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A PAIR OF CHARACTERISTIC SUBGROUPS FOR PUSHING-UP. II

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Dedicated to Ronald Solomon on his sixtieth birthday

Abstract Many problems about local analysis in a finite group G reduce to a special case in which G has a large normal p-subgroup satisfying several restrictions. In 1983, R. Niles and G. Glauberman showed that every finite p-group S of nilpotence class at least 4 must have two characteristic subgroups S_1 and S_2 such that, whenever S is a Sylow p-subgroup of a group G as above, S_1 or S_2 is normal in G. In this paper, we prove a similar theorem with a more explicit choice of S_1 and S_2.

Keywords: Sylow p-subgroups; characteristic subgroup

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1. Introduction and notation

Let p be a prime and let S be a finite p-group. Let $J_R(S)$ be the subgroup of S generated by the abelian subgroups of largest rank. In 1964, John G. Thompson introduced the subgroup $J_R(G)$ and used it to prove the following result [**7**, p. 118].

Suppose p is odd and S is a Sylow p-subgroup of a finite group G. Assume that $C_G(Z(S))$ and $N_G(J_R(S))$ both have normal p-complements. Then G has a normal p-complement.

This theorem led to further work by Thompson and others that used subgroups similar to $J_R(S)$ and local information about Sylow subgroups to obtain global information about finite groups, particularly simple groups [**14**, pp. 225–282]. Much of this work reduced to the following minimal situation:

(E_0) G is a nonidentity finite group;
p is a prime;
S is a Sylow p-subgroup of G;
$C_G(\mathrm{O}_p(G)) \leqslant \mathrm{O}_p(G)$;
S is contained in a unique maximal subgroup of G; and
for some normal subgroup K of G and some natural number n, $G/K \cong \mathrm{PSL}(2, p^n)$.

Here, one needs to show that some non-identity characteristic subgroup of S is a normal subgroup of G.

There are examples (below) in which no such characteristic subgroup exists, even though S has nilpotence class precisely 2 and is thus almost abelian. Thus, it seems surprising that there must exist such a subgroup if S has nilpotence class precisely 4 or larger (or precisely 3 or larger, if $p \neq 3$), by results of Niles [**19**] (in 1977) and Baumann [**2**] (in 1979). In 1983, Niles and the author managed to extend these results as follows [**12**, Theorem A].

Theorem. *Suppose p is a prime and S is a finite p-group. Assume that S has nilpotence class at least 3; if $p = 3$, assume that S has nilpotence class at least 4. Then there exist non-identity characteristic subgroups S_1, S_2 of S satisfying the following condition: whenever a group G satisfies (E_0), $S_1 \lhd G$ or $S_2 \lhd G$.*

This result is useful when G ranges over a family of subgroups of a group, such as a simple group [**14**, pp. 273–279].

In this article we extend this theorem in two ways. First, we find further sufficient conditions under which some pair S_1, S_2 satisfies the conclusion of the theorem (Theorems A, B, D and E). Second, motivated by a question about the results of [**12**], we focus on a different particular pair and find sufficient conditions for it to satisfy the conclusion of the theorem (Theorem C). These results may shed light on a conjecture of Thompson (below).

Just as the results of [**12**] used characteristic subgroups similar to $J_R(S)$, our new results involve characteristic subgroups arising from a recent article [**11**] using work of Chermak and Delgado [**5**].

Some results related to [**12**] (and to this paper) appear in [**1**] and [**3**]. (For these articles, $J(S)$ is defined to be generated by the elementary abelian subgroups of maximal order in S, and so may be different from the subgroup called $J(S)$ in this paper. Similarly, the Baumann subgroup is defined differently in these articles.)

The results of [**12**] are divided into cases, and this article was inspired by a question about one case. In every case of [**12**], the subgroup S_1 is relatively small and is contained in the centre of S, while the subgroup S_2 is relatively large and contains its centralizer in S, just like the pair $Z(S)$, $J_R(S)$ in Thompson's theorem. Moreover, in all except one case, S_2 has the additional property that no subgroup of S other than S_2 is isomorphic to S_2. (This property is clearly satisfied by $J_R(S)$, which is one of the reasons that $J_R(S)$ is useful.) Hence, in these cases, whenever (E_0) is satisfied and S_2 is contained in $\mathrm{O}_p(G)$, then S_2 is normal in G.

The exceptional case of [**12**] (which occurs in part (c) of Theorem D of [**12**] and occupies most of the proof in [**12**]) is somewhat mysterious. Here, S_2 is defined as the intersection of some subgroups of S, and the author suspected that some subgroup S^* of S_2 defined more explicitly would satisfy the additional property above. After obtaining the results of [**11**], our suspicion fell in particular on the subgroup S_{MCL} defined below, which clearly satisfies the additional property.

Example 7.1 below shows that these suspicions were incorrect in general. However, in Theorem C we use [**11**] to prove them under some restrictions on G. In part of the proof, we are able to prove that $S_{\mathrm{MCL}} \lhd G$ in a situation in which a variation of $J_R(S)$ (namely, the subgroup $J(S)$ defined below) may not be normal in G. In Theorems B and D, we apply [**11**] to obtain new sufficient conditions on S for S_1 and S_2 to exist. This yields Theorems A and E, which extend the theorem of [**12**] above.

To state Theorem A, we use notation from [**14**, pp. 227, 274] for two subgroups similar to $J_R(S)$. As before, S denotes an arbitrary finite p-group. Let $\mathscr{A}(S)$ be the set of all abelian subgroups of S of maximal order and let $J(S)$ be the *Thompson subgroup* of S, which is generated by $\mathscr{A}(S)$. Let $\tilde{J}(S)$ be the *Baumann subgroup* of S, given by $C_S(Z(J(S)))$. As usual, for any group G, let $\Phi(G)$ denote the Frattini subgroup of G and let $Z_2(G)$ denote the subgroup given by $Z_2(G)/Z(G) = Z(G/Z(G))$. In this article, we call the elements of $\mathscr{A}(S)$ the *large abelian subgroups* of S.

Consider the following hypothesis:

(P) (i) S_1 is a subgroup of $Z(S)$ and S_2 is a characteristic subgroup of $\tilde{J}(S)$,
(ii) whenever (E_0) is satisfied for some group G, then $S_1 \lhd G$ or $S_2 \lhd G$.

Theorem A. *Suppose p is a prime and S is a non-identity finite p-group. Then there exist non-identity characteristic subgroups S_1 and S_2 of S satisfying the hypothesis (P), except possibly when S satisfies the following conditions:*

(a) *S is not abelian;*

(b) *$J(S) = S$;*

(c) *$Z(S)$ and $\Phi(S)$ are elementary abelian;*

(d) (i) *if $p = 2$, then $\Phi(S) \leqslant Z(S)$,*
(ii) *if $p = 3$, then $\Phi(S) \leqslant Z_2(S)$, and*
(iii) *if $p > 3$, then $\Phi(S) \leqslant Z(S)$ and S has exponent p;*

(e) *some large abelian subgroup of S is elementary abelian; and*

(f) *for all large abelian subgroups A, B of S and all subgroups Q of S,*

$$|A|^2 = |S|\,|Z(S)| \geqslant |Q|\,|Z(Q)| \quad \text{and} \quad \langle A, B\rangle = AB = BA = C_S(A \cap B).$$

Note that conditions (a) and (d) yield that S has nilpotence class precisely 2 if $p \neq 3$ and precisely 2 or 3 if $p = 3$. Parts (a)–(d) come mainly from [**12**], while parts (e) and (f) come from Theorem B below, and thus mainly from [**11**].

To describe some examples in which S has nilpotence class 2, consider a group H that is isomorphic to $\mathrm{SL}(2, p^n)$ for some natural number n and acts faithfully on an elementary abelian group V of order p^{2n}. We say that V is a *standard module* for H if there exists a field F such that V is a two-dimensional vector space over F and $\mathrm{SL}(V, F)$ is the group of all automorphisms of V induced by H.

Now, suppose that S is a Sylow p-subgroup of the semi-direct product VH in the situation above. In the simplest case, when $n = 1$, S is a dihedral group of order 8 if

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$p = 2$ and a non-abelian group of order p^3 and exponent p if p is odd. It is well known that, for every n, no non-identity characteristic subgroup of S is normal in G. We show this in Example 7.6 for $n = 1$ and give references for $n > 1$. Hence, S satisfies conditions (a)–(f) of Theorem A, as one may easily verify.

For $p = 3$, we give in Example 7.7 a family of examples in which S has nilpotence class 3 and no non-identity characteristic subgroup of S is normal in G.

We need additional notation from [**11**] and [**12**] for our other results:

$$\begin{aligned} d(S) &= \max\{|A| \mid A \leqslant S \text{ and } A \text{ is abelian}\}, \\ f(S) &= \max\{|R| \cdot |Z(R)| \mid R \leqslant S\}, \\ f_1(S) &= \max\{|R| \cdot |C_S(R)| \mid R \leqslant S\}, \\ \mathscr{F}(S) &= \{R \leqslant S \mid |R| \cdot |Z(R)| = f(S)\}, \\ \mathscr{F}_1(S) &= \{R \leqslant S \mid |R| \cdot |C_S(R)| = f_1(S)\}, \\ S_{\mathrm{CL}} &= \langle \mathscr{F}(S) \rangle, \\ S' &= [S, S]. \end{aligned}$$

We call elements of $\mathscr{F}(S)$ *centrally large subgroups*, or CL-subgroups, of S.

By Proposition 2.4 of [**11**], $f(S) = f_1(S)$ and $\mathscr{F}(S)$ is a subset of $\mathscr{F}_1(S)$. A CL-subgroup of S that is minimal under inclusion in $\mathscr{F}(S)$ is called a *minimal CL-subgroup* of S. Let S_{MCL} denote the subgroup of S generated by all the minimal CL-subgroups of S.

For a finite group G and a prime p, we also let $\mathrm{O}^p(G)$ be the subgroup generated by all the p'-elements of G.

Now we may state our other main results.

Theorem B. *Assume* (E_0)*, and suppose* $\tilde{J}(S) = S$*. Let*

$$mz(S) = \max\{|\Omega_1(Z(Q))| \mid Q \text{ is a minimal CL-subgroup of } S\}$$

and

$$S_\Phi = \langle \Phi(Q) \mid Q \text{ is a minimal CL-subgroup of } S \text{ and } |\Omega_1(Z(Q))| = mz(S) \rangle.$$

Then

(a) $Z(S) \lhd G$ *or* $S_\Phi \lhd G$*, and*

(b) *if* $S_\Phi = 1$*, then the minimal CL-subgroups of* S *coincide with the large abelian subgroups of* S*, and at least one of them is elementary abelian.*

Remark 1.1. Note that S_Φ contains $\mho^1(Z(S))$. Whenever (E_0) is satisfied, $Z(S) \lhd G$ if and only if $Z(S) = Z(G)$, by Lemma 2.19 below.

Theorem B will follow easily from results in [**11**]. We show in § 3 that in case (b) of Theorem B and case (c) of Theorem D (below), some large abelian subgroup of S is normal in S and, for all large abelian subgroups A, B of S and all subgroups Q of S,

$$|A|^2 = |S|\,|Z(S)| \geqslant |Q|\,|Z(Q)| \quad \text{and} \quad AB = BA = C_S(A \cap B)$$

(as in condition (f) of Theorem A).

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Theorem C. *Assume* (E_0), *and suppose* $\tilde{J}(S) = S$. *Let*

$$T = \mathrm{O}_p(G), \quad \hat{G} = \mathrm{O}^p(G), \quad \hat{S} = S \cap \hat{G}, \quad \hat{T} = \mathrm{O}_p(\hat{G}), \quad L = C_G(Z(T)) \quad \text{and} \quad q = p^n.$$

Then $Z(S) \lhd G$ *or* $S_{\mathrm{MCL}} \lhd G$, *except possibly if* G *satisfies the following conditions.*

(a) $\hat{S}$ *is a Sylow* p*-subgroup of* $\hat{G}$ *of nilpotence class at most* 3.

(b) *The commutator subgroup* Q' *is the same for each minimal CL-subgroup* Q *of* S *and is a characteristic subgroup of* S, T *and* G, *and* $G = TC_G(Q')$.

(c) $\hat{T}$ *has nilpotence class at most* 2, $\hat{T}/Z(\hat{T})$ *is elementary abelian, and* $\hat{T}' \leqslant Z(\hat{G}) < Z(\hat{T}) \leqslant \hat{T} = [\hat{T}, \hat{G}]$.

(d) $\hat{T}$ *has exponent* p *if* p *is odd, and* $\hat{S}$ *has exponent* p *if* $p \geqslant 5$.

(e) $G/L \cong \mathrm{SL}(2, q)$ *and* $Z(T)/Z(G)$ *is a standard module for* G/L.

(f) *A chief factor* U/V *of* G *for which* $U \leqslant T$ *is central if* $U \leqslant Z(\hat{G})$ *or* $\hat{T} \leqslant V < U \leqslant T$ *and is not central if* $Z(\hat{G}) \leqslant V < U \leqslant \hat{T}$.

(g) *If* $q = 2$, *then* G/T *is a dihedral group of order* $2 \cdot 3^k$ *for some natural number* k.

(h) *If* $q > 2$, *then* $L = T$ *and every non-central chief factor* U/V *of* G *satisfying* $U \leqslant T$ *is a standard module for* G/T.

(i) *If* $q \geqslant 4$, *then there exists a normal subgroup* R *of* $N_G(S)$ *such that*

$$R \leqslant \hat{S}, \quad S = TR, \quad [S, R] \leqslant \hat{S}' Z(\hat{G}) \quad \text{and} \quad [S, R, R, R] = 1.$$

By Theorem 2.10, the condition that $Q' = R'$ for all minimal CL-subgroups Q, R of S is satisfied for all groups S, and does not depend on the hypothesis of Theorem C.

While S_{MCL} has the advantage of being defined more explicitly than the group S_2 in the exceptional case in [**12**], there are cases (Examples 7.1–7.3) in which $S_2 \lhd G$, but neither $Z(S)$ nor S_{MCL} is normal in G. (Thus, G satisfies conditions (a)–(i) of Theorem C.)

Consider the following condition:

$$(P') \quad \text{condition } (P) \text{ is satisfied and } f(S_2) = f(\tilde{J}(S)).$$

Remark 1.2. Condition (P') says that S_2 contains a CL-subgroup Q of $\tilde{J}(S)$. By Theorem 3.1 of [**11**], Q contains some large abelian subgroup A of $\tilde{J}(S)$. Then A is a large abelian subgroup of S. Therefore, $d(S_2) = d(S)$ and $C_S(S_2) \leqslant C_S(A) = A \leqslant S_2$.

We also obtain the following analogues of Theorems A and B.

Theorem D. *Assume* (E_0) *and suppose* $\tilde{J}(S) = S$. *Let* Q *be any minimal CL-subgroup of* S. *Then*

(a) Q' *is a characteristic subgroup of* S;

(b) $Z(S) \cap Q' \lhd G$ *or* $S_{\mathrm{MCL}} \lhd G$; *and*

(c) *if* $Q' = 1$, *then the minimal CL-subgroups of* S *coincide with the large abelian subgroups of* S.

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Note that in case (c), S satisfies the conditions of Remark 1.1.

Theorem E. *Suppose p is a prime and S is a non-identity finite p-group. Then there exist non-identity characteristic subgroups S_1 and S_2 of S satisfying condition (P'), except possibly if S satisfies the following conditions:*

(a) *S is not abelian;*

(b) *$J(S) = S$;*

(c) *$Z(S)$ and $\Phi(S)$ are elementary abelian;*

(d) (i) *if $p = 2$, then $\Phi(S) \leqslant Z(S)$,*

(ii) *if $p = 3$, then $\Phi(S) \leqslant Z_2(S)$, and*

(iii) *if $p > 3$, then $\Phi(S) \leqslant Z(S)$ and S has exponent p; and*

(e) *for all large abelian subgroups A, B of S and all subgroups Q of S,*

$$|A|^2 = |S|\,|Z(S)| \geqslant |Q|\,|Z(Q)| \quad \text{and} \quad \langle A, B\rangle = AB = BA = C_S(A \cap B).$$

Rather than alternating between two subgroups S_1 and S_2, it would be ideal to find a single characteristic subgroup S_3 of S that is normal in every group satisfying (E_0). However, examples (as in [**12**, pp. 412–413]) show that S_3 need not exist, even for S of arbitrarily large class.

Despite this, there are results that give some global information about a group G from information about the normalizer $N_G(S_3)$ of a single non-identity characteristic subgroup S_3 of S. These results generally reduce to showing that $S_3 \lhd G$ in a group G that satisfies conditions like (E_0) as well as additional conditions, such as commutator conditions on the chief factors U/V of G for U contained in $\mathrm{O}_p(G)$ [**9**, §§ 7 and 12].

As mentioned in [**12**, p. 413], John G. Thompson has asked whether, for p odd, there exists a characteristic subgroup S_3 such that $S_3 \lhd G$ for every group G that satisfies (E_0) and the conditions that $G/\mathrm{O}_p(G) \cong \mathrm{SL}(2, p^n)$ and some non-central chief factor U/V of G with $U \leqslant \mathrm{O}_p(G)$ is not a standard module for $G/\mathrm{O}_p(G)$. From Theorem 2.15 below, the latter condition is equivalent to the commutator condition $[U/V, S, S] > 1$. This is related to the condition of p-stability, which yields $Z(J(S)) \lhd G$ [**9**, pp. 22, 23, 41], and, indeed, Thompson has conjectured [**12**, p. 452] that one can take $S_3 = Z(J(S))$ under his conditions as well.

By Remark 1.2 of [**12**], every group G satisfying Thompson's conditions falls into one of the cases of [**12**], and hence satisfies $S_1 \lhd G$ or $S_2 \lhd G$ for the corresponding pair S_1, S_2. If it also satisfies $\tilde{J}(S) = S$, then $Z(S) \lhd G$ or $S_{\mathrm{MCL}} \lhd G$, by part (h) of Theorem C. These observations may shed light on Thompson's question.

Section 2 consists of preliminary results. Theorems A, B, D and E are proved in § 3. The proofs come mainly from [**12**] and [**11**] and do not require most of the results of § 2. Thus, most of this paper is devoted to the proof of Theorem C.

Starting before Proposition 3.4, we assume the following additional hypothesis and notation:

$$(H)\quad \begin{aligned} &G, p, S, K \text{ and } n \text{ satisfy } (E_0),\\ &T = \mathrm{O}_p(G),\\ &Z(S) \neq Z(G) \text{ and } S = \tilde{J}(S). \end{aligned}$$

Note that (H) is the hypothesis of case (c) of Theorem D of [**12**], except that there one denotes $\mathrm{O}_p(G)$ by M and one also assumes that $\mho^1(Z(S)) = 1$. Note also that if (H) holds, then $Z(S) = Z(J(S))$.

In §§ 3–5, we reduce the proof of Theorem C to the special case in which the minimal CL-subgroups of S are large abelian subgroups and G is generated by two large abelian subgroups from different Sylow subgroups. We complete the proof in § 6, and we give examples in § 7.

All groups in this paper will be finite. In addition to the notation already defined, most of our notation is standard and taken from [**13**]. In particular, for subgroups X, Y, Z of a group,

$$[X, Y, Z] = [[X, Y], Z], \quad [X, Y; 1] = [X, Y],$$
$$[X, Y; i+1] = [[X, Y; i], Y] \quad \text{for } i = 1, 2, 3, \ldots.$$

Throughout this paper, p denotes a fixed but arbitrary prime, and S denotes a fixed but arbitrary p-group.

2. Preliminary results

Here we state several previous results, mainly from [**11**]. Theorem 2.7 and Proposition 2.8 will be used very frequently, as will Dedekind's Law: if H, K, L and HK are subgroups of a group and $H \leqslant L$, then $HK \cap L = H(K \cap L)$. Therefore, we will usually apply them without quoting them.

