (ii) f has an orbit which is dense in Ω and (iii) Ω has a neighbourhood N consisting of points whose orbits tends asymptotically to i.e., $\lim_{n\to\infty} f^{(n)}(N) \subset \Omega$. Orbit in or near such limit set behave in an essentially chaotic manner.

Hénon's attractor [7] and Lorenz's attractors [23] are examples of strange attractors.

6. CONCLUDING REMARKS

What has been revealed through this brief review indicates that to study nonlinear phenomena in more realistic way one must use the chaotic theory. Finally, let us recall the statement of Ian Stewart,[54],that chaos is (i) exciting as it provides tools for simplifying complicated phenomena, (ii) worrying because it introduces new doubts about the traditional model building procedure, (iii) fascinating for its interplay of mathematics, science and technology and (iv) beautiful, above all.

ACKNOWLEDGEMENTS

The author wishes to present his sincere thanks to Prof. K.B. Bhatnagar who extended his help and encouragement in preparation of this paper.

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