Most of the results in this section are used only for Theorem C. The other main theorems are proved in § 3 and require only Theorems 2.7 and 2.10, Proposition 2.8 and Lemmas 2.12 and 2.19 from this section.

In this section, P denotes a fixed, but arbitrary, p-group. (Some of these results remain valid when P is an arbitrary finite group.)

Lemma 2.1.

(a) *If H and K are subgroups of a group G, then $[H, K] \lhd \langle H, K\rangle$.*

(b) *(Frattini argument.) If H is a normal subgroup of a group G and P is a Sylow subgroup of H, then $G = N_G(P)H$.*

(c) *If A is a p'-group of automorphisms of P, then*

$$P = C_P(A)[P, A] \quad \textit{and} \quad [P, A, A] = [P, A],$$

and, if P is abelian, $P = C_P(A) \times [P, A]$.

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(d) *If N is a normal A-invariant subgroup of P in (c), then $C_{P/N}(A) = C_P(A)N/N$.*

(e) *If A centralizes P/N and N in (d), then A centralizes P.*

(f) *If P is a Sylow subgroup of a group G, then $P \cap G' \cap Z(G) \leqslant P'$.*

Proof. Parts (a)–(d) are proved in [**13**] (part (a) on p. 18, part (b) on p. 12 and parts (c) and (d) on pp. 177–181). Part (e) follows from (d). Part (f) follows from Theorem 10.8 in [**21**]. □

Theorem 2.2. *Suppose that A is a group acting on a p-group P. Let B be a Sylow p-subgroup of A.*

(a) *(Thompson.) Assume $A = B \times C$ for some p'-subgroup C of A, and C centralizes $C_P(B)$. Then C centralizes P.*

(b) *(Gaschütz.) Assume P is abelian and $P = Q \times R$ for some A-invariant subgroup Q and some B-invariant subgroup R of P. Then $P = Q \times R^\star$ for some A-invariant subgroup $R^\star$ of P.*

Proof. (a) This is proved in [**13**, pp. 179–180].

(b) Let X be the semi-direct product of P by A. We embed P and A in X in the usual manner. Then

$$P \lhd X, \quad PB \text{ is a Sylow } p\text{-subgroup of } X, \quad PB \cap Q = Q,$$

and RB is a complement to Q in PB, i.e. PB splits over $PB \cap Q$.

For any prime q other than p, a Sylow q-subgroup of A is a Sylow q-subgroup of X and intersects Q trivially, and hence obviously splits over this intersection. Thus, for every prime q, including p, X possesses a Sylow q-subgroup that splits over its intersection with Q. It follows from [**16**, Theorem 15.8.6] that X is a splitting extension of Q by some subgroup Y.

Let $R^\star = P \cap Y$. Then $P = Q \times R^\star$ and $R^\star \lhd QY = X$. Therefore, $R^\star$ is invariant under A, as desired. □

Theorem 2.3 (Noboru Itô). *Suppose A and B are abelian subgroups of a group and $AB = BA$. Then $(AB)'$ is abelian.*

Proof. This is proved in [**17**, p. 674]. □

Theorem 2.4. *Suppose P has nilpotence class at most $p - 1$. Then*

(a) *every element of $\Omega_1(P)$ has order 1 or p and*

(b) *if $x, y \in P$ and $x^p = y^p$, then $(xy^{-1})^p = 1$.*

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Proof. This follows easily from Hall's theory of regular p-groups, since P is a regular p-group by [**16**, Corollary 12.3.1, p. 182]. Specifically, (a) and (b) follow from [**16**, p. 186].

Alternatively, these results follow easily from Lazard's correspondence between p-groups of class at most $p-1$ and finite nilpotent Lie rings of p-power order and class at most $p-1$ [**18**, Chapter 10]. □

Lemma 2.5. *Suppose p is a prime, n is a natural number and H is an abelian group of order dividing p^n-1 acting irreducibly on an elementary abelian p-group V.*

Then $|V| = p^k$ for some natural number k dividing n.

Proof. Let $H^\star$ be the group of automorphisms of V induced by the elements of H, and let E be the ring of endomorphisms of V generated by $H^\star$. Since E centralizes H, E is an integral domain by Schur's Lemma. As E is finite, it is a finite field $\mathrm{GF}(p^k)$. Hence, $H^\star$ is cyclic.

We may regard V as a vector space over E. As H is irreducible on V, the dimension of V over E is 1. Since the order of $H^\star$ divides p^n-1, the theory of finite fields shows that k is a divisor of n. Then $|V| = |E| = p^k$. □

Theorem 2.6 (Richard Niles). *Suppose n is a natural number, K is a normal p'-subgroup of a group H, A is a non-identity p-subgroup of H, and V is an elementary abelian p-group on which H operates. Assume that*

(i) $H/K \simeq \mathrm{PSL}(2,p^n)$,

(ii) *some Sylow p-subgroup of H lies in a unique maximal subgroup of H,*

(iii) $[V,A,A]=1$ *and*

(iv) $|V/C_V(A)| \leqslant |A|$ *and* $C_V(A) \neq C_V(H)$.

Then

(a) *A is a Sylow p-subgroup of H,*

(b) $H/C_H(V) \simeq \mathrm{SL}(2,p^n)$ *and*

(c) *$V/C_V(H)$ is a standard module for $H/C_H(V)$.*

Proof. This is proved in Lemma 2.8 of [**19**] (and is part of Lemma 2.3 of [**12**]). □

Theorem 2.7 (Chermak and Delgado). *Suppose $Q,R \in \mathfrak{F}_1(P)$. Then*

(a) $QR = RQ$ *and* $QR, Q\cap R \in \mathfrak{F}_1(P)$,

(b) $C_P(Q) \in \mathfrak{F}_1(P)$ *and* $Q = C_P(C_P(Q))$, *and*

(c) $C_P(Q\cap R) = C_P(Q)C_P(R)$.

Proof. This is part of Theorem 2.1 and Proposition 2.3 of [**11**] (and follows from Lemmas 1.1 and 3.1 of [**5**]). □

Proposition 2.8. *Suppose Q is a subgroup of P. Then*

(a) *if Q is a CL-subgroup of P, then $Q \in \mathfrak{F}_1(P)$ and $C_P(Q) = Z(Q)$;*

(b) *if $Q \in \mathfrak{F}_1(P)$, then Q is a CL-subgroup of P if and only if $Q \geqslant C_P(Q)$;*

(c) *if Q and R are CL-subgroups of R, then $QR = RQ$ and QR is a CL-subgroup of P; and*

(d) *P_{CL} and P_{MCL} are CL-subgroups of P.*

Proof. Parts (a) and (b) come from Proposition 2.4 and Corollary 2.6 of [**11**]. Then (c) follows from (a) and (b) and Theorem 2.7, and (d) follows from (c). □

Theorem 2.9. *Suppose Q is a CL-subgroup of P and A is a large abelian subgroup of P. Then*

(a) *$QA = AQ$ and QA is a CL-subgroup of P,*

(b) *$C_{QA}(Q \cap A) = Z(Q)A = AZ(Q)$ and*

(c) *P_{CL} contains $\tilde{J}(P)$.*

Proof. Theorem 3.1 and Corollary 3.2 of [**11**] give (a) and (b) and the containment $P_{\mathrm{CL}} \geqslant J(P)$. Then $Z(P_{\mathrm{CL}}) \leqslant C_P(J(P)) = Z(J(P))$. By Theorem 2.7,

$$P_{\mathrm{CL}} = C_P(Z(P_{\mathrm{CL}})) \geqslant C_P(Z(J(P))) = \tilde{J}(P).$$

□

Theorem 2.10. *Suppose Q and R are minimal CL-subgroups of P. Then*

(a) *$Q = (Q \cap R)Z(Q)$,*

(b) *$Q' = R'$,*

(c) *$|Q| = |R|$ and $|Z(Q)| = |Z(R)|$ and*

(d) *if Q is abelian, then $\mathscr{A}(P)$ is the set of all minimal CL-subgroups of P.*

Proof. Parts (a)–(c) are part of Corollary 4.2 and Theorem 4.5 of [**11**].

For (d), assume Q is abelian. By (b) and (c), every minimal CL-subgroup of P is abelian of the same order as Q. By the definition of a CL-subgroup,

$$|Q|^2 = |Q|\,|Z(Q)| \geqslant |A|\,|Z(A)| = |A|^2$$

for every abelian subgroup A of P. This gives (d). □

Our next result uses the methods of Lemma 4.3 of [**11**] to extend the lemma.

Lemma 2.11. *Suppose $K, L \lhd P = KL$ and $L = C_P(K)$. Assume that K is contained in some minimal CL-subgroup of P. Let $Z = K \cap L$.*

Then $Z = Z(K)$ and there is a bijection between

the set of all minimal CL-subgroups Q of P containing K

and

the set of all minimal CL-subgroups $Q^\star$ of L,

given by $Q^\star = Q \cap L$ and $Q = KQ^\star$. In this bijection, $|Q| = |K/Z|\,|Q^\star|$.

Proof. Since $L = C_P(K)$, $Z = K \cap C_P(K) = Z(K)$. Clearly, there is a bijection between the set of all subgroups T of P that contain K and the set of all subgroups $T^\star$ of L that contain Z, given by

$$T^\star = T \cap L \quad \text{and} \quad T = T \cap KL = K(T \cap L) = KT^\star.$$

In this bijection, we have $Z = K \cap L = (K \cap T) \cap L = K \cap (T \cap L) = K \cap T^\star$ and

$$|T| = |KT^\star| = |K|\,|T^\star|/|K \cap T^\star| = |K/Z|\,|T^\star|,$$
$$Z(T) = C_T(KT^\star) = C_T(K) \cap C_T(T^\star) = L \cap T \cap C_T(T^\star) = Z(T^\star).$$

Therefore, $|T|\,|Z(T)| = |K/Z|\,|T^\star|\,|Z(T^\star)|$. It is now clear that this bijection restricts to the desired bijection for minimal CL-subgroups. □

Lemma 2.12.

(a) *If Q is a CL-subgroup of P, then $QJ(P) \geqslant \tilde{J}(P)$.*

(b) *Some minimal CL-subgroup of P is normalized by $J(P)$ and P_{MCL}.*

(c) *If $P = J(P)$ and $d(P)^2 = |P|\,|Z(P)|$, then every minimal CL-subgroup of P is abelian.*

(d) *If every minimal CL-subgroup of P is abelian, then $\tilde{J}(P) = J(P)$.*

Proof. (a) Let $R = QJ(P)$. Then $Z(R) \leqslant C_P(J(P)) = Z(J(P))$.
By Theorems 2.7 and 2.9 and a short argument, R is a CL-subgroup of P and

$$R = C_P(Z(R)) \geqslant C_P(Z(J(P))) = \tilde{J}(P).$$

(b) This follows from Theorem 5.7 of [**11**].

(c) By Proposition 2.8 and Theorem 2.9, $P_{\mathrm{CL}} \geqslant \tilde{J}(P) \geqslant J(P) = P$ and P_{CL} is a CL-subgroup of P. Hence, $P = P_{\mathrm{CL}}$ and $f(P) = |P|\,|Z(P)| = d(P)^2$. Let A be a large abelian subgroup of P. Then $f(P) = d(P)^2 = |A|\,|Z(A)|$, and A is a CL-subgroup of P. Apply Theorem 2.10.

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(d) Here, $J(P) = P_{\mathrm{MCL}}$ by part (d) of Theorem 2.10. By Theorem 2.7 and Proposition 2.8, $J(P) = C_P(Z(J(P))) = \tilde{J}(P)$. □

Definition 2.13. Suppose Q is a subgroup of P and $\mathcal{C}$ is a central series

$$1 = Q_0 \leqslant Q_1 \leqslant \cdots \leqslant Q_k = Q$$

of Q. We define a partial ordering $\prec_{\mathcal{C}}$ on the set of all subgroups of Q as follows: $A \prec_{\mathcal{C}} B$ if $|A| = |B|$ and

(a) $|A \cap Q_i| \leqslant |B \cap Q_i|$ for $i = 1, 2, \ldots, k$ and

(b) $|A \cap Q_i| < |B \cap Q_i|$ for some i, $1 \leqslant i \leqslant k$.

Theorem 2.14. *Suppose Q is a minimal CL-subgroup of P and $x \in P$. Assume that $[x, Z(Q)]$ is abelian.*

Let

$$Z = Z(Q), \quad M = [x, Z], \quad Y = MC_Z(M) \quad \text{and} \quad T = (Q \cap Q^x)Y.$$

Then

(a) *T is a minimal CL-subgroup of P,*

(b) *$Y = Z(T)$ and $T = C_P(Y)$, and*

(c) *if x does not normalize Q, then $Z \prec_{\mathcal{C}} Y$ for every central series $\mathcal{C}$ of P.*

Proof. This is Theorem 5.5 of [**11**]. □

Theorem 2.15. *Let n be a natural number, let G be $\mathrm{SL}(2, p^n)$ and let V be an elementary abelian p-group on which G acts irreducibly. Suppose S is a Sylow p-subgroup of G and $V_0 = \{v \text{ in } V \mid S \text{ fixes } v\}$.*

Assume that G does not centralize V and that

(a) *$[V, S, S] = 0$ or*

(b) *$|V| \leqslant |V_0|^2$.*

Then V is a standard module for G.

Proof. Let F be the set of all endomorphisms of V that commute with the action of each element of G:

$$F = \mathrm{Hom}_G(V, V).$$

By Schur's Lemma, F is a division ring. Since F is finite, it is a field, by Wedderburn's Theorem. Then V is a vector space over F and it is an absolutely irreducible module for G over F, and V_0 is an F-subspace of V. Let $d = \dim_F V$. By a special case of a result of Curtis and Richen (see [**22**, Theorem 44(b), pp. 231–232] or [**20**, Theorem 3.9(b), p. 446]), $\dim_F V_0 = 1$. Since G is generated by conjugates of S and G does not centralize V,

$$d \geqslant 2. \tag{2.1}$$

We first assume (a). Then $|V| = |V_0|^d \leqslant |V_0|^2$, so that $d = 2$ and $\dim_F V/V_0 = 1$. Since S is a p-group and F has characteristic p, S centralizes V/V_0 and

$$[V, S, S] \leqslant [V_0, S] = 0,$$

which gives (b).

Thus, we may assume (b) for the rest of the proof. Let us regard V as a vector space over $\boldsymbol{F}_p$ rather than F. Set $H = N_G(S)$ and $q = p^n$. Then V_0 is a subspace of V under H. Let W be an irreducible subspace of V_0 under H. Then H/S acts irreducibly on W. From the structure of $\mathrm{SL}(2, q)$, H/S is a cyclic group of order $q - 1$, i.e. $p^n - 1$. By Lemma 2.5,

$$|W| \leqslant q. \tag{2.2}$$

Since V is irreducible under G, the subspace

$$\sum_{g \in G} W^g$$

of V is equal to V. Take an element u of G outside H. By the structure of $\mathrm{SL}(2, q)$, G is the set-theoretic union of H and the double coset HuS. Note that

$$W^x = W \quad \text{and} \quad W^{xuy} = (W^u)^y \quad \text{for all } x \text{ in } H \text{ and } y \text{ in } S.$$

Therefore,

$$V = \sum_{g \in G} W^g = W + \sum_{y \in S} (W^u)^y. \tag{2.3}$$

Recall that $W \leqslant V_0$ and $[V, S, S] = 0$, by (2.1). Therefore, for each v in W^u and y in S,

$$v^y = v + (v^y - v) = v + [v, u] \in W^u + C_V(S) = W^u + V_0,$$

and by (2.3), (2.1) and (2.2),

$$V = V_0 + W^u \quad \text{and} \quad |F| \leqslant |F|^{d-1} = |V/V_0| \leqslant |W^u| = |W| \leqslant q = |S|. \tag{2.4}$$

Then $|F| = |V_0| \geqslant |W| \geqslant |F|^{d-1}$, and $d = 2$.

Now the theorem follows from Theorem 2.6. Alternatively, let $|F| = p^k$. Since G is generated by p-elements, which act by determinant 1 on V over F, the action of G on V induces a homomorphism of G into an irreducible subgroup of $\mathrm{SL}(2, p^k)$. It is easy to see that the homomorphism has trivial kernel, so that

$$|\,\mathrm{SL}(2, q)| = |G| \leqslant |\,\mathrm{SL}(2, p^k)|.$$

Since $|F| = p^k \leqslant q$ by (2.4), $q = p^k = |F|$ and V is a standard module for G. □

Theorem 2.16. *Suppose S is a Sylow p-subgroup of a group G, K and L are normal p'-subgroups of G, and n is a natural number. Assume that G acts on an elementary*

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abelian p-group M and

(i) $G/L \cong \mathrm{SL}(2,p^n)$, $K \geqslant L$ *and* $K/L = Z(G/L)$,

(ii) $L = [L,G]$ *and* $K = \Phi(G)$,

(iii) $[M,S,S,S] = 1$,

(iv) $|M| = |C_M(S)|^2$ *and*

(v) *for each* x *in* $S^{\#}$, $C_M(x) = C_M(S)$.

Then L *centralizes* M *except possibly if* $p^n = 2$ *or* 3.

Proof. Assume that L does not centralize M. Note that S is isomorphic to a Sylow p-subgroup of $\mathrm{SL}(2,p^n)$, and hence is elementary abelian of order p^n.

Since $L \lhd G$, the kernel $C_L(M)$ of L on M is normal in G. Assume first that S centralizes $L/C_L(M)$. Let $C = C_G(L/C_L(M))$. Then C is a normal subgroup of G that contains S. So CK/K is a normal subgroup of G/K that contains SK/K. Since G/K is isomorphic to $\mathrm{PSL}(2,p^n)$, which is generated by its p-elements,

$$CK/K = G/K \quad \text{and} \quad G = CK = C\Phi(G).$$

As $\Phi(G)$ is the Frattini subgroup of G, we obtain

$$G = C \quad \text{and} \quad L = [L,G] \leqslant C_L(M).$$

This is a contradiction because L does not centralize M. Thus,

$$S \text{ does not centralize } L/C_L(M). \tag{2.5}$$

We regard M as a vector space over $\boldsymbol{F}_p$. Let $\bar{G} = G/C_G(M)$. For every element x and subgroup H of G, let $\bar{x}$ and $\bar{H}$ be the images under the canonical homomorphism of G onto $\bar{G}$. By (2.5), $\bar{S}$ does not centralize $\bar{L}$.

We show first that $p < 5$. Let y be an element of S that does not centralize $\bar{L}$. Since S is elementary abelian, y has order p. Therefore, $\bar{y}$ has order p and $\mathrm{O}_p(\bar{L}\langle\bar{y}\rangle) = 1$. By a theorem of Philip Hall and Graham Higman (see [**13**, Theorem 11.1.1, p. 359]), the linear transformation t of M over $\boldsymbol{F}_p$ induced by the action of $\bar{y}$ has minimal polynomial $(x-1)^p$ or $(x-1)^{p-1}$. Therefore, $(t-1)^{p-2} \neq 0$, which gives

$$[M,y;p-2] > 1.$$

By (iii), $[M,S;3] = 1$. Consequently, $p-2 < 3$, and $p < 5$, as desired.

To complete the proof, we assume that $n \geqslant 2$ and derive a contradiction. Since S is elementary abelian of order p^n, S is not cyclic. By [**13**, Theorem 6.2.4],

$$L = \langle C_L(u) \mid u \in S^{\#} \rangle.$$

For each u in $S^{\#}$, $C_L(u)$ preserves $C_M(u)$, which is equal to $C_M(S)$, by (v). Therefore, $C_M(S)$ is preserved by L and hence by LS.

Let $L^* = [L, S]$. Since LS preserves $C_M(S)$, the centralizer of $C_M(S)$ in LS is a normal subgroup of LS that contains S and, therefore, L^*. So

$$C_M(S) \leqslant C_M(L^*).$$

By (2.5), $[M, L^*] > 1$ because L^* does not centralize M. By Lemma 2.1, $M = C_M(L^*) \times [M, L^*]$. Hence,

$$[M, L^*] \cap C_M(S) \leqslant [M, L^*] \cap C_M(L^*) = 1.$$

However, $[M, L^*]$ is a non-trivial S-invariant subgroup of M, and so must contain non-identity fixed elements under S. This contradiction completes the proof of Theorem 2.16. □

Lemma 2.17. *Assume the hypothesis of Theorem 2.16, and suppose also that*

(i) *G acts faithfully and irreducibly on M,*

(ii) *$L > 1$ and $p^n = 3$, and*

(iii) *$G = \mathrm{SO}_2(G)$ and $K = \Phi(\mathrm{O}_2(G))$.*

Regard M as a module for G over $\boldsymbol{F}_p$. Then

(a) *the restriction of M to KS contains a unique irreducible submodule N subject to being also irreducible for K,*

(b) *the representation of G on M is induced from the representation of KS on N,*

(c) *the restriction of M to K is the direct sum of N and three other irreducible submodules N_1, N_2, N_3,*

(d) *no two of N, N_1, N_2, N_3 are isomorphic as K-modules,*

(e) *the modules N_1, N_2, N_3 are cyclically permuted by S,*

(f) *S acts trivially on N, and*

(g) *M is the only K-submodule of M that contains $C_M(S)$.*

Proof. Here, $|G/\mathrm{O}_2(G)| = |S| = 3$. Let $Q = \mathrm{O}_2(G)$. From (iii) and Theorem 2.16, $K = \Phi(Q) \geqslant L$ and $G/L \cong \mathrm{SL}(2,3)$. From the structure of $\mathrm{SL}(2,3)$,

$$G/L = (SL/L)(G/L)' = SG'L/L \quad \text{and} \quad G = SG'L.$$

Assume first that K is cyclic. Then the automorphism group of K is an abelian 2-group. So K is centralized by S, G' and itself. As $G = SG'L \leqslant SG'K$, Theorem 2.16 yields

$$1 = [K, G] \geqslant [L, G] = L,$$

contrary to (ii). Thus, K is not cyclic.

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If every characteristic abelian subgroup of Q is cyclic, then a theorem of Philip Hall (see [**13**, p. 198]) asserts that Q is a central product of two subgroups E and R, where $E = 1$ or E is an extra-special 2-group, and $R = 1$ or R is a 2-group of maximal class. Then $\Phi(Q)$ is abelian, hence cyclic. But $\Phi(Q) = K$, which is not cyclic, which is a contradiction. Thus, there exists a non-cyclic abelian characteristic subgroup A of Q.

Since Q is normal in G, A is normal in G. As M is irreducible under G, we may decompose it as a direct sum

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_r$$

of homogeneous A-modules transitively permuted by G. Moreover, M_1 is irreducible under the stabilizer $N_G(M_1)$ in G, and M is induced from the representation of $N_G(M_1)$ on M_1.

Now, M_1 is a direct sum of isomorphic irreducible A-modules. As A is abelian, this forces $A/C_A(M_1)$ to be cyclic. Hence, $C_A(M_1) > 1$, and $M_1 < M$ by (i). Let H be a maximal subgroup of G containing $N_G(M_1)$, and let N be the sum of M_1^h as h ranges over H. Then N is an irreducible H-module that is induced from the irreducible $N_G(M_1)$-module M_1, and M is induced from the representation of H on N. Therefore, H is the stabilizer of N in G, and M is the direct sum

$$M = \bigoplus \sum_{g \in T} N^g \tag{2.6}$$

as g ranges over a transversal T to H in G (i.e. $HT = G$ and $Hu \neq Hv$ for $u \neq v$ in T).

Let u be a generator of S. If S does not fix any subspace N^g in (2.6), then it permutes these subspaces in cycles of length 3, and

$$M = M^* \oplus M^{*u} \oplus M^{*u^2}$$

for some subspace M^* of M. Then

$$C_M(S) = C_M(u) = \{x + x^u + x^{u^2} \mid x \in M^*\}$$

and $|M| = |M^*|^3 = |C_M(S)|^3 > |C_M(S)|^2$. But $|M| = |C_M(S)|^2$ from Theorem 2.16, which is a contradiction. Thus, S fixes some subspace N^g in (2.6).

By replacing M_1 by $M_1^{g^{-1}}$, we may replace N^g by N. Then S is contained in the stabilizer of N in G, which is the maximal subgroup H of G. Since $\Phi(G)$ is the intersection of all the maximal subgroups of G and $K = \Phi(G)$, we have $K \leqslant H$. So $SK \leqslant H$.

Now H/K is a maximal subgroup of G/K that contains the Sylow 3-subgroup SK/K of G/K. From Theorem 2.16, G/K is isomorphic to $\mathrm{PSL}(2,3)$ and thus to the alternating group of degree 4. Therefore, SK/K itself is a maximal subgroup of G/K. Hence,

$$H/K = SK/K, \quad H = SK, \quad |G : H| = |G/K : H/K| = 4,$$

and the transversal T has cardinality 4.

Since $K \lhd G$ and K preserves N, K preserves N^g for every g in G. Thus, G/K acts as a permutation group on the four summands N^g in (2.6), and the group H/K of order 3 is the stabilizer of N in G/K. It is easy to see that S permutes the other three summands cyclically. Let N_1 be one of them. Then $N_1 \oplus N_1^u \oplus N_1^{u^2}$ is irreducible under SK,

$$C_M(S) = C_N(S) \oplus \{x + x^u + x^{u^2} \mid x \in N_1\} \quad \text{and} \quad M = N \oplus (N_1 \oplus N_1^u \oplus N_1^{u^2}). \quad (2.7)$$

Now we obtain (a), (b), (c) and (e).

Consider the dimensions of various subgroups of M as vector spaces over the prime field $\boldsymbol{F}_p$. Since $|N|^4 = |M| = |C_M(S)|^2$ and $|N_1| = |N|$, (2.7) gives

$$4 \dim N = \dim M = 2 \dim C_M(S) = 2(\dim C_N(S) + \dim N) \leqslant 4 \dim N.$$

Therefore, $\dim C_N(S) = \dim N$, and S centralizes N, which gives (f).

As KS is irreducible on N and S centralizes N, K acts irreducibly on N and $[K, S]$ centralizes N. As $K \lhd G$, we see that K acts irreducibly on N^g for every g in G. Since $M_1 \leqslant N$ and $A \lhd G$ and M_1 is a homogeneous component of M as an A-module, none of the summands N_1, N_1^u, $N_1^{u^2}$ is isomorphic to N as an A-module, or, *a fortiori*, as a K-module. Thus, no two of the four distinct summands of M in (2.7) are isomorphic as K-modules, as claimed in (d).

Suppose M^* is a K-submodule of M that contains $C_M(S)$. Then $M^* \geqslant N$. If $M^* < M$, then we may assume that M^* is a maximal K-submodule of M. By the Jordan–Hölder Theorem for modules, M/M^* is isomorphic as a K-module to N_1, N_1^u or $N_1^{u^2}$. If $M/M^* \cong N_1$, then M^* contains N, N_1^u and $N_1^{u^2}$, and hence (by (2.7)),

$$M^* \text{ contains } (N \oplus N_1^u \oplus N_1^{u^2}) + C_M(S), \text{ which is } M.$$

This is a contradiction. Similar contradictions for the other cases show that $M^* = M$. This proves (g) and completes the proof of the lemma. □

Lemma 2.18. *Suppose p, G, S, K and L satisfy conditions (i) and (ii) of Theorem 2.16 for $n = 1$, and p is 2 or 3. Let G act on elementary abelian p-subgroups M_1, M_2 and M. Regard M_1, M_2 and M as vector spaces over the prime field $\boldsymbol{F}_p$. Assume that f is an $\boldsymbol{F}_p$-bilinear function on $M_1 \times M_2$ into M and*

(i) *$f(u^g, v^g) = f(u, v)^g$ for all u in M_1, v in M_2, and g in G, and*

(ii) *$f(u, v) \neq 0$ for some u in M_1 and v in M_2.*

Assume also that

(iii) *G acts irreducibly on M_1 and M_2, and L centralizes M,*

(iv) *for all u in $C_{M_1}(S)$ and v in $C_{M_2}(S)$, $f(u, v) = 0$,*

(v) *for $i = 1, 2$, $|M_i| = |C_{M_i}(S)|^2$ and L does not centralize M_i,*

(vi) *if* $p = 2$, *then* G *is a dihedral group of order* $2 \cdot 3^k$ *for some natural number* k, *and*

(vii) *if* $p = 3$, *then* $G = \mathrm{SO}_2(G)$ *and* $K = \Phi(\mathrm{O}_2(G))$.

Then $p = 2$ *and* G *centralizes the image of* f.

Proof. Here, $|S| = p^n = p$. Let x be a generator of S. Take i to be 1 or 2. By (v), S acts faithfully on M_i. We embed S in the endomorphism ring of M_i. Since $p \leqslant 3$ and M_i has characteristic p,

$$(x-1)^p = x^p - 1 = 0 \quad \text{and} \quad 0 = (x-1)^3 = (x^j-1)(x^k-1)(x^l-1)$$

for all natural numbers j, k and l. Therefore,

$$[M_i, S, S, S] = 0 \quad \text{for } i = 1, 2.$$

Assume first that $p = 3$. We work towards a contradiction. By Lemma 2.17, $C_{M_1}(S)$ contains a non-zero K-submodule N of M_1, and $C_{M_2}(S)$ contains a non-zero K-submodule N^* of M_2.

Let X be the set of all u in M_1 such that

$$f(u, v) = 0 \quad \text{for all } v \text{ in } N^*.$$

By (i) and (iv), X is a K-submodule of M_1 that contains $C_{M_1}(S)$. By Lemma 2.17, $X = M_1$. Similarly, the set Y of all v in M_2 satisfying

$$f(u, v) = 0 \quad \text{for all } u \text{ in } M_1$$

is a G-submodule of M_2 containing N^*. As G acts irreducibly on M_2, we have $Y = M_2$. Thus, f is identically zero, contrary to (ii). This contradiction shows that $p = 2$.

Let F be a finite field extension of $\boldsymbol{F}_2$ that is a splitting field for all of the subgroups of G. Let

$$M_i^* = F \otimes_{\boldsymbol{F}_2} M_i \quad \text{for each } i$$

and let

$$M^* = F \otimes_{\boldsymbol{F}_2} M.$$

Then f extends uniquely to a bilinear function over F on $M_1^* \times M_2^*$ into M^*, which we also call f for convenience. Part (i) of the hypothesis is still valid, but M_1^* and M_2^* need not be irreducible. However, by [**6**, pp. 471–472],

$$\text{each of } M_1^* \text{ and } M_2^* \text{ is a direct sum of irreducible modules.} \tag{2.8}$$

It is easy to see that $C_{M_i^*}(S) = F \otimes_{\boldsymbol{F}_2} C_{M_i}(S)$ for each i, and hence, from (iv), that

$$\text{for all } u \text{ in } C_{M_1^*}(S) \text{ and } v \text{ in } C_{M_2^*}(S), \quad f(u, v) = 0. \tag{2.9}$$

To complete the proof, we wish to show that G centralizes the image of f. By (2.8), it suffices to show that, for arbitrary irreducible summands N_1 of M_1 and N_2 of M_2, G centralizes $f(u,v)$ for every u in N_1 and v in N_2.

By (vi), G is a dihedral group of order $2\cdot 3^k$ for some natural number k. Let H be the Sylow 3-subgroup of G, so that $|G/H| = 2$. Let h be a generator of H. By Theorem 2.16, G/L is isomorphic to $\mathrm{SL}(2,2)$, the dihedral group of order 6. Hence, $L < H$.

Now we take i to be 1 or 2 in order to choose notation. By (v), L does not centralize M_i. So $C_{M_i}(L) < M_i$. As G is irreducible on M_i and $L \lhd G$, the subspace $C_{M_i}(L)$ of M_i is invariant under G and must be zero. Therefore,

$$C_{N_i}(L) \leqslant C_{M_i^*}(L) = F \otimes_{\boldsymbol{F}_2} C_{M_i}(L) = 0,$$

and $G/C_G(N_i)$ is a dihedral group of order $2\cdot 3^m$ for some natural number m. Since F is a splitting field for H and N_i is irreducible under G, it is easy to see that N_i is induced from a one-dimensional representation of H. Thus, N_i has dimension 2 and $C_{N_i}(S)$ has dimension 1. Let u_i be a non-zero vector in $C_{N_i}(S)$ and $v_i = u_i^h$.

We continue with the assumption that i is 1 or 2. Then u_i, v_i is a basis of N_i. Since S^{h^2} is different from S and S^h when taken modulo $C_G(N_i)$, the subspace $C_{N_i}(S^{h^2})$ is different from $\langle u_i\rangle$ and $\langle v_i\rangle$. So

$$C_{N_i}(S^{h^2}) = \langle u_i^{h^2}\rangle = \langle u_i + \lambda_i v_i\rangle \text{ for some non-zero element } \lambda_i \text{ in } F.$$

Now we apply the notation chosen above for $i = 1$ and $i = 2$. By (2.9), $f(u_1, u_2) = 0$. Therefore,

$$0 = 0^g = f(u_1^g, u_2^g) = f(v_1, v_2),$$

and similarly,

$$0 = f(u_1 + \lambda_1 v_1, u_2 + \lambda_2 v_2) = \lambda_2 f(u_1, v_2) + \lambda_1 f(v_1, u_2).$$

Hence,

$$f(v_1, u_2) = \lambda_1^{-1}\lambda_2 f(u_1, v_2).$$

This shows that the image of f on $N_1 \times N_2$ into M^* is spanned by $f(u_1, v_2)$ and is either one dimensional or zero. Since M^* has characteristic 2, S centralizes this image. As G is generated by S and S^h, G centralizes this image. As mentioned above, this suffices to prove the lemma. □

Lemma 2.19. *Assume* (E_0). *Then*

(a) $G = \langle S, S^y\rangle$ *for every element* y *in* $G \setminus N_G(SK)$ *and*

(b) $Z(S) \lhd G$ *if and only if* $Z(S) = Z(G)$.

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Proof. (a) This is part of Lemma 2.7 of [**12**].

(b) Obviously, $Z(S) \lhd G$ if $Z(S) = Z(G)$.

Assume conversely that $Z(S) \lhd G$. Take some element y in $G \setminus N_G(SK)$. Since $C_G(Z(S))$ is a normal subgroup of G that contains S, it contains S^y. Hence, by (a), $C_G(Z(S)) = G$, and $Z(S) \leqslant Z(G)$. Since

$$Z(G) \leqslant C_G(\mathrm{O}_p(G)) \leqslant \mathrm{O}_p(G) \leqslant S$$

by (E_0), we obtain $Z(S) = Z(G)$. □

3. Proof of Theorems A, B, D and E

Let $T = \mathrm{O}_p(G)$. In this section, we prove Theorems A, B, D and E and Remark 1.1. Then we reduce part of Theorem C to studying the chief factors within a particular subgroup of T.

Recall conditions (E_0) and (H) from § 1. Assume condition (E_0). Let

$$q = p^n, \quad Z = Z(T) \quad \text{and} \quad L = C_G(Z).$$

Theorem 3.1. *Assume (H). Then*

(a) *$Z(G) \leqslant Z(S) \leqslant Z$ and $T \leqslant L \leqslant K$,*

(b) *$G/L \simeq \mathrm{SL}(2, q)$ and $Z/Z(G)$ is a standard module for G/L,*

(c) *$Z(S)/Z(G) = C_{Z/Z(G)}(S/Z(G))$,*

(d) *$\mathscr{A}(T)$ is a proper subset of $\mathscr{A}(S)$,*

(e) *whenever $A \in \mathscr{A}(S) - \mathscr{A}(T)$, then $AT = S$ and $(A \cap T)Z \in \mathscr{A}(T)$,*

(f) *$Z \leqslant Z_2(S)$,*

(g) *if p is odd or $n = 1$, then $Z = [Z, G] \times Z(G)$,*

(h) *$K/L = Z(G/L)$, and*

(i) *$L/T = [L/T, G/T] = [L, G]T/T$ and $K/T = \Phi(G/T)$.*

Moreover, let W_1 be the subgroup of T that contains $Z(G)$ and satisfies $W_1/Z(G) = Z(T/Z(G))$. Then

(j) *if $q > 2$, then $L = TC_L(W_1)$,*

(k) *if $q = 2$, then G/T is a dihedral group and $\frac{1}{2}|L/T|$ is a power of 3, and*

(l) *if $q = 3$, then $G/T = (S/T)\,\mathrm{O}_2(G/T)$ and $K/T = \Phi(\mathrm{O}_2(G/T))$.*

Proof. Obviously, $T \leqslant C_G(Z(T)) = L$. Since (H) includes condition (E) of [**12**], parts (a)–(g) of the theorem follow from Lemma 2.9 of [**12**]. Part (h) follows from (H) and part (b). Parts (i)–(l) follow from Lemmas 3.5 and 2.2 in [**12**]. □

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Lemma 3.2. *Assume (H). Then*

(a) $Z(G) < Z(S) < Z = \Omega_1(Z)Z(G)$ *and* $|Z/Z(S)| = |S/T| = q$,

(b) $[Z, S] \leqslant Z(S)$, *and*

(c) *for each* x *in* $Z - Z(S)$, $C_S(x) = T$.

Proof. This follows from Theorem 3.1 above and Lemma 3.1 of [**12**]. □

Theorem 3.3. *Suppose* G *satisfies (H) and* S_{MCL} *is not normal in* G. *Then some minimal CL-subgroup* Q *of* S *is not contained in* T. *For any such subgroup,*

(a) $S = QT = Z(Q)T$ *and* $Q \cap Z = Z(S)$,

(b) $(Q \cap T)Z$ *is a minimal CL-subgroup of* S *and of* T,

(c) Q' *is a characteristic subgroup of* T *and of* S,

(d) $S = TC_S(Q')$ *and* $G = TC_G(Q')$,

(e) $Q = (Q \cap T)Z(Q)$,

(f) $|Q/(Q \cap T)| = q$, *and*

(g) $f(S) = f(T)$ *and the CL-subgroups of* T *are the CL-subgroups of* S *that are contained in* T.

Proof. Suppose every minimal CL-subgroup of S is contained in T. Then $f(S) = f(T)$ and the minimal CL-subgroups of S and T coincide. So

$$S_{\mathrm{MCL}} = T_{\mathrm{MCL}} \lhd G,$$

contrary to hypothesis. This contradiction shows that Q exists.

Now, (a)–(c) and the first part of (d) follow directly from Theorem 4.7 and Corollary 4.8 of [**11**], and (g) follows from (b). Hence, $Q' \lhd G$.

Take y in $G \setminus N_G(SK)$. Since Q' is normal in G, so are $C_G(Q')$ and $TC_G(Q')$. Since $S = TC_S(Q') \leqslant TC_G(Q')$, we also have $S^y \leqslant TC_G(Q')$. By Lemma 2.19, $G = \langle S, S^y \rangle \leqslant TC_G(Q')$. So $G = TC_G(Q')$, which completes the proof of (d).

Let $R = (Q \cap T)Z$. By (b) and Theorem 2.10,

$$Q = (Q \cap R)Z(Q) \leqslant (Q \cap T)Z(Q) \leqslant Q,$$

which yields (e). By (a) and Lemma 3.2,

$$|Q/(Q \cap T)| = |QT/T| = |S/T| = q.$$

Thus, (f) is valid. □

Now we can prove most of our main results. Note first that Remark 1.1 follows from Theorems 2.7 and 2.10, Proposition 2.8 and Lemma 2.12.

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Proof of Theorem B. Define T_Φ by analogy with the definition of S_Φ. Then T_Φ is characteristic in T and hence normal in G. If $Z(S)$ is not normal in G, then $Z(S) \neq Z(G)$ and we obtain condition (H). By Lemma 3.2 above and Remark 4.9 of [**11**], the theorem follows. □

Proof of Theorem D. Theorem 2.10 gives (a) and (c). To prove (b), assume S_{MCL} is not normal in G. If $Z(S) \triangleleft G$, then Lemma 2.19 yields

$$Z(S) \cap Q' \leqslant Z(S) = Z(G) \quad \text{and} \quad Z(S) \cap Q' \triangleleft G.$$

So assume $Z(S)$ is not normal in G. Then (H) holds. By Theorem 3.3,

$$G = TC_G(Q') \leqslant N_G(Z(S) \cap Q') \quad \text{and} \quad Z(S) \cap Q' \triangleleft G,$$

as desired. □

Proof of Theorem E. As in Theorem B of [**12**], let

$$S_0 = \begin{cases} [\Phi(S), S]\Phi(\Phi(S)) & \text{if } p = 2, \\ [[\Phi(S), S], S]\Phi(\Phi(S)) & \text{if } p = 3, \\ [\Phi(S), S]\mho^1(S) & \text{if } p > 3. \end{cases}$$

We wish to find a pair of characteristic subgroups S_1, S_2 that satisfies (P) and the condition that $f(S_2) = f(\tilde{J}(S))$. By Theorem D of [**12**], we can satisfy (P) by taking

$$S_1 = [ZJ(S), S] \cap Z(S) \quad \text{and} \quad S_2 = \tilde{J}(S) \quad \text{if } S \neq \tilde{J}(S)$$

and

$$S_1 = \mho^1(Z(S)) \quad \text{and} \quad S_2 = S \quad \text{if } S = \tilde{J}(S) \text{ and } \mho^1(Z(S)) > 1.$$

Since we have $f(S_2) = f(\tilde{J}(S))$ in both cases, we may assume that $S = \tilde{J}(S)$ and $\mho^1(Z(S)) = 1$. So $Z(S)$ is elementary abelian.

Let Q be any minimal CL-subgroup of S. If $Q' > 1$, then Theorem D yields that we can satisfy (P) by taking $S_1 = Z(S) \cap Q'$ and $S_2 = S_{\mathrm{MCL}}$. Since $f(S_{\mathrm{MCL}}) = f(S)$ and $S = \tilde{J}(S)$, this pair satisfies (P'). Hence, we may assume that $Q' = 1$. By Theorem 2.10, the minimal CL-subgroups of S coincide with the large abelian subgroups of S. Thus, we will have $f(S_2) = f(\tilde{J}(S))$ if and only if $d(S_2) = d(S)$.

Now we return to Theorem D of [**12**]. Assume $S_0 > 1$. Then we are in case (c) of Theorem D of [**12**], in which $S_1 = Z(S) \cap S_0$ and S_2 is an intersection of subgroups $\mathrm{O}_p(G^*)$ for a family of groups G^* that satisfy (E_0).

Take a large abelian subgroup A of S for which $|A \cap S_2|$ is as large as possible. If $A \leqslant S_2$, then $d(S_2) = d(S)$, as desired. We assume that A is not contained in S_2 and work towards a contradiction.

Clearly, A is not contained in $\mathrm{O}_p(G_1)$ for some group G_1 in the family of groups G^* above. Let $P = \mathrm{O}_p(G_1)$ and $B = (A \cap P)Z(P)$. By Lemma 2.9 of [**12**], B is a large abelian subgroup of S. Since $B \leqslant P$, we have $B \neq A$. Therefore, $Z(P)$ is not contained in A.

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By Theorem C of [**12**], $Z(P) \leqslant \mathrm{O}_p(G^*)$ for every group G^* above. Therefore,

$$B \cap S_2 \geqslant (A \cap S_2)Z(P) > A \cap S_2,$$

contrary to the choice of A. This contradiction shows that $A \leqslant S_2$, as desired.

This leaves us with the case in which $S = \tilde{J}(S)$ and $Q' = S_0 = 1$. Since $Q' = 1$, Theorem 2.10 and Lemma 2.12 give parts (b) and (e) of Theorem E. Since $S_0 = 1$, we obtain parts (c) and (d). Finally, since $C_G(\mathrm{O}_p(G)) \leqslant \mathrm{O}_p(G) < S$, we obtain part (a). □

Proof of Theorem A. Assume that there exists no pair of non-identity characteristic subgroups of S satisfying condition (P). Since condition (P') includes condition (P), Theorem E yields conditions (a), (b), (c), (d) and (f) of Theorem A. In particular, $\tilde{J}(S) = S$.

By Theorem B, $Z(S) \lhd G$ or $S_\Phi \lhd G$ for every group G satisfying (E_0). Since $\tilde{J}(S) = S$, the subgroup S_Φ is a characteristic subgroup of $\tilde{J}(S)$. Therefore, the pair $Z(S)$, S_Φ satisfies (P). Since $Z(S) > 1$, we must have $S_\Phi = 1$. Now Theorem B gives us condition (e) of Theorem A. □

We have now proved Remark 1.1 (after Theorem 3.3) and Theorems A, B, D and E. So we turn our attention to Theorem C.

3.0.1. *Henceforth in this article, we assume the hypothesis of Theorem C.*

Then $\tilde{J}(S) = S$. Clearly, we may assume $Z(S) \neq Z(G)$. Then G satisfies condition (H).

Take a central series $\mathcal{C}$ of S. Define a partial ordering $\prec = \prec_{\mathcal{C}}$ on the set of all subgroups of S as in Definition 2.13. Consider the centres $Z(Q)$ for all the minimal CL-subgroups Q that are not contained in T. By Theorem 2.10, the order $|Z(Q)|$ is the same for all the choices of Q. Choose Q_0 so that $Z(Q_0)$ is maximal under $\prec$, that is, no choice of Q satisfies $Z(Q_0) \prec Z(Q)$.

Proposition 3.4. *Take Q_0 as above. Then*

(a) *K/T is a p'-group,*

(b) *$N_G(SK)$ is the unique maximal subgroup of G that contains S,*

(c) *$S = Q_0T = Z(Q_0)T$, and*

(d) *for every element y in $G - N_G(SK)$,*

$$G = \langle S, S^y \rangle = \langle Q_0, Q_0^y \rangle T = \langle Z(Q_0), Z(Q_0)^y \rangle T.$$

Proof. Lemma 2.7 of [**12**] gives (a) and (b), and gives $\langle S, S^y \rangle = G$ for (d). Theorem 3.3 above gives (c). Then (c) gives

$$G = \langle S, S^y \rangle = \langle (Q_0T)^y, (Q_0T)^y \rangle = \langle Q_0, Q_0^y \rangle T.$$

Similarly, $G = \langle Z(Q_0), Z(Q_0)^y \rangle T$. □

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Now we obtain our first main reduction.

Proposition 3.5. *Let $\hat{Z}$ be the subgroup of T generated by the subgroups $Z(R)$ as R ranges over all of the minimal CL-subgroups of T. Then $[Z(Q_0), S] \leqslant \hat{Z}$.*

Proof. Let $W = Z(Q_0)$ and $Q_1 = (Q_0 \cap T)Z$. Note that $\hat{Z}$ is a characteristic subgroup of T and hence a normal subgroup of G. We must show that W centralizes the quotient group $S/\hat{Z}$.

By Proposition 3.3, Q_1 is a minimal CL-subgroup of T (and of S). So, by Lemma 2.12, $\tilde{J}(S) \leqslant Q_1 J(S)$. Since $S = \tilde{J}(S)$,

$$S = Q_1 J(S). \tag{3.1}$$

Since $Z = Z(T) \leqslant Z(Q_1) \leqslant \hat{Z}$,

$$Q_1 = (Q_0 \cap T)Z \leqslant (Q_0 \cap T)\hat{Z}.$$

As W centralizes Q_0,

$$W \text{ centralizes } Q_1\hat{Z}/\hat{Z}. \tag{3.2}$$

Now take any large abelian subgroup A of S and any element x of A. By Theorems 2.9 and 2.3, WA is a subgroup of S, and $(WA)'$ is abelian. Hence, $[x, W]$ is abelian. Let

$$M = [x, W], \quad Y = MC_W(M) \quad \text{and} \quad R = (Q_0 \cap Q_0^x)Y.$$

If x normalizes Q_0, then x normalizes W and

$$[x, W] \leqslant W \cap T = Z(Q_0) \cap T \leqslant Z(Q_1) \leqslant \hat{Z}.$$

Assume x does not normalize Q_0. By Theorem 2.14, R is a minimal CL-subgroup of S and $Y = Z(R)$; moreover, $W \prec Y$. Therefore, $R \leqslant T$ by our choice of Q_0, and

$$[x, W] = M \leqslant Y = Z(R) \leqslant \hat{Z}.$$

This shows that in all cases, $[x, W] \leqslant \hat{Z}$. Since x was chosen arbitrarily in A, we see that W centralizes $A\hat{Z}/\hat{Z}$. As $J(S)$ is generated by all the large abelian subgroups A of S,

$$W \text{ centralizes } J(S)\hat{Z}/\hat{Z}.$$

By (3.1) and (3.2), W centralizes $S/\hat{Z}$, as desired. □

Theorem 3.6. *For $\hat{Z}$ as in Proposition 3.5, $[\mathrm{O}^p(G), T] \leqslant \hat{Z}$.*

Proof. As in the proof of Proposition 3.5, we let $W = Z(Q_0)$ and consider the action of G on $T/\hat{Z}$ by conjugation. Let C be the kernel of this action, i.e. $C = C_G(T/\hat{Z})$, the centralizer of $T/\hat{Z}$ in G. We must show $\mathrm{O}^p(G) \leqslant C$.

Clearly, $C \lhd G$. By Proposition 3.5, W centralizes $S/\hat{Z}$ and hence $T/\hat{Z}$. So $W \leqslant C$. Take y in $G - N_G(SK)$. By Proposition 3.4,

$$G = \langle W, W^y \rangle T \leqslant CT,$$

whence $G = CT$. Therefore, G/C is a p-group, and $\mathrm{O}^p(G) \leqslant C$. □

Theorem 3.6 gives our first reduction. It shows that G centralizes all of the chief factors U/V of G for which $\hat{Z} \leqslant V < U \leqslant T$, so that we need to consider only the chief factors for which $U \leqslant \hat{Z}$.

4. The second reduction

Take Q_0 as in § 3. We fix a p'-element f in $G - N_G(SK)$ for the rest of this paper. Recall that $q = p^n$, $Z = Z(T)$ and $L = C_G(Z)$. Let

$$R_0 = Q_0^f, \quad G_0 = \langle Q_0, R_0 \rangle, \quad T_0 = G_0 \cap T,$$
$$Q_1 = (Q_0 \cap T)Z \quad \text{and} \quad R_1 = Q_1^f = (R_0 \cap T)Z.$$

We define $G^\star$, $T^\star$ and $S^\star$ after Proposition 4.5.

In § 3, we showed that $[\mathrm{O}^p(G), T]$ is contained in the group $\hat{Z}$ of Proposition 3.5. In this section, we show that it is contained in $G_0 \cap \hat{Z}$ and that $\mathrm{O}^p(G)$ is contained in G_0.

Lemma 4.1. *The following conditions are satisfied.*

(a) *Q_1 and R_1 are minimal CL-subgroups of T and S.*

(b) *$Q_0 \cap Q_1 = Q_0 \cap T$ and $|Q_0 : Q_0 \cap T| = q$.*

(c) *$Z \cap Z(Q_0) = Z \cap Q_0 = Z(S)$.*

(d) *$Q_0 \cap R_0 = Q_0 \cap Q_1 \cap R_0 \cap R_1 \leqslant T$.*

(e) *$T = C_S(Z)$.*

(f) *$Z = Z(S)Z(S)^f = (Z \cap Q_0)(Z \cap R_0) = (Z \cap Z(Q_0))(Z \cap Z(R_0))$.*

(g) *T_0 contains Q_1 and R_1.*

Proof. By Theorem 3.3, Q_1 is a minimal CL-subgroup of T, and the CL-subgroups of T are merely the CL-subgroups of S that are contained in T; moreover,

$$Q_0 \cap Z = Z(S) \quad \text{and} \quad |Q_0/(Q_0 \cap T)| = q. \tag{4.1}$$

Conjugation by f shows that R_1 is a minimal CL-subgroup of T. Thus, we obtain (a).

Since $Q_0 \cap T \leqslant Q_0 \cap Q_1 \leqslant Q_0 \cap T$, we have $Q_0 \cap T = Q_0 \cap Q_1$. So (4.1) gives (b). As $Z(S) \leqslant Z(Q_0)$, (4.1) also gives $Z(S) = Z(Q_0) \cap Z$ and (c).

By Proposition 3.4, the quotient groups Q_0K/K and R_0K/K generate G/K and hence are distinct Sylow p-subgroups of $\mathrm{PSL}(2, q)$, which must intersect in the identity subgroup. Therefore, $Q_0 \cap R_0 \leqslant S \cap K = T$ and, by (b),

$$Q_0 \cap R_0 = (Q_0 \cap T) \cap (R_0 \cap T) = Q_0 \cap Q_1 \cap R_0 \cap R_1,$$

which gives (d).

Part (e) follows from Lemma 3.2. Part (f) follows from Lemma 3.1 of [**12**] and part (c). Part (g) follows from (f) and the definition of Q_1 and R_1. □

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Part (d) of the following result shows that G_0 is smaller than one might expect.

Proposition 4.2. *The following conditions are satisfied.*

(a) $Z(Q_1)$ *and* $Z(R_1)$ *are contained in* $\langle Z(Q_0), Z(R_0)\rangle$.

(b) $Z(Q_1) \cap Z(R_1) = (Z(Q_0) \cap Z(R_0))Z$.

(c) $Q_1 \cap R_1 = (Q_0 \cap R_0)Z$.

(d) $T_0 = Q_1 R_1 = (Q_0 \cap T)(R_0 \cap T)$.

Proof. By Lemma 4.1, Q_1 and R_1 are minimal CL-subgroups of T and of S. Therefore, by Theorems 2.7 and 2.10 and Proposition 2.8,

$$\langle Q_1, R_1\rangle = Q_1 R_1, \quad Q_0 = (Q_0 \cap Q_1)Z(Q_0), \quad Z \leqslant C_S(Q_1) = Z(Q_1), \tag{4.2}$$

and $Q_1 R_1$ is a CL-subgroup of T and of S.

Since $Q_1 = (Q_0 \cap T)Z$ and $Z \leqslant Z(Q_1)$,

$$Z(Q_1) = Z(Q_1) \cap (Q_0 \cap T)Z = (Z(Q_1) \cap Q_0 \cap T)Z = (Z(Q_1) \cap Q_0)Z.$$

Clearly, $Z(Q_1) \cap Q_0$ centralizes $Q_0 \cap Q_1$ and $Z(Q_0)$. Hence, by (4.2), $Z(Q_1) \cap Q_0 \leqslant C_S(Q_0) = Z(Q_0)$. Therefore,

$$Z(Q_1) \cap Q_0 = Z(Q_1) \cap Z(Q_0) \quad \text{and} \quad Z(Q_1) = (Z(Q_1) \cap Z(Q_0))Z. \tag{4.3}$$

Let $J = Q_0 \cap R_0$. Conjugation of (4.3) by f yields $Z(R_1) \cap R_0 = Z(R_1) \cap Z(R_0)$ and $Z(R_1) = (Z(R_1) \cap Z(R_0))Z$. Therefore,

$$Z(Q_1) \cap Z(R_1) \cap J = Z(Q_1) \cap Z(R_1) \cap Z(Q_0) \cap Z(R_0) \tag{4.4}$$

and Lemma 4.1 (f) gives (a).

By Lemma 4.1 and Theorem 2.10, $J \leqslant Q_1 \cap R_1 \leqslant T$, $Q_0 \cap Q_1 = Q_0 \cap T$ and $|Q_0| = |Q_1|$. Therefore,

$$q = |Q_0 : Q_0 \cap T| = |Q_0 : Q_0 \cap Q_1| = |Q_1 : Q_0 \cap Q_1|.$$

Conjugation by f gives $|R_1 : R_0 \cap R_1| = q$. Consequently,

$$\begin{aligned} |Q_1 \cap R_1 : J| &= |Q_1 \cap R_1 : Q_1 \cap R_1 \cap J| \\ &= |Q_1 \cap R_1 : Q_1 \cap Q_0 \cap R_1 \cap R_0| \\ &= |Q_1 \cap R_1 : Q_1 \cap Q_0 \cap R_1|\,|Q_1 \cap Q_0 \cap R_1 : Q_1 \cap Q_0 \cap R_1 \cap R_0| \\ &\leqslant |Q_1 : Q_1 \cap Q_0|\,|R_1 : R_1 \cap R_0| \\ &= q^2. \end{aligned} \tag{4.5}$$

Now let $I_i = Z(Q_i) \cap Z(R_i)$ for $i = 0, 1$. Then $Z = Z(T) \leqslant I_1$. Since

$$I_0 \leqslant J \leqslant T \quad \text{and} \quad Q_1 = (Q_0 \cap T)Z = (Q_0 \cap T)Z(T),$$

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we have $I_0 \leqslant Z(Q_0) \cap T \leqslant C_S(Q_1) = Z(Q_1)$. Similarly, $I_0 \leqslant Z(R_1)$. So $I_0 \leqslant I_1$. By (4.4), $I_1 \cap J = I_1 \cap I_0 = I_0$.

By Proposition 3.4, $G = \langle Z(Q_0), Z(R_0)\rangle T$. Hence,

$$Z \cap J = Z(T) \cap Q_0 \cap R_0 \leqslant Z(G).$$

By Theorem 3.1, $Z(G) \leqslant Z$. Therefore, by (4.5),

$$q^2 = |Z/Z(G)| \leqslant |Z/(Z \cap J)| \leqslant |I_1/(I_1 \cap J)| \leqslant |Q_1 \cap R_1 : J| \leqslant q^2.$$

Since $I_1 \cap J = I_0$, we have $Z(G) = Z \cap J$ and we obtain (b) and (c).

By (b) and Theorem 2.7,

$$C_S(Q_1R_1) = C_S(Q_1) \cap C_S(R_1) = Z(Q_1) \cap Z(R_1) = (Z(Q_0) \cap Z(R_0))Z$$

and

$$Q_1R_1 = C_S(C_S(Q_1R_1)) \geqslant C_S((Z(Q_0) \cap Z(R_0))Z) \geqslant T \cap \langle Q_0, R_0\rangle = T_0.$$

Since $Q_1R_1 \leqslant T_0$ and $Z = (Z \cap Q_0)(Z \cap R_0)$ by Lemma 4.1, we obtain (d). □

Lemma 4.3. *Let P be a CL-subgroup of T. Then G_0 normalizes T_0P.*

Proof. By Proposition 4.2, $T_0 = Q_1R_1$, which is a CL-subgroup of T and of S. So T_0P is a CL-subgroup of S, and so is Q_0T_0P. Since $T_0P \leqslant T$,

$$T_0P \leqslant Q_0T_0P \cap T = (Q_0 \cap T)T_0P \leqslant Q_1T_0P = T_0P.$$

Therefore,

$$T_0P = Q_0T_0P \cap T \lhd Q_0T_0P \quad \text{and} \quad Q_0 \text{ normalizes } T_0P.$$

Similarly, R_0T_0P is a CL-subgroup of S^f, and R_0 normalizes T_0P. Since Q_0 and R_0 generate G_0, it follows that G_0 normalizes T_0P. □

Proposition 4.4. *There exists a series of subgroups*

$$T_0 = U_0 \leqslant U_1 \leqslant \cdots \leqslant U_n = T_{\mathrm{MCL}}$$

such that, for $i = 1, 2, \ldots, n$,

$$U_{i-1} \lhd U_i, \quad G_0 \text{ normalizes } U_i \quad \text{and} \quad [U_i, G_0] \leqslant U_{i-1}. \tag{4.6}$$

Proof. Consider the CL-subgroups X of T_{MCL} containing T such that

$$G_0 \text{ normalizes } X$$

and there exists a series of CL-subgroups

$$T_0 = U_0 \leqslant U_1 \leqslant \cdots \leqslant U_n = X$$

satisfying (4.6).

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Trivially, T_0 is such a subgroup. Take X of maximal order among these subgroups. We show by contradiction that $X = T_{\mathrm{MCL}}$.

Assume $X < T_{\mathrm{MCL}}$. Since T_{MCL} is generated by all minimal CL-subgroups P of T, some P is not contained in X. As X and P are CL-subgroups, $XP = PX$. Choose P such that the order of XP is as small as possible. Since G_0 normalizes T_0P by Lemma 4.3 and $X(T_0P) = XP$, G_0 normalizes XP.

Since T is nilpotent and G_0 normalizes X and XP, there exists a series of subgroups of XP, $X = V_0 < V_1 < \cdots < V_k = XP$ such that $V_{i-1} \lhd V_i$ and G_0 normalizes V_i, for $i = 1, \dots, k$. By our assumptions, there exists i such that

$$[V_i, G_0] \text{ is not contained in } V_{i-1},$$

i.e. G_0 does not centralize V_i/V_{i-1}.

As G_0 is generated by Q_0 and R_0, at least one of Q_0 and R_0 does not centralize V_i/V_{i-1}. We assume that Q_0 does not centralize V_i/V_{i-1}, as the argument for the other case is similar because

$$Q_0^f = R_0 \leqslant S^f \leqslant G_0.$$

Since Q_0 and P are minimal CL-subgroups of S, Theorem 2.10 gives

$$P = (Q_0 \cap P)Z(P) \quad \text{and} \quad XP = X(Q_0 \cap P)Z(P) = XZ(P).$$

Similarly, since $Q_0 \cap T \leqslant Q_1 \leqslant X$,

$$Q_0 = (Q_0 \cap P)Z(Q_0) \quad \text{and} \quad XQ_0 = XZ(Q_0). \tag{4.7}$$

Thus, $X \leqslant V_{i-1} < V_i \leqslant XZ(P)$. Since Q_0 does not centralize V_i/V_{i-1}, there exists w in $Z(P)$ such that

$$w \text{ lies in } V_i \quad \text{and} \quad Q_0 \text{ does not centralize the element } V_{i-1}w \text{ of } V_i/V_{i-1}.$$

By (4.7), $Z(Q_0)$ does not centralize $V_{i-1}w$. Therefore,

$$[w, Z(Q_0)] \text{ is contained in } XP \text{ but not in } V_{i-1}. \tag{4.8}$$

Let $Y = Z(Q_0)$ and $W = Z(P)$. Then $w \in W$. We now argue as in the proof of Proposition 3.5. By Theorem 2.7 and Proposition 2.8, $\mathscr{F}_1(S)$ contains Y, W and YW. Therefore, by Theorem 2.3,

$$(YW)' \text{ is abelian.}$$

So $[w, Y]$ is abelian. Let

$$M = [w, Y], \quad L = MC_Y(M) \quad \text{and} \quad R = (Q_0 \cap Q_0^w)L.$$

Since $[w, Y]$ is not contained in V_{i-1}, it is not contained in $Q_0 \cap T$, and hence it is not contained in Q_0. Therefore, w does not normalize Q_0. As in the proof of Proposition 3.5, R is a minimal CL-subgroup of S and $R \leqslant T$. Since

$$(Q_0 \cap Q_0^w)C_Y(M) \leqslant Q_0 \cap R \leqslant Q_0 \cap T \leqslant T_0 \leqslant X,$$

we have

$$R = (Q_0 \cap Q_0^w)L = (Q_0 \cap Q_0^w)C_Y(M)M \leqslant XM \leqslant XR.$$

Hence, $XR = XM$ and $V_{i-1}R = V_{i-1}M$.

Recall that $M = [w, Y]$ and that w lies in V_i but Y does not centralize w, modulo V_{i-1}. As V_iY/V_{i-1} is a p-group and Y normalizes V_i,

$$1 < V_{i-1}M/V_{i-1} \leqslant [V_i/V_{i-1}, V_iY/V_{i-1}] < V_i/V_{i-1}.$$

Therefore, $X \leqslant V_{i-1} < V_{i-1}M = V_{i-1}R < V_i \leqslant XP$, which yields $X < XR < XP$ and $|XR| < |XP|$. This contradicts our choice of P and proves the proposition. □

Proposition 4.5. *Let* $G^\star = \langle Z(Q_0), Z(R_0)\rangle$ *and* $T^\star = \langle Z(Q_1), Z(R_1)\rangle$. *Then*

(a) $G = G^\star T$,

(b) $T^\star = Z(Q_1)Z(R_1)$,

(c) $T^\star \lhd G_0$,

(d) $[G^\star, T_0] \leqslant T^\star$, *and*

(e) $G^\star = C_G(Q_0 \cap R_0)$ *and* $T^\star = G^\star \cap T = \mathrm{O}_p(G^\star)$.

Proof. Proposition 3.4 gives (a). By Theorem 2.7, $\mathfrak{F}_1(S)$ contains $Z(Q_1)$ and $Z(R_1)$ and (b) is valid. Note that, similarly, $\mathfrak{F}_1(S)$ contains $T^\star$ and $\langle T^\star, Z(Q_0)\rangle = T^\star Z(Q_0)$.

Recall that $Q_1 = (Q_0 \cap T)Z(T)$. Hence, $Z(Q_0) \cap T \leqslant Z(Q_1) \leqslant T^\star \leqslant T$. Therefore,

$$T^\star = T^\star(Z(Q_0) \cap T) = T^\star Z(Q_0) \cap T \lhd T^\star Z(Q_0),$$

whence $Z(Q_0)$ normalizes $T^\star$.

By Theorem 2.10, $Q_1 = (Q_1 \cap R_1)Z(Q_1)$. Since $Z(Q_1) \leqslant T^\star$ and $Q_1 \cap R_1$ centralizes $T^\star$, Q_1 normalizes $T^\star$. By Theorem 3.3,

$$Q_0 = (Q_0 \cap T)Z(Q_0) \leqslant \langle Q_1, Z(Q_0)\rangle.$$

So Q_0 normalizes $T^\star$. Similarly, R_0 normalizes $T^\star$. Hence, $T^\star \lhd G_0$, which is (c).

Recall that $T_0 = Q_1R_1$. By Theorem 2.10,

$$Q_1 = (Q_1 \cap R_0)Z(Q_1) \leqslant (Q_1 \cap R_0)T^\star.$$

Hence, $Z(R_0)$ centralizes $Q_1T^\star/T^\star$. Similarly, $Z(R_0)$ centralizes $R_1T^\star/T^\star$, and $Z(Q_0)$ centralizes $Q_1T^\star/T^\star$ and $R_1T^\star/T^\star$. Therefore, $G^\star$ centralizes $T_0/T^\star$, which gives (d).

Let $C = C_G(Q_0 \cap R_0)$. Clearly, $G^\star = \langle Z(Q_0), Z(R_0)\rangle \leqslant C$. By (a), $G = G^\star T$. Hence,

$$C = C \cap G^\star T = G^\star(C \cap T).$$

By Proposition 4.2, $T^\star \leqslant G^\star$ and $Q_1 \cap R_1 = (Q_0 \cap R_0)Z$. Therefore,

$$C \cap T = C_T(Q_0 \cap R_0) = C_T(Q_1 \cap R_1),$$

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and Theorem 2.7 yields

$$C \cap T = C_T(Q_1)C_T(R_1) = Z(Q_1)Z(R_1) = T^\star \quad \text{and} \quad C = G^\star(C \cap T) = G^\star T^\star = G^\star.$$

Thus, $T^\star = C \cap T = G^\star \cap T$.

Since $G^\star/T^\star = G^\star/(G^\star \cap T) \simeq G^\star T/T = G/T$ and $T = \mathrm{O}_p(G)$, we obtain

$$1 = \mathrm{O}_p(G/T) \quad \text{and} \quad \mathrm{O}_p(G^\star/T^\star) = 1.$$

Hence, $T^\star = \mathrm{O}_p(G^\star)$, which completes the proof of (e) and of the proposition. □

Henceforth, we define $G^\star$ and $T^\star$ as in Proposition 4.5, and let $S^\star$ be $S \cap G^\star$.

Theorem 4.6. *Take $G^\star$, $S^\star$ and $T^\star$ as above. Then*

(a) *$S^\star = Z(Q_0)T^\star$ and $S^\star$ is a Sylow p-subgroup of $G^\star$,*

(b) *$Z(Q_0)T_0$ is a Sylow p-subgroup of G_0,*

(c) *$\mathrm{O}^p(G) = \mathrm{O}^p(G^\star)$, and*

(d) *$[T, \mathrm{O}^p(G)] \leqslant T^\star$.*

Proof. Let $Q = Q_0$. Since $Z(Q) \leqslant G^\star$ and $T^\star = G^\star \cap T$ (by Proposition 4.5), we have $Z(Q) \cap T^\star = Z(Q) \cap T$. Therefore,

$$Z(Q)T^\star/T^\star \simeq Z(Q)/(Z(Q) \cap T^\star) = Z(Q)/(Z(Q) \cap T) \simeq Z(Q)T/T = S/T.$$

This shows that $Z(Q)T^\star/T^\star$ is a Sylow p-subgroup of $G^\star/T^\star$ and $Z(Q)T^\star$ is a Sylow p-subgroup of $G^\star$. Since $Z(Q)T^\star \leqslant S$, we obtain $S^\star = Z(Q)T^\star$ and (a). A similar proof yields (b) because $S = Z(Q)T$ and $T_0 = G_0 \cap T$.

Let x be any p'-element of $G^\star$. By Lemma 2.1,

$$[T, \langle x \rangle, \langle x \rangle] = [T, \langle x \rangle]. \tag{4.9}$$

By Theorem 3.6, $[T, \langle x \rangle] \leqslant \hat{Z}$ for

$$\hat{Z} = \langle Z(P) \mid P \text{ is a minimal CL-subgroup of } T \rangle.$$

Since

$$\hat{Z} \leqslant \langle P \mid P \text{ is a minimal CL-subgroup of } T \rangle = T_{\mathrm{MCL}},$$

we have $[T, \langle x \rangle] \leqslant T_{\mathrm{MCL}}$.

Take $U_0, \dots, U_n$ as in Proposition 4.4, i.e.

$$T_0 = U_0 \leqslant U_1 \leqslant \cdots \leqslant U_n = T_{\mathrm{MCL}} \quad \text{and} \quad [U_i, G_0] \leqslant U_{i-1} \quad \text{for } i = 1, \dots, n.$$

Obviously, $G^\star \leqslant G_0$. Then $[T, \langle x \rangle] \leqslant U_n$ and, by (4.9), $[T, \langle x \rangle] = [T, \langle x \rangle, \langle x \rangle] \leqslant [U_n, \langle x \rangle] \leqslant U_{n-1}$. Similar further arguments give $[T, \langle x \rangle] \leqslant U_0 = T_0$. Since $[T_0, \langle x \rangle] \leqslant T^\star$ by Proposition 4.5, we obtain similarly

$$[T, \langle x \rangle, \langle x \rangle] = [T, \langle x \rangle] \leqslant T^\star. \tag{4.10}$$

Let

$$T_1 = \langle [T, \langle x \rangle] \mid x \text{ is a } p'\text{-element of } G^\star \rangle.$$

Then $T_1 \leqslant T^\star$. By Lemma 2.1, $[T, \langle x \rangle] \lhd T$ for every p'-element x of $G^\star$. Therefore, $T_1 \lhd T$. The definition of T_1 shows that $G^\star$ normalizes T_1. Hence, by Proposition 4.5,

$$T_1 \lhd G^\star T = G.$$

Let C be the centralizer of T/T_1 in G. Clearly, C contains every p'-element of $G^\star$, and hence contains $\mathrm{O}^p(G^\star)$. So

$$[\mathrm{O}^p(G^\star), T] \leqslant T_1. \tag{4.11}$$

Let $H = \mathrm{O}^p(G^\star)$. By Proposition 4.5, $G^\star \geqslant T^\star \geqslant T_1$. For every p'-element x in $G^\star$, (4.10) gives

$$[T, \langle x \rangle] = [T, \langle x \rangle, \langle x \rangle] \leqslant [T_1, \langle x \rangle] \leqslant [G^\star, H] \leqslant H.$$

Therefore, $T_1 \leqslant H$ and, by (4.11), $[H, T] \leqslant T_1 \leqslant H$. It follows that T normalizes H. Since H is obviously normal in $G^\star$,

$$H \lhd G^\star T = G.$$

Now, G/H is the product of the p-group $G^\star/H$ and the normal p-subgroup TH/H, and so must be a p-group. Consequently, $\mathrm{O}^p(G) \leqslant H = \mathrm{O}^p(G^\star)$. This and (4.11) give (c) and (d). □

5. Reduction to $G^\star$

In this section, we reduce the proof of Theorem C to the case in which $G = G^\star$. (We take $G^\star$, $T^\star$ and $S^\star$ as defined before Theorem 4.6.)

Lemma 5.1. *Let $I = Q_0 \cap R_0$. Then*

(a) *$Q_0 = Z(Q_0)I$ and $R_0 = Z(R_0)I$,*

(b) *$G_0 = IG^\star$ and $I \lhd G_0$,*

(c) *$G_0 \cap S = IS^\star = Z(Q_0)T_0$ and $G_0 \cap S$ is a Sylow p-subgroup of G_0, and*

(d) *$S^\star = Z(Q_0)Z(Q_1)Z(R_1)$.*

Proof. Let $Q = Q_0$ and $R = R_0$. By Proposition 4.5 and Lemma 4.1, $G^\star = C_G(I)$ and $T^\star = G^\star \cap T$, and $I \leqslant T$ and $Z = Z(S)Z(S)^f$. Therefore,

$$R_1 = (R \cap T)Z = (R \cap T)Z(S)^f Z(S) = (R \cap T)Z(S). \tag{5.1}$$

Since Q and R_1 are minimal CL-subgroups of S,

$$Q = (Q \cap R_1)Z(Q). \tag{5.2}$$

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Since $Z(S) \leqslant Z(Q)$ and $I \leqslant T$, (5.1) yields

$$Q \cap R_1 = Q \cap ((R \cap T)Z(S)) = (Q \cap R \cap T)Z(S) = IZ(S).$$

So, by (5.2), $Q = (IZ(S))Z(Q) = IZ(Q)$. Similarly, $R = IZ(R)$. Since $G^\star = C_G(I)$, this gives (a) and shows that

$$G_0 = \langle Q, R \rangle = \langle IZ(Q), IZ(R) \rangle \leqslant \langle I, G^\star \rangle = IG^\star \leqslant G_0,$$

whence $G_0 = IG^\star$ and $I \lhd G_0$. Now we have (b) and

$$G_0 \cap S = IG^\star \cap S = I(G^\star \cap S) = IS^\star. \tag{5.3}$$

By Theorem 4.6, $S^\star = Z(Q_0)T^\star$, and $Z(Q_0)T_0$ is a Sylow p-subgroup of G_0. Since $Z(Q_0)T_0 \leqslant S$, we have $Z(Q_0)T_0 = G_0 \cap S$. This and (5.3) give (c). Since $T^\star = Z(Q_1)Z(R_1)$ by Proposition 4.5, we obtain (d). □

Recall that, for a p-group P, $\mathscr{A}(P)$ is the set of all large abelian subgroups of P, i.e. all abelian subgroups of maximal order in P.

Lemma 5.2. *Let $Q = Q_0$. Then*

(a) *$Z(Q)$ is in $\mathscr{A}(S^\star)$ and*

(b) *$\mathscr{A}(S^\star)$ is the set of all minimal CL-subgroups of $S^\star$.*

Proof. As in the proof of Lemma 5.1, let $R = R_0$ and $I = Q_0 \cap R_0$.

Then $Q = IZ(Q)$ by Lemma 5.1. Thus, $C_Q(I)$ lies in the centre of Q, which it obviously contains. So

$$C_Q(I) = Z(Q). \tag{5.4}$$

Let $P = G_0 \cap S$. Then $Q_0 \leqslant P$. By Lemma 5.1, $P = IS^\star$. Since $S^\star = G^\star \cap S = C_G(I) \cap S$,

$$I, S^\star \lhd P \quad \text{and} \quad S^\star = G^\star \cap P = C_P(I). \tag{5.5}$$

Moreover, I is contained in Q, which is a minimal CL-subgroup of S and hence of P. Therefore, the hypothesis of Lemma 2.11 is satisfied with I and $S^\star$ in place of K and L, and the conclusion of the lemma tells us that $Q \cap S^\star$ is a minimal CL-subgroup of $S^\star$. By (5.4) and (5.5), $Q \cap S^\star = C_Q(I) = Z(Q)$. This gives (a), and Theorem 2.10 gives (b). □

Lemma 5.3. *The following conditions are satisfied.*

(a) $G/T = G^\star T/T \cong G^\star/(G^\star \cap T) = G^\star/T^\star$.

(b) $Z(\mathrm{O}^p(G)) \leqslant T \cap \mathrm{O}^p(G) = \mathrm{O}_p(\mathrm{O}^p(G))$.

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Proof. By Proposition 4.5, $G = G^\star T$. This gives (a).

Let $H = \mathrm{O}^p(G)$ and $W = Z(\mathrm{O}^p(G))$. Then $W = \mathrm{O}_p(W) \times Y$ for the subgroup Y of all p'-elements of W, and H, W and Y are characteristic, hence normal, subgroups of G. Since $T = \mathrm{O}_p(G)$,

$$\mathrm{O}_p(W) \leqslant T \quad \text{and} \quad Y \cap T = 1.$$

Therefore, $[Y, T] \leqslant Y \cap T = 1$. But then $Y \leqslant C_G(T) \leqslant T$, which gives $Y = 1$. Hence, $W = \mathrm{O}_p(W) \leqslant T$. Thus, $W \leqslant T \cap H$.

Since $T \cap H$ is a normal p-subgroup of H, and $\mathrm{O}_p(H)$ is a normal p-subgroup of G,

$$T \cap H \leqslant \mathrm{O}_p(H) \leqslant \mathrm{O}_p(G) \cap H = T \cap H.$$

This completes the proof of (b) and of the lemma. □

Lemma 5.4. *Assume $q \geqslant 4$ and $L = T$. Then*

(a) *$G = \mathrm{O}^p(G)T$ and $S = (S \cap \mathrm{O}^p(G))T$, and*

(b) *there exists a non-identity cyclic p'-subgroup M of $\mathrm{O}^p(G)$ and an element x of $(\mathrm{O}^p(G) \cap S) \setminus T$ such that x normalizes M and $x^p \in C_T(M)$.*

Proof. (a) Let $H = \mathrm{O}^p(G)$. Since we have assumed $L = T$, Theorem 3.1 yields $G/T \cong \mathrm{SL}(2, q)$.

As $q \geqslant 4$, $\mathrm{SL}(2, q)$ is generated by its p'-elements. Therefore,

$$G/T = \mathrm{O}^p(G/T) = \mathrm{O}^p(G)T/T = HT/T \cong H/(H \cap T).$$

Hence,

$$G = HT \quad \text{and} \quad S = S \cap HT = (S \cap H)T.$$

(b) Assume first that $p = 2$. Then $\mathrm{SL}(2, q)$ has non-trivial cyclic Sylow 3-subgroups. Let $H_3/(H \cap T)$ be a Sylow 3-subgroup of $H/(H \cap T)$.

Let $H_1/(H \cap T)$ be the normalizer of $H_3/(H \cap T)$ in $H/(H \cap T)$ and let M be a Sylow 3-subgroup of H_3. Then M is cyclic and $H_1/(H \cap T)$ is a dihedral group. By the Frattini argument (part of Lemma 2.1),

$$H_1 = H_3 N_{H_1}(M) = ((H \cap T)M)N_{H_1}(M) = (H \cap T)N_{H_1}(M).$$

As $H_1/(H \cap T)$ is dihedral, $N_{H_1}(M)$ contains an element x of 2-power order that lies outside T such that x^2 lies in T. Since H is normal in G, $H \cap S$ is a Sylow 2-subgroup of H. Therefore, we may replace H_1, H_3 and x by conjugates, if necessary, so that x lies in $(H \cap S) \setminus T$. Then

$$x^2 \in T \cap N_G(M) \leqslant C_T(M),$$

as desired.

If p is odd, we obtain x by a similar argument in which we let $H_3/(H \cap T)$ be the centre of $H/(H \cap T)$ (of order 2) and we let $H_1/(H \cap T)$ be the direct product of $H_3/(H \cap T)$ with a subgroup of order p in $H/(H \cap T)$. □

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Now we present the first step in the reduction of Theorem C from G to $G^\star$.

Proposition 5.5. *Condition* (H) *and the hypothesis of Theorem C are satisfied with* $G^\star$, $S^\star$ *and* $G^\star \cap K$ *in place of* G, S *and* K. *Moreover,* $(S^\star)_{\mathrm{MCL}} = S^\star$.

Proof. We first check condition (E_0) of § 1 with $G^\star$, $S^\star$ and $G^\star \cap K$ in place of G, S and K. Recall (from before Theorem 4.6) that $S^\star = S \cap G^\star$. By Theorem 4.6, $S^\star$ is a Sylow p-subgroup of $G^\star$. By Proposition 4.5, $G = G^\star T$ and $T^\star = G^\star \cap T = \mathrm{O}_p(G^\star)$. Therefore,

$$S = S \cap G^\star T = (S \cap G^\star)T = S^\star T \quad \text{and} \quad G^\star/T^\star \cong G^\star T/T = G/T. \tag{5.6}$$

Since S is contained in a unique maximal subgroup of G, (5.6) shows that the same is true for S/T in G/T, for $S^\star/T^\star$ in $G^\star/T^\star$ and for $S^\star$ in $G^\star$.

As $K \geqslant T$ and $G = G^\star T$, we have

$$(K \cap G^\star) \cap T = G^\star \cap T = T^\star, \quad K = K \cap G^\star T = (K \cap G^\star)T \quad \text{and} \quad G = G^\star K.$$

Hence, the isomorphism of $G^\star/T^\star$ onto G/T in (5.6) takes $(K \cap G^\star)T^\star/T^\star$ onto K/T. Consequently, by (E_0),

$$G^\star/(G^\star \cap K) \cong G/K \cong \mathrm{PSL}(2, q).$$

Let $H = C_{G^\star}(T^\star)$. Then $H \lhd G^\star$. To finish the proof of (E_0) for $G^\star$, $S^\star$ and $G^\star \cap K$, we must show that $H \leqslant T^\star$.

Let x be a p'-element of H. As in Lemma 5.1, let $I = Q_0 \cap R_0$. By Proposition 4.5, $G^\star = C_G(I)$. So $T^\star = C_T(I)$ and x centralizes I and $C_T(I)$. Thus,

$$\langle x, I\rangle = \langle x\rangle \times I.$$

Now $\langle x\rangle \times I$ acts on T by conjugation, and x centralizes $C_T(I)$. By Theorem 2.2, $\langle x\rangle$ centralizes T. Since x is a p'-element and $C_G(T) \leqslant T$ by (E_0), $x = 1$. This shows that H is a p-group. As $H \lhd G^\star$, we have $H \leqslant \mathrm{O}_p(G^\star) = T^\star$, as desired.

Next, we check the hypothesis (H) of § 1 for $G^\star$, $S^\star$, $G^\star \cap K$ and $T^\star$ in place of G, S, K and T. We saw above that $T^\star = \mathrm{O}_p(G^\star)$. Since $Z(S) \leqslant S \cap C_S(I) = S \cap G^\star = S^\star$, we have $Z(S) \leqslant Z(S^\star)$. By Lemma 3.2,

$$Z(G) < Z(S) < Z = Z(T).$$

As $G = G^\star T$, $G^\star$ does not centralize $Z(S)$ and hence does not centralize $Z(S^\star)$. Thus, $Z(S^\star) \neq Z(G^\star)$.

The final condition needed for (H) and the hypothesis of Theorem C is that $S^\star = \tilde{J}(S^\star)$. By Lemma 5.2, $Z(Q_0)$ is a large abelian subgroup of $S^\star$ and is a minimal CL-subgroup of $S^\star$. By Theorem 2.10, $Z(Q_1)$ and $Z(R_1)$ have the same order as $Z(Q_0)$, and hence are large abelian subgroups of $S^\star$. By Lemma 5.1,

$$S^\star = Z(Q_0)Z(Q_1)Z(R_1).$$

Therefore, $S^\star = J(S^\star) = \tilde{J}(S^\star) = (S^\star)_{\mathrm{MCL}}$, as desired.

Since $Z(Q_0)$ is a minimal CL-subgroup of $S^\star$ and is not contained in $T^\star$ (by Theorem 4.6), $(S^\star)_{\mathrm{MCL}}$ is not normal in $G^\star$. This completes the hypothesis of Theorem C for $G^\star$, $S^\star$ and $G^\star \cap K$ in place of G, S and K. □

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5.1. Reduction for Theorem C

By Proposition 5.5, condition (H) and the hypothesis of Theorem C are satisfied with $G^\star$, $S^\star$ and $G^\star \cap K$ in place of G, S and K, and $(S^\star)_{\mathrm{MCL}} = S^\star$.

Now assume that the conclusion of Theorem C is valid for $G^\star$, $S^\star$ and $G^\star \cap K$ in place of G, S and K. By (H) and Lemma 2.19, $Z(S^\star)$ is not normal in $G^\star$. Since $(S^\star)_{\mathrm{MCL}} = S^\star$, $(S^*)_{\mathrm{MCL}}$ is not normal in $G^\star$. Therefore, conditions (a)–(i) of Theorem C are valid for $G^\star$, $S^\star$ and $G^\star \cap K$ in place of G, S and K. Since $Z(S)$ and S_{MCL} are not normal in G, we must show that (a)–(i) are valid for G, S and K.

Parts (b), (e) and (g) follow from Theorems 2.10, 3.1 and 3.3. By Theorem 4.6, $\mathrm{O}^p(G^\star) = \mathrm{O}^p(G)$. Recall that we define $\hat{G} = \mathrm{O}^p(G)$ and $\hat{T} = \mathrm{O}_p(\hat{G})$ for Theorem C. Therefore, parts (a)–(d) carry over immediately from $G^\star$ to G.

Clearly,

$$\hat{T},\ \hat{G} \text{ and } Z(\hat{G}) \text{ are characteristic, hence normal, subgroups of } G. \tag{5.7}$$

By Lemma 5.3,

$$G = G^\star T, \quad G/T \cong G^\star/T^\star \quad \text{and} \quad Z(\hat{G}) \leqslant T \cap \hat{G} = \hat{T}. \tag{5.8}$$

Hence, by parts (e) and (h) of Theorem C for $G^\star$ and Theorem 3.1,

$$\text{if } q > 2, \text{ then } G/T \cong \mathrm{SL}(2,q) \text{ and } L = T. \tag{5.9}$$

To prove (f) and (h), we consider a chief series of G containing the series

$$1 \leqslant Z(\hat{G}) \leqslant \hat{T} \leqslant T \leqslant G.$$

Let U/V be a chief factor coming from successive terms in the chief series such that $U \leqslant T$. Then we have one of the following cases:

(i) $\hat{T} \leqslant V < U \leqslant T$;

(ii) $Z(\hat{G}) \leqslant V < U \leqslant \hat{T}$;

(iii) $V < U \leqslant Z(\hat{G})$.

In case (i), (5.7) gives

$$[U, \hat{G}] \leqslant T \cap \hat{G} = \hat{T} \leqslant U.$$

Thus, $\hat{G}$ centralizes U/V. Since conjugation by G induces an irreducible action of G on the module U/V, we see that $G/\hat{G}$ acts irreducibly on U/V. As $\hat{G} = \mathrm{O}^p(G)$, $G/\hat{G}$ is a p-group. Hence, U/V is a central chief factor of G.

A similar argument shows that U/V is a central chief factor in case (iii).

Now assume case (ii). Here, $U \leqslant \hat{T} < \hat{G} = \mathrm{O}^p(G) = \mathrm{O}^p(G^\star) \leqslant G^\star$. Again, G acts irreducibly on U/V. Since $T = \mathrm{O}_p(G)$ and $G = G^\star T$, T centralizes U/V and $G^\star$ acts irreducibly on U/V. Therefore, U/V is a chief factor of $G^\star$ such that $U \leqslant \mathrm{O}_p(G^\star)$. Since

$G^\star$ satisfies Theorem C, (5.8) and (5.9) and parts (f) and (h) of Theorem C show that U/V is not a central chief factor and that

$$\text{if } q > 2, \text{ then } G/T \cong G^\star/T^\star \cong \mathrm{SL}(2,q) \text{ and } U/V \text{ is a standard module for } G^\star/T^\star, \text{ and hence for } G/T.$$

This proves part (f) of Theorem C and shows that U/V satisfies the conditions in part (h) for cases (i)–(iii) above. By the Jordan–Hölder Theorem for chief series (see [**16**, Theorem 8.44], where they are called principal series), this proves part (h) in general.

To finish the proof, we must obtain part (i) of Theorem C. We may assume that $q \geqslant 4$. By (5.9),

$$L = T \quad \text{and} \quad G/T \cong \mathrm{SL}(2,q).$$

We take x and M as in Lemma 5.4, so that

$$S = \hat{S}T, \quad x \in \hat{S} \setminus T \quad \text{and} \quad M \text{ is a non-trivial } p'\text{-subgroup of } \hat{G} \text{ normalized by } x. \tag{5.10}$$

Then

$$[M,T] \leqslant [\hat{G},T] \leqslant \hat{G} \cap T \leqslant \hat{T}, \tag{5.11}$$

and, by Lemma 2.1, $T = [M,T]C_T(M) = \hat{T}C_T(M)$. Therefore, by (5.10),

$$S = \hat{S}T = \hat{S}\hat{T}C_T(M) = \hat{S}C_T(M). \tag{5.12}$$

By (f) and (h), each chief factor U/V of G satisfying $Z(\hat{G}) \leqslant V < U \leqslant \hat{T}$ is a standard module for G/T, and hence (by (5.10)) has no non-zero fixed points under M. Therefore, $C_{\hat{T}}(M) \leqslant Z(\hat{G})$ and, by (5.10) and (5.11),

$$Z(\hat{G}) \geqslant C_{\hat{T}}(M) \geqslant C_T(M) \cap [\hat{G},T] \geqslant [\langle x\rangle, C_T(M)]. \tag{5.13}$$

Since $\hat{S} = S \cap \hat{G}$, (5.7) and (5.8) show that $\hat{S}$, $Z(\hat{G})$ and $\hat{S}'Z(\hat{G})$ are normal subgroups of S and $N_G(S)$. Therefore, by (5.13),

$$[\langle x\rangle, C_T(M)] \leqslant Z(\hat{G}) \leqslant \hat{S}'Z(\hat{G}),$$

and x centralizes $C_T(M)$, module $\hat{S}'Z(\hat{G})$. Since $[\langle x\rangle, \hat{S}] \leqslant \hat{S}' \leqslant \hat{S}'Z(\hat{G})$, (5.12) shows that x centralizes S, modulo $\hat{S}'Z(\hat{G})$.

By (5.10), x lies in $\hat{S} \setminus T$. Let

$$R = C_{\hat{S}}(S/\hat{S}'Z(\hat{G})).$$

Then $R \leqslant \hat{S}$ and R is normal in $N_G(S)$. Therefore, RT/T is a normal subgroup of $N_G(S)/T$ that contains the non-identity element xT. By (5.9), $G/T \cong \mathrm{SL}(2,q)$. Note that $N_G(S)/T = N_G(S/T)$. Therefore, from the structure of $\mathrm{SL}(2,q)$, S/T is the only non-identity normal subgroup of $N_{G/T}(S/T)$ contained in S/T. Consequently,

$$RT/T = S/T \quad \text{and} \quad RT = S. \tag{5.14}$$

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By definition, $[S,R] \leqslant \hat{S}' Z(\hat{G})$. Since G satisfies (a),

$$[S,R,R] \leqslant [\hat{S}' Z(\hat{G}), R] \leqslant [\hat{S}', \hat{S}] \leqslant Z(\hat{S}).$$

So $[S,R,R,R] = 1$. This completes the proof of (i) and the reduction of Theorem C to the case in which $G = G^\star$.

Remark 5.6. The reduction above did not use the assumption that $G^\star$ satisfies parts (b), (e), (g) and (i) of Theorem C. Moreover, the only parts of (f) and (h) for $G^\star$ that were needed were the following statements:

$$\text{if } U/V \text{ is a chief factor of } G^\star \text{ and } Z(\hat{G}) \leqslant V < U \leqslant \hat{T}, \text{ then } U/V \text{ is not a central chief factor} \tag{5.15}$$

and

$$\text{if } q > 2, \text{ then } L = T, \text{ and every chief factor } U/V \text{ of } G^\star \text{ as in (5.15) is a standard module for } G^\star/T^\star. \tag{5.16}$$

Therefore, to prove Theorem C, we need only check parts (a), (c) and (d), and (5.15) and (5.16) when $G = G^\star$. Note also that the p'-element f from the beginning of § 4 lies in $G^\star$ because $\mathrm{O}^p(G) = \mathrm{O}^p(G^\star)$.

6. Proof of Theorem C

In this section we complete the proof of Theorem C. We continue with the assumptions stated at the beginning of § 4. By § 5, we may assume that $G = G^\star = \langle Z(Q_0), Z(R_0) \rangle$ and that the minimal CL-subgroups of S are the large abelian subgroups of S. To remind us of this, we change notation. Let

$$A = Q_0 = Z(Q_0), \quad B = R_0 = Z(R_0), \quad A^\star = Q_1 \quad \text{and} \quad B^\star = R_1.$$

We also let $\tilde{T} = \langle [A, B^\star], [B, A^\star] \rangle$. Recall that $B = A^f$ and $T' = [T,T]$.

Lemma 6.1. *The following conditions are satisfied.*

(a) $T = (A \cap T)(B \cap T)$.

(b) $[A, B^\star]$ *and* $[B, A^\star]$ *are abelian.*

(c) $T' = [A \cap T, B \cap T] \leqslant [A, B^\star] \cap [B, A^\star] \leqslant Z(\tilde{T})$.

(d) $\tilde{T} = [T, G] \lhd G$.

(e) $T = (A \cap T)\tilde{T} = (B \cap T)\tilde{T}$.

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Proof. Recall that $T = A^\star B^\star = (A \cap T)(B \cap T)$ from Proposition 4.2. This gives (a). Let $U = [A \cap T, B \cap T]$. Then $U \lhd \langle A \cap T, B \cap T\rangle = T$ and $U \leqslant T'$. Since $A \cap T$ and $B \cap T$ are abelian and centralize each other modulo U, we have $T' \leqslant U$. Thus,

$$T' = U = [A \cap T, B \cap T]. \tag{6.1}$$

Since A and $B^\star$ are CL-subgroups of S, $AB^\star$ is a CL-subgroup of S. As A and $B^\star$ are abelian, Itô's Theorem (Theorem 2.3) yields that $[A, B^\star]$ is abelian. By (6.1),

$$T' = [A \cap T, B \cap T] \leqslant [A, B^\star].$$

Similarly, $[B, A^\star]$ is abelian and $T' \leqslant [B, A^\star]$. Now we obtain (b) and (c).

As $T' \leqslant \tilde{T}$, we have $\tilde{T} \lhd T$. By (a),

$$[\tilde{T}, A] \leqslant [T, A] = [(A \cap T)(B \cap T), A] = [B \cap T, A] \leqslant [B^\star, A] \leqslant \tilde{T}.$$

Therefore, A normalizes $\tilde{T}$ and centralizes $T/\tilde{T}$. Similarly, B normalizes $\tilde{T}$ and centralizes $T/\tilde{T}$. Since A and B generate G,

$$G \text{ normalizes } \tilde{T} \quad \text{and} \quad [T, G] \leqslant \tilde{T}.$$

But clearly $\tilde{T} \leqslant [T, G]$. This gives (d).

Finally, recall that $B = A^f$. Hence,

$$B \cap T = A^f \cap T = (A \cap T)^f.$$

By (a) and (d),

$$T = (A \cap T)(B \cap T)\tilde{T} = (A \cap T)(A \cap T)^f \tilde{T} \leqslant (A \cap T)[A \cap T, f]\tilde{T} = (A \cap T)\tilde{T}.$$

So $T = (A \cap T)\tilde{T}$. Similarly, $T = (B \cap T)\tilde{T}$. This proves (e) and completes the proof of the lemma. □

For this section only, we say that a subgroup U of T is an *F-subgroup* of T (factorizable subgroup of T) if

$$U \lhd G \quad \text{and} \quad U = (U \cap A)(U \cap B).$$

Lemma 6.2. *Suppose N is a normal subgroup of T. Let*

$$N^\star = \langle a, b \mid a \text{ is in } A \cap T,\ b \text{ is in } B \cap T \text{ and } ab \text{ is in } N\rangle.$$

Then

(a) *$N \leqslant N^\star$ and $N^\star/N$ is contained in the centre of G/N,*

(b) *$N^\star = (A \cap N^\star)N = (B \cap N^\star)N = (A \cap N^\star)(B \cap N^\star)$, and*

(c) *$N^\star$ is an F-subgroup of T.*

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Proof. By Lemma 6.1,

$$T = (A \cap T)(B \cap T). \tag{6.2}$$

Since $N \lhd G$,

$$(A \cap N^\star)N \text{ is a subgroup of } G.$$

For each a in $A \cap T$ and b in $B \cap T$ such that ab lies in N,

$$(A \cap N^\star)N \text{ contains } a \text{ and } ab, \text{ and hence contains } b.$$

Therefore, $N^\star \leqslant (A \cap N^\star)N$. By (6.2) and the definition of $N^\star$, we have $N \leqslant N^\star$. So $(A \cap N^\star)N = N^\star$. Similarly, we obtain

$$(B \cap N^\star)N = N^\star = (A \cap N^\star)N. \tag{6.3}$$

By (6.3), AN/N and BN/N centralize $N^\star/N$. Since A and B generate G, we obtain (a). Note that this also shows that $N^\star$ is a normal subgroup of G.

Consider the subset $(A \cap N^\star)(B \cap N^\star)$ of $N^\star$. By (6.2) and the definition of $N^\star$, this set contains N. Clearly, it is closed under left multiplication by $A \cap N^\star$. So it contains $(A \cap N^\star)N$. By (6.3), it is equal to $N^\star$, and we obtain (b) and (c). □

Recall that $Z = Z(T)$.

Proposition 6.3. *The group T satisfies $Z(G/Z) \cap (T/Z) = 1$.*

Proof. Let N be the subgroup of G that contains Z and satisfies

$$N/Z = Z(G/Z) \cap (T/Z).$$

We must show that $N = Z$.

Let $\bar{G} = G/Z$ and let $\bar{H} = HZ/Z$ for every subgroup H of G. Define $N^\star$ as in Lemma 6.2. Then

$$\bar{N} = Z(\bar{G}) \cap \bar{T} \quad \text{and} \quad N^\star = (A \cap N^\star)N = (B \cap N^\star)N.$$

So $\overline{N^\star} = (\overline{A \cap N^\star})(Z(\bar{G}) \cap \bar{T}) = (\overline{B \cap N^\star})(Z(\bar{G}) \cap \bar{T})$. Therefore, $\overline{N^\star}$ is centralized by $\bar{A}$ and by $\bar{B}$, and hence by $\bar{G}$. So

$$Z(\bar{G}) \cap \bar{T} \geqslant \overline{N^\star} \geqslant \bar{N} = Z(\bar{G}) \cap \bar{T}.$$

This shows that $N^\star = N$ and, by Lemma 6.2,

$$N = (A \cap N^\star)(B \cap N^\star) = (A \cap N)(B \cap N). \tag{6.4}$$

Recall that $A^f = B$. Therefore,

$$B \cap N = A^f \cap N = (A \cap N)^f.$$

Since $\bar{N} \leqslant Z(\bar{G})$, (6.4) yields

$$\bar{N} = (\overline{A \cap N})(\overline{A \cap N})^f = \overline{A \cap N} \quad \text{and} \quad N = (A \cap N)Z = (A \cap N)Z(T).$$

It follows that $A \cap T$ centralizes N. Similarly, $B \cap T$ centralizes N. By Lemma 6.1, $T = (A \cap T)(B \cap T)$. Consequently, $N \leqslant Z(T) = Z$. As $Z \leqslant N$, we obtain $N = Z$, as desired. □

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Now we show that G has no central chief factors between Z and the subgroup T_1 of T determined by $T_1/Z = Z(T/Z)$.

Proposition 6.4. *Suppose $N \lhd G$ and*

$$Z \leqslant N \quad \text{and} \quad N/Z \leqslant Z(T/Z).$$

Then

(a) $N = [N, G]Z$,

(b) $N = (N \cap A)(N \cap B)$.

Proof. As in the previous proof, let $\bar{H} = HZ/Z$ for every subgroup H of G. Let

$$M = [N, G]Z.$$

The hypothesis and the definition of M yield that

$$G \text{ centralizes } N/M \quad \text{and} \quad \bar{N} \leqslant Z(\bar{T}). \tag{6.5}$$

Define $N^\star$ as in Lemma 6.2, so that

$$N^\star = (A \cap N^\star)N \quad \text{and} \quad \overline{N^\star} = (\overline{A \cap N^\star})\bar{N} \leqslant (\overline{A \cap N^\star})Z(\bar{T}).$$

Obviously, $\overline{N^\star}$ is centralized by $\overline{A \cap T}$. Similarly, $\overline{N^\star}$ is centralized by $\overline{B \cap T}$. Since $T = (A \cap T)(B \cap T)$,

$$\overline{N^\star} \leqslant Z(\bar{T}). \tag{6.6}$$

By Lemma 6.2,

$$N^\star/N \text{ is centralized by } G. \tag{6.7}$$

Now we prove (a) and (b) separately.

(a) We use induction on $|N|$. Assume first that $\bar{N}$ is not elementary abelian. Let

$$N_1/Z = \Omega_1(\bar{N}) = \{x \in N \mid x^p \in Z\}/Z.$$

Then $|N_1| < |N|$. By induction,

$$N_1 = [N_1, G]Z \leqslant [N, G]Z = M \quad \text{and} \quad \bar{N}_1 \leqslant \bar{M}. \tag{6.8}$$

Continuing from the previous paragraph, let ϕ be the mapping on $\bar{N}$ given by $\phi(x) = x^p$. Since $\bar{N}$ is abelian, ϕ is a homomorphism. Clearly, ϕ commutes with the action of each element of G under conjugation, and the kernel of ϕ is $\bar{N}_1$. By (6.8), $\bar{N}_1 \leqslant \bar{M}$. Therefore, by (6.5),

$$\phi(\bar{N})/\phi(\bar{M}) \text{ is isomorphic to } \bar{N}/\bar{M} \quad \text{and} \quad [\phi(\bar{N}), \bar{G}] \leqslant \phi(\bar{M}) \leqslant \phi(\bar{N}). \tag{6.9}$$

By induction, $[\phi(\bar{N}), \bar{G}] = \phi(\bar{N})$. Hence, by (6.9),

$$\phi(\bar{M}) = \phi(\bar{N}) \quad \text{and} \quad \bar{N} = \bar{M},$$

which shows that $N = M$, as desired. Thus, we may assume that

$$\bar{N} \text{ is elementary abelian.} \tag{6.10}$$

Define a mapping $\phi^\star$ on $\overline{N^\star}$ by $\phi^\star(x) = x^p$. By (6.10), $\phi^\star(\bar{N}) = 1$. Hence, by (6.7), $\phi^\star(\overline{N^\star})$ is centralized by $\bar{G}$. Thus,

$$\phi^\star(\overline{N^\star}) \leqslant Z(\bar{G}) \cap \bar{T}.$$

By Proposition 6.3, $\phi^\star(\overline{N^\star}) = 1$. This says that $\overline{N^\star}$ is elementary abelian.

We regard $\overline{N^\star}$ as a vector space over the prime field $\mathbb{F}_p$ and as a module for G over $\mathbb{F}_p$. By Lemma 6.2, $N^\star = (A \cap N^\star)N$. Therefore, there exists a subgroup W of $N^\star$ such that

$$Z \leqslant W \leqslant (A \cap N^\star)Z \quad \text{and} \quad \overline{N^\star} = \bar{W} \times \bar{N}. \tag{6.11}$$

Then $\bar{N}$ is a G-submodule of $\overline{N^\star}$ and $\bar{W}$ is a vector space complement to $\bar{N}$ in $\overline{N^\star}$. By (6.6) and (6.11), $\bar{W}$ is invariant (in fact, centralized) under T and under A. Since $S = TA$ (by Theorem 3.3), $\bar{W}$ is invariant under S. By Theorem 2.2, there exists a complement $\bar{V}$ to $\bar{N}$ in $\overline{N^\star}$ that is invariant under G.

By (6.7), G centralizes $\bar{V}$. Therefore,

$$\bar{V} \leqslant Z(\bar{G}) \cap \bar{T}.$$

By Proposition 6.3, $\bar{V} = 1$. Consequently, $\overline{N^\star} = \bar{N}$. So $N^\star = N$. By Lemma 6.2,

$$N = (A \cap N)(B \cap N) = (A \cap N)(A \cap N)^f \leqslant (A \cap N)[N, G]Z = (A \cap N)M.$$

Hence, $\bar{N} = \overline{(A \cap N)}\bar{M}$.

Since $\bar{N}$ is elementary abelian and G centralizes N/M (by (6.10) and (6.5)), a small variation on our proof that $N^\star = N$ shows that $\bar{N} = \bar{M}$, whence $N = M$, as desired.

(b) By (6.6) and (6.7), $\overline{N^\star} \leqslant Z(\bar{T})$ and G centralizes $\overline{N^\star}/\bar{N}$. Therefore, by part (a),

$$\overline{N^\star} = [\overline{N^\star}, G] \leqslant \bar{N} \leqslant \overline{N^\star}.$$

So $\bar{N} = \overline{N^\star}$ and $N^\star = N$. By Lemma 6.2, $N = (N \cap A)(N \cap B)$, as desired. □

Proposition 6.5. *The group T satisfies*

$$T' \leqslant C_T(\tilde{T}) = Z.$$

Proof. Clearly, $Z = Z(T) \leqslant C_T(\tilde{T})$. By Lemma 6.1, $T' \leqslant Z(\tilde{T}) \leqslant C_T(\tilde{T})$. So we need only prove that $C_T(\tilde{T}) = Z$.

As in the proofs of Propositions 6.3 and 6.4, let $\bar{H} = HZ/Z$ for every subgroup H of G.

Let $C = C_T(\tilde{T})$. We will assume that $C > Z$ and aim for a contradiction.

Here, $1 < \bar{C} \leqslant \bar{T}$ and $\bar{C} \lhd \bar{G}$. Therefore,

$$\bar{C} \cap Z(\bar{T}) > 1.$$

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Take the subgroup W of T for which

$$W \geqslant Z \quad \text{and} \quad \bar{W} = \bar{C} \cap Z(\bar{T}).$$

Then $1 < \bar{W} \lhd \bar{G}$.

By Proposition 6.4 and Lemma 6.1,

$$W = (W \cap A)(W \cap B) \quad \text{and} \quad T = (A \cap T)\tilde{T} = (B \cap T)\tilde{T}.$$

Since $W \leqslant C = C_T(\tilde{T})$, it follows that $\tilde{T}$ and $A \cap T$ both centralize $W \cap A$, and

$$W \cap A \leqslant Z(T) = Z.$$

Similarly, $W \cap B \leqslant Z$. Hence, $W \leqslant Z$ and $\bar{W} = 1$, a contradiction. This completes the proof of Proposition 6.5. □

Proposition 6.6. *The following conditions are satisfied.*

(a) *T/Z is abelian.*

(b) *Whenever $U \lhd G$ and $Z \leqslant U \leqslant T$, then*

$$U = [U, G]Z \quad \text{and} \quad U = (U \cap A)(U \cap B).$$

(c) *Whenever $U, V \lhd G$ and $Z \leqslant V < U \leqslant T$, then in the action of G induced on U/V by conjugation,*

$$C_{U/V}(A) = (A \cap U)V/V, \quad C_{U/V}(B) = (B \cap U)V/V$$

and

$$U/V = C_{U/V}(A) \times C_{U/V}(B), \quad C_{U/V}(G) = 1.$$

(d) *In the situation of (c),*

$$T \text{ centralizes } U/V \quad \text{and} \quad C_{U/V}(A) = C_{U/V}(x) \quad \text{for every } x \text{ in } A \setminus T.$$

(e) *$T = [T, \mathrm{O}^p(G)]Z(G)$.*

Proof. (a) This follows from Proposition 6.5.

(b) This follows from (a) and Proposition 6.4.

(c) Let $F = U/V$, $\hat{A} = (A \cap U)V/V$ and $\hat{B} = (B \cap U)V/V$. Since A and B are abelian, we can use (b) to obtain

$$\hat{A} \leqslant C_F(A), \quad \hat{B} \leqslant C_F(B) \quad \text{and} \quad F = \hat{A}\hat{B} \leqslant C_F(A)C_F(B) \leqslant F. \tag{6.12}$$

Let $C_F(A) \cap C_F(B) = U^\star/V$. Since $\langle A, B\rangle = G$, we have

$$U^\star/V = C_F(G), \quad U^\star \lhd G \quad \text{and} \quad [U^\star, G] \leqslant V.$$

But $Z \leqslant V \leqslant U^\star \leqslant T$, and (b) gives

$$U^\star = [U^\star, G] \leqslant VZ = V \leqslant U^\star.$$

So $U^\star = V$ and

$$1 = U^\star/V = C_F(A) \cap C_F(B) = C_F(G).$$

Now (6.12) gives $F = \hat{A} \times \hat{B}$ and (c).

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(d) Take U and V as in (c) and $x \in A \setminus T$. Recall that $A^f = B$. From the structure of $\mathrm{PSL}(2,q)$, $x^{f^{-1}}$ lies outside S and $N_G(S)$. Therefore, by condition (E_0) in § 1,

$$G = \langle S, x^{f^{-1}} \rangle \quad \text{and} \quad G = G^f = \langle S^f, x \rangle = \langle B, T, x \rangle.$$

By (a), $[U,T] \leqslant Z \leqslant V$. So T centralizes F. Hence, $1 = C_F(G) = C_F(B) \cap C_F(x)$. Since $C_F(A) \leqslant C_F(x)$, part (c) gives

$$C_F(x) = C_F(x) \cap (C_F(A)C_F(B)) = C_F(A)(C_F(x) \cap C_F(B)) = C_F(A),$$

as desired.

(e) Let

$$H = \mathrm{O}^p(G), \quad R = [T,H], \quad Y = Z(G) \quad \text{and} \quad Q = RY.$$

Then, $H, R, Y, Q \lhd G$.

By Theorem 3.1, Z/Y is a standard module for G/L, and hence is irreducible under G and is not centralized by H. As $[Z,H]Y/Y$ is a submodule of Z/Y,

$$[Z,H]Y/Y = Z/Y \quad \text{and} \quad Z = [Z,H]Y \leqslant RY = Q.$$

Let $\bar{G} = G/Q$, and let $\bar{X} = XQ/Q$ for every subgroup X of G. Then $\bar{H}$ centralizes $\bar{T}$ because $[T,H] \leqslant Q$. By (c), $T = [T,G]Z = [T,G]Q$. Since $G = \mathrm{O}^p(G)S = HS$,

$$\bar{T} = [\bar{T}, \bar{G}] = [\bar{T}, \bar{H}\bar{S}] = [\bar{T}, \bar{S}].$$

As $\bar{S}$ is nilpotent, this shows that $\bar{T} = 1$, i.e. $Q = T$, as desired. □

Recall that $Z(G) \leqslant C_G(T) \leqslant T$, so that $Z(G) \leqslant Z(S)$.

Proposition 6.7. *In the situation of Proposition 6.6 (c),*

(a) *$[U,A,A] \leqslant V$ if $p = 2$ and U/V is elementary abelian, and*

(b) *$[U,A;3] \leqslant V$ and $[T,A;3] \leqslant Z$ if p is odd.*

Proof. As in the proof of Proposition 6.6, let $F = U/V$. By Proposition 6.6 (d),

$$T \text{ centralizes } F. \tag{6.13}$$

(a) Assume that $p = 2$ and that F is elementary abelian, and thus a vector space over $\boldsymbol{F}_2$. Take any x in A. Then x^2 lies in T because S/T is elementary abelian. Therefore, by (6.13), the linear transformation t induced on F over $\boldsymbol{F}_2$ by conjugation by x satisfies

$$0 = t^2 - 1 = (t-1)^2,$$

which gives $[F,x,x] = 0$. Thus, $[F,x] \leqslant C_F(x)$. By Proposition 6.6,

$$[F,x] \leqslant C_F(A).$$

As this is true for all x in A,

$$[F,A] \leqslant C_F(A) \quad \text{and} \quad [F,A,A] = 0,$$

which gives (a).

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(b) Assume that p is odd. By Theorem 3.1, $Z = [Z, G] \times Z(G)$ and $Z/Z(G)$ is a standard module for G/L. Therefore, $[Z/Z(G), A, A] = 1$ and

$$[Z, A, A] = 1. \tag{6.14}$$

Take any elements y in $A \cap T$, a in A and w in T. Since $T' \leqslant Z(T) = Z$,

$$[y, w] \in Z \quad \text{and} \quad [y, w]^a = [y^a, w^a] = [y, w^a],$$
$$[y, w, a] = [y, w]^{-1}[y, w]^a = [y, w^{-1}][y, w^a] = [y, w^{-1}w^a].$$

Thus,

$$[y, w, a] = [y, [w, a]].$$

Similarly, for a' in A,

$$[y, w, a, a'] = [y, [w, a], a'] = [y, [[w, a], a']] = [y, [w, a, a']].$$

By (6.14), we obtain

$$[y, [w, a, a']] = [y, w, a, a'] \in [T', A, A] \leqslant [Z, A, A] = 1.$$

As y can be any element of $A \cap T$,

$$[w, a, a'] \in C_T(A \cap T) = C_T((A \cap T)Z) = C_T(A^*) = A^*.$$

Thus, $[T, A, A] \leqslant A^* = (A \cap T)Z$ and

$$[T, A; 3] \leqslant [(A \cap T)Z, A] \leqslant Z.$$

Since $Z \leqslant V < U \leqslant T$, we also have $[U, A; 3] \leqslant V$, as desired. □

Proposition 6.8. *The subgroup L contains T and satisfies the following conditions.*

(a) *L/T is a p'-group.*

(b) *$T/Z = C_{T/Z}(L) \times [T, L]Z/Z$.*

(c) *Whenever $U, V \lhd G$ and $Z \leqslant V < U \leqslant [T, L]Z$, U/V is centralized by T, but not by L.*

(d) *If $L > T$, then q is 2 or 3.*

Proof. (a) By Theorem 3.1 and Proposition 3.4, $L \leqslant K$ and K/T is a p'-group. Hence, L/T is a p'-group.

(b), (c) Let $T^* = [T, L]Z$. By Proposition 6.6, T/Z is abelian. Therefore, conjugation by L on T induces an action of L/T on T/Z. By (a) and Lemma 2.1,

$$T/Z = C_{T/Z}(L/T) \times [T/Z, L/T] = C_{T/Z}(L) \times [T/Z, L] = C_{T/Z}(L) \times (T^*/Z),$$

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which gives (b). Moreover,

$$C_{T^*/Z}(L) = (T^*/Z) \cap C_{T/Z}(L) = 1.$$

For U and V as in (c), T centralizes U/V because T centralizes T/Z. Moreover, $C_{U/Z}(L) \leqslant C_{T^*/Z}(L) = 1$. Therefore, Lemma 2.1 with $P = U/Z$, $A = L/T$ and $N = V/Z$ gives

$$C_{P/N}(L) = C_{P/N}(L/T) = C_P(L/T)N/N = C_{U/Z}(L)N/N = N/N.$$

Thus,

$$C_{U/V}(L) \cong C_{(U/Z)/(V/Z)}(L) = C_{P/N}(L) = 1,$$

which gives (c).

(d) Suppose $L > T$. By (a) and Cauchy's Theorem, L contains a subgroup X of prime order other than p.

Assume first that X centralizes T/Z. Since $L = C_G(Z)$ (defined before Theorem 3.1), X centralizes Z. Therefore, Lemma 2.1 yields that X centralizes T. However, by condition (H), $C_G(T) \leqslant T$. As $|X|$ does not divide $|T|$, this is a contradiction. Thus,

$$X \text{ does not centralize } T/Z.$$

Now we have $T^* = [T, L]Z \geqslant [T, X]Z > Z$. Clearly, Z and T^* are normal in G. Let U/V be a chief factor of G such that

$$Z \leqslant V < U \leqslant T^*.$$

Let $M = U/V$. Then (c) shows that G/T acts on M and that L/T acts non-trivially on M in this action. Since $S = AT$, Proposition 6.7 gives

$$[M, S; 3] = 1. \tag{6.15}$$

Let $\bar{G} = G/T$ and let $\bar{H} = HT/T$ for every subgroup H of G. By Theorem 3.1,

$$\bar{K} = \Phi(\bar{G}), \quad \bar{K}/\bar{L} = Z(\bar{G}/\bar{L}) \quad \text{and} \quad \bar{L} = [\bar{L}, \bar{G}].$$

Hence, by (6.15) and Theorem 3.1 and Proposition 6.6, the hypothesis of Theorem 2.16 is satisfied. As $\bar{L}$ does not centralize M, Theorem 2.16 yields that $q = 2$ or 3. □

Recall from Theorem 3.1 that $G/L \cong \mathrm{SL}(2, q)$.

Proposition 6.9. *Suppose U/V is a chief factor of G such that $Z \leqslant V < U \leqslant T$ and L centralizes U/V.*

Then U/V is a standard module for G/L.

Proof. Since $S = AT$ and $T \leqslant L$,

$$C_{U/V}(S) = C_{U/V}(A).$$

Then, by Proposition 6.6, $|C_{U/V}(S)|^2 = |U/V|$. By Theorem 2.15 with G/L, U/V, $C_{U/V}(S)$ and SL/L in place of G, V, V_0 and S, we see that U/V is a standard module for G/L. □

Proposition 6.10. *The group $T/Z(G)$ is abelian.*

Proof. Assume otherwise. Recall that $Z = Z(T)$ and, by Proposition 6.6, T/Z is abelian. Let C and D be subgroups of T containing Z such that

$$C/Z = C_{T/Z}(L) \quad \text{and} \quad D/Z = [T, L]Z/Z.$$

Then $C, D \lhd G$. By Proposition 6.8,

$$T/Z = (C/Z) \times (D/Z). \tag{6.16}$$

So $T = CD$.

Let $Y = Z(G)$. By Theorem 3.1,

$$G/L \cong \mathrm{SL}(2, q), \quad Z/Y \text{ is a standard module for } G/L \tag{6.17}$$

and $K/L = Z(G/L)$. Hence, Z/Y is irreducible under G/L. As T/Z is abelian, $T' \leqslant Z$. Thus, $T'Y/Y \leqslant Z/Y$ and

$$\text{if } T/Y \text{ is not abelian, then } (T/Y)' = T'Y/Y = Z/Y. \tag{6.18}$$

In any case, since T has nilpotence class 2, the commutator mapping $T \times T \to Z$ induces a bi-additive mapping of abelian groups

$$T/Z \times T/Z \to Z/Y$$

that takes (xZ, yZ) to $[x, y]Y$.

We consider the action of G on its chief factors induced by conjugation. By Proposition 6.6,

$$C_X(A) = (A \cap U)V/V \quad \text{and} \quad X = C_X(A) \times C_X(B) \tag{6.19}$$

whenever $U, V \lhd G$ and $Z \leqslant V < U \leqslant T$ and $X = U/V$. Since $B = A^f$, (6.19) also gives

$$|U/V| = |C_{U/V}(A)|^2 \tag{6.20}$$

in this situation.

We prove the result in three steps:

1. C/Y is abelian;

2. D/Y is abelian;

3. D/Y centralizes C/Y.

Since $T = CD$, this suffices.

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Step 1. C/Y is abelian.

Proof. Assume first that p is odd. Then $\mathrm{SL}(2,q)$ contains a unique element of order 2. Therefore, by (6.17), there exists a 2-element g of G such that gL is the unique element of order 2 in G/L.

Now g^2 is a p'-element of L. So g^2 centralizes C/Z. By (6.17), g^2 centralizes Z/Y. Hence, by Lemma 2.1, g^2 centralizes C/Y, and g induces an automorphism of order 2 on C/Y.

By (6.17), g acts as the -1 transformation of Z/Y. So $C_{Z/Y}(g) = 1$, and $C_Z(g) \leqslant Y$. Similarly, by Proposition 6.9,

$$C_{U/V}(g) = 1$$

whenever U/V is a chief factor of G and $Z \leqslant U < V \leqslant C$. Therefore, g induces an automorphism of order 2 on C/Y that fixes only the identity element. By an elementary result, C/Y is an abelian group inverted by g.

Next, assume that $p = 2$. Then, by (6.17) and Theorem 3.1 (h),

$$K/L = Z(G/L) \cong Z(\mathrm{SL}(2,q)) = 1 \quad \text{and} \quad K = L.$$

Now, $\mathrm{SL}(2,q)$ contains a subgroup H/L isomorphic to the symmetric group of degree 3. Since S is a Sylow 2-subgroup of G, we may replace H by a conjugate, if necessary, so that $H \cap S$ is a Sylow 2-subgroup of H. Let g be a 3-element of H such that gL is an element of order 3 in H/L. Then g does not normalize S because gL does not normalize SL/L.

We chose f (at the beginning of § 4) to be an arbitrary p'-element of $G \setminus N_G(SK)$. Since $SL = SK$, we may assume for this part of the proof that $f = g$. Hence, $B = A^f = A^g$.

By an argument similar to our argument above for p odd,

$$C_{C/Z}(g) = 1 \quad \text{and} \quad C_{Z/Y}(g) = 1. \tag{6.21}$$

We write C/Z and Z/Y as additive groups and let

$$\phi\colon (C/Z) \times (C/Z) \to Z/Y$$

be the bi-additive mapping induced by the commutator mapping. For any x in C/Z, g centralizes $x + x^g + x^{g^2}$, so that $x + x^g + x^{g^2} = 0$, by (6.21); and similarly for x in Z/Y.

By Proposition 4.5 and the definitions at the beginning of § 6,

$$G = G^* = \langle Z(Q_0), Z(R_0)\rangle = \langle A, B\rangle. \tag{6.22}$$

By (6.19) and (6.20) with $U = C$ and $V = Z$,

$$C_{C/Z}(A) = (A \cap C)Z/Z \quad \text{and} \quad |C/Z| = |C_{C/Z}(A)|^2$$

and

$$C/Z = C_{C/Z}(A) \times C_{C/Z}(B). \tag{6.23}$$

Therefore,

$$\phi(a, a') = 0 \text{ whenever } a, a' \text{ lie in } C_{C/Z}(A). \tag{6.24}$$

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Take any a in $C_{C/Z}(A)$ and $b' = C_{C/Z}(B)$. Let $b = a^g$ and $a' = b'^{g^2}$. Then $a' \in C_{C/Z}(A)$ and $b' \in C_{C/Z}(B)$. By (6.24),

$$\phi(a, a') = 0, \qquad \phi(b, b') = \phi(a^g, a'^g) = \phi(a, a')^g = 0$$

and

$$\begin{aligned} 0 &= \phi(a^{g^2}, a'^{g^2}) \\ &= \phi(-a - a^g, -a' - a'^g) \\ &= \phi(a^g, a') + \phi(a, a'^g) \\ &= \phi(b, a') + \phi(a, b'). \end{aligned}$$

Therefore,

$$\phi(a, b')^g = \phi(b, a'^{g^2}) = \phi(b, -a' - b') = -\phi(b, a') = \phi(a, b').$$

However, $C_{Z/Y}(g) = 1$, by (6.21). Thus, $\phi(a, b') = 0$. As $[b', a] = -[a, b']$, $\phi(b', a) = -\phi(a, b') = 0$. Since a and b' are arbitrary elements of $C_{C/Z}(A)$ and $C_{C/Z}(B)$, (6.23) and (6.24) and the argument above show that ϕ is identically zero. By (6.18), we are done. □

Step 2. The group D/Y is abelian.

Proof. Assume that D/Y is not abelian. We work towards a contradiction.

Recall that $D = [T, L]Z$. Since Z/Y is abelian, $[T, L]$ is not contained in Z. Since $T' \leqslant Z$ and $L \geqslant T$, we have $L > T$. By Proposition 6.8, q is 2 or 3.

Consider a chief series for G that contains the series

$$1 \leqslant Y < Z < D < G.$$

Let

$$Y = W_0 < W_1 < \cdots < W_k = D$$

be the portion of the chief series from Y to D.

Take i maximal such that $1 \leqslant i \leqslant k$ and W_i/Y is contained in the centre of D/Y. Since D/Y is not abelian, $1 \leqslant i \leqslant k - 1$.

Now, W_{i+1}/Y is not contained in the centre of D/Y. Take j maximal such that $0 \leqslant j \leqslant k$ and W_j/Y centralizes W_{i+1}/Y. Then $j \leqslant k - 1$ and W_{j+1}/Y does not centralize W_{i+1}/Y. To summarize:

Y contains $[W_i, D]$ (and hence $[W_i, W_{j+1}]$) and $[W_{i+1}, W_j]$, but not $[W_{i+1}, W_{j+1}]$.

By (6.17) and (6.18), $[D, D]Z/Z = Y/Z$. The previous paragraph shows that the bi-additive mapping $(T/Z) \times (T/Z) \to Y/Z$ induced by the commutator mapping restricts to a bi-additive surjective mapping

$$f\colon (W_{i+1}/W_i) \times (W_{j+1}/W_j) \to Y/Z$$

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such that

$$f(u^g, v^g) = f(u, v)^g \quad \text{for all } u \text{ in } W_{i+1}/W_i,\ v \text{ in } W_{j+1}/W_j \text{ and } g \text{ in } G.$$

Let $M_1 = W_{i+1}/W_i$, $M_2 = W_{j+1}/W_j$ and $M = Y/Z$. Since T centralizes every chief p-factor of G, conjugation induces action of G/T on M_1, M_2 and M. By Proposition 6.8 and (6.17), L/T acts non-trivially on M_1 and M_2 and trivially on M. By (6.19) and (6.20) applied to $U/V = M_k$ for $k = 1, 2$,

$$|M_k| = |C_{M_k}(A)|^2 = |C_{M_k}(S)|^2,$$
$$C_{M_1}(A) = (W_{i+1} \cap A)W_i/W_i,$$
$$C_{M_2}(A) = (W_{j+1} \cap A)W_j/W_j.$$

Therefore,

$$f(u, v) = 0 \quad \text{for all } u \text{ in } C_{M_1}(A) \text{ and } v \text{ in } C_{M_2}(A),$$

and, by Theorem 3.1, the hypothesis of Lemma 2.18 is satisfied with G/T, K/T and L/T in place of G, K and L. Therefore, G/T centralizes the image of f. However, f is a surjective mapping onto Z/Y, which is a standard module for G/L. This contradiction shows that D/Y is abelian. □

Step 3. D/Y centralizes C/Y.

Proof. Since L/T is a p'-group, there exists a complement, L_0, to T in L, by the Schur–Zassenhaus Theorem. Then $L = L_0T$. As $L = C_G(Z)$ and L centralizes C/Z, L_0 centralizes C/Z and Z. By Lemma 2.1, L_0 centralizes C.

Clearly, $C \lhd G$ and $L_0 \leqslant C_G(C) \lhd G$. Therefore,

$$[T, L_0] \leqslant C_G(C) \leqslant C_G(C/Y).$$

As T/Z is abelian and $Z = Z(T)$,

$$C_G(C/Y) \geqslant [T, L_0]Z \geqslant [T, L_0T]Z = [T, L]Z = D.$$

Thus, D/Y centralizes C/Y, as desired. □

As mentioned at the beginning of the proof, Steps 1–3 complete the proof of the proposition. □

Corollary 6.11. *The group S satisfies*

(a) $S' \leqslant (A \cap T)Z = A^\star$ *and*

(b) $\gamma_3(S) \leqslant [Z, A]T' \leqslant Z(S)$ *and* $\gamma_4(S) = 1$.

Proof. Take x in $A \cap T$, y in T, and a in A. By Proposition 6.10, $T' \leqslant Z(G)$. Hence,

$$[x, y] = [x, y]^a = [x^a, y^a] = [x, y^a] \quad \text{and} \quad [x, y^{-1}y^a] = [x, y]^{-1}[x, y^a] = 1.$$

Thus, $[y, a] = y^{-1}y^a \in C_T(A \cap T) = C_T((A \cap T)Z) = C_T(A^\star) = A^\star$. Since y and a were chosen arbitrarily, $[T, A] \leqslant A^\star$.

Now, $[T, A] \lhd AT = S$. So $[T, A]Z(G) \lhd S$. As $T' \leqslant Z(G)$ and A is abelian, $S/[T, A]Z(G)$ is abelian. Therefore, since $Z(G) \leqslant Z \leqslant A^\star$,

$$S' \leqslant [T, A]Z(G) \leqslant A^\star,$$

which proves (a).

By Lemma 3.2, $[Z, S] \leqslant Z(S)$. Hence, by (a),

$$\gamma_3(S) = [S', S] \leqslant [(A \cap T)Z, AT] \leqslant T'[Z, A] \leqslant Z(G)Z(S) = Z(S).$$

Then $\gamma_4(S) \leqslant [Z(S), S] = 1$. This proves (b). □

Proposition 6.12. *The subgroup L contains T and satisfies the following conditions.*

(a) $G/L \simeq \mathrm{SL}(2, q)$.

(b) *If $q > 2$, then $L = T$.*

(c) *If $q = 2$, then G/T is a dihedral group of order $2 \cdot 3^k$ for some positive integer k.*

(d) *$Z/Z(G)$ is a standard module for G/L.*

Proof. By (E_0), $C_G(T) \leqslant T$. By Proposition 6.10, $T/Z(G)$ is abelian, and thus is the centre of itself. Therefore, the group W_1 in Theorem 3.1 is equal to T, and all of this proposition follows from Theorem 3.1. □

Proposition 6.13. *Let $H = \mathrm{O}^p(G)$, $P = H \cap S$ and $R = H \cap T$. Then*

(a) *T/Z is elementary abelian,*

(b) *if p is odd, then $T/Z(G)$ is elementary abelian,*

(c) $[R, H] = R$,

(d) *if p is odd, then R has exponent p, and*

(e) *if $p \geqslant 5$, then P has exponent p and $S = PZ(G)$.*

Proof. Recall that $Z(G) < Z(S) < Z$, by Lemma 3.2. Let $Y = Z(G)$.

(a) Let $\bar{T} = T/Z$. By Proposition 6.6, $\bar{T}$ is abelian. Let $T_1/Z = \Omega_1(\bar{T})$. Then $T_1 \lhd G$.

Take any element a of A and let α be the automorphism of $\bar{T}$ induced by conjugation by a. We regard the operation of $\bar{T}$ as addition, and α as an invertible element of the endomorphism ring of $\bar{T}$. Let $\delta = \alpha - 1$. Since

$$[T, A, A] \leqslant \gamma_3(S) \leqslant Z(S) < Z,$$

by Corollary 6.11, $\delta^2 = (\alpha - 1)^2 = 0$.

As S/T is elementary abelian, a^p lies in T and hence centralizes T/Z. Therefore,

$$1 = \alpha^p = (1 + \delta)^p = 1 + p\delta,$$

whence $p\delta = 0$. Thus, $[T, a]^p \leqslant Z$ and $[T, a] \leqslant T_1$. This shows that A centralizes T/T_1. Since $T' \leqslant Z \leqslant T_1$ and $S = AT$,

$$S \text{ centralizes } T/T_1.$$

As $T, T_1 \lhd G$, we see that $C_G(T/T_1)$ is a normal subgroup of G that contains S and hence $\langle S^G \rangle$, which is G, by Proposition 3.4. Thus, $[T, G] \leqslant T_1$. However, by Proposition 6.6,

$$T = [T, H]Y = [T, G]Z. \tag{6.25}$$

Since $[T, G]Z \leqslant T_1 \leqslant T$, we obtain $T_1 = T$, i.e.

$$T/Z \text{ is elementary abelian.}$$

(b) Assume p is odd. We follow the proof of (a) with a few changes.

Recall that $Y = Z(G)$. We take $\bar{T}$ to be T/Y instead of T/Z. By Proposition 6.10, $\bar{T}$ is abelian.

Take any element a of A. Define α and δ as in the proof of (a), but acting on $\bar{T}$ instead of T/Z. It is possible that $\delta^2 \neq 0$. But since $[T, A, A, A] = 1$ by Corollary 6.11, $\delta^3 = 0$. Let $k = (p - 1)/2$. Then

$$1 = \alpha^p = (1 + \delta)^p = 1 + p\delta + pk\delta^2 \quad \text{and} \quad 0 = p\delta + pk\delta^2 = p\delta(1 + k\delta).$$

Then $0 = 0(1 - k\delta) = p\delta(1 + k\delta)(1 - k\delta) = p\delta(1 - k^2\delta^2) = p\delta$ because $\delta^3 = 0$.

As in the proof of (a), we obtain $[T, G] \leqslant T_1$, where $T_1/T = \Omega_1(T/Y)$. Then Proposition 6.6 yields $T = [T, H]Y \leqslant T_1$. Consequently, $T = T_1$, and T/Y is elementary abelian.

(c) Here, p is arbitrary. Let $Q = [T, H]$. Since $T, H \lhd G$, we see that $Q \lhd G$ and $Q \leqslant T \cap H = R \lhd G$, and P is a Sylow p-subgroup of H.

Let $\bar{G} = G/Q$. For every subgroup X of G, let $\bar{X} = XQ/Q$. By (6.25), $T = QY$ and $\bar{T} = \bar{Y} \leqslant Z(\bar{G})$. Since $S = TA$ and A is abelian, $\bar{S}$ is abelian and $\bar{R} \leqslant Z(\bar{H})$.

As H is generated by p'-elements, so is $\bar{H}$. So $\bar{H}/\bar{H}'$ is a p'-group, and $\bar{P} \leqslant \bar{H}'$. By Lemma 2.1,

$$\bar{R} \leqslant \bar{P} \cap Z(\bar{H}) = \bar{P} \cap \bar{H}' = Z(\bar{H}) \leqslant \bar{P}' = 1 \quad \text{and} \quad R = Q = [T, H].$$

By Proposition 6.6, $T = [T, H]Y = RY$. Hence, $R = [T, H] = [RY, H] = [R, H]$, as desired.

(d) Assume p is odd. Since T has nilpotence class at most 2, $\Omega_1(T)$ has exponent p, by Theorem 2.4.

Take any elements u of T and g of G. Let $v = u^g$. By (b), $u^p \in Y = Z(G)$. Hence, $v^p = (u^g)^p = (u^p)^g = u^p$. By Theorem 2.4, $(uv^{-1})^p = 1$, and $uv^{-1} \in \Omega_1(T)$. Thus,

$$[T, G] \leqslant \Omega_1(T).$$

So $R = [T, H] \leqslant \Omega_1(T)$, and R has exponent p.

(e) Assume $p \geqslant 5$. Let $W = H \cap Y$. By Corollary 6.11 and Theorem 2.4,

$$S \text{ has nilpotence class at most 3 and } \Omega_1(S) \text{ has exponent } p. \tag{6.26}$$

Similarly,

$$S/W \text{ has nilpotence class at most 3 and } \Omega_1(S/W) \text{ has exponent } p. \tag{6.27}$$

By Proposition 6.12, $L = T$ and $G/L \cong \mathrm{SL}(2, q)$. Since $q \geqslant p \geqslant 5$, we may take x and M as in Lemma 5.4. Then x lies in $P \setminus T$, M is a non-identity p'-subgroup of G normalized by x, and x^p lies in $C_T(M) \cap H$.

By Proposition 6.9, every chief factor U/V of G such that $Y \leqslant V < U \leqslant T$ is a standard module for G/L. Thus, $C_{U/V}(M) = 1$ for every such chief factor. By arguing as in Step 1 of the proof of Proposition 6.10, we see that $C_T(M) \leqslant Y$. Hence,

$$x^p \in C_T(M) \cap H \leqslant Y \cap H = W. \tag{6.28}$$

For each element g and subgroup G^* of G, let $\bar{g}$ and $\overline{G^\star}$ be the element gW and subgroup G^*W/W of G/W. Let $F = N_H(P)$. Since $W \leqslant H \cap T = R \leqslant P$,

$$F/R = N_{H/R}(P/R) \quad \text{and} \quad \bar{F}/\bar{R} = N_{\bar{H}/\bar{R}}(\bar{P}/\bar{R}).$$

By (d) and (6.28),

$$\Omega_1(\bar{P}) \geqslant \langle \bar{x}, \bar{R} \rangle > \bar{R}.$$

So $\Omega_1(\bar{P})/\bar{R}$ is a non-identity normal subgroup of $\bar{F}/\bar{R}$ contained in $\bar{P}/\bar{R}$. However, from the structure of $\mathrm{SL}(2, q)$ for $q \geqslant 4$,

$$G/T = \mathrm{O}^p(G/T) = \mathrm{O}^p(G)T/T = HT/T \cong H/(H \cap T) = H/R \cong \bar{H}/\bar{R},$$
$$\bar{P}/\bar{R} \text{ is a minimal normal subgroup of } \bar{F}/\bar{R},$$
$$\bar{P}/\bar{R} = [\bar{F}/\bar{R}, \bar{P}/\bar{R}]. \tag{6.29}$$

Therefore, $G = HT$, $S = PT$, $\bar{P}/\bar{R} = \Omega_1(\bar{P})/\bar{R}$ and $\bar{P} = \Omega_1(\bar{P})$. By (6.25) and (6.27),

$$S = PRY = PY \quad \text{and} \quad \bar{P} \text{ has exponent } p. \tag{6.30}$$

Since P is a normal Hall subgroup of F, it has a normal complement F_0, which is a Hall p'-subgroup of F. Then $F = F_0P$. As $\bar{P}/\bar{R}$ is abelian, (6.29) yields

$$\bar{P}/\bar{R} = [\bar{F}/\bar{R}, \bar{P}/\bar{R}] = [\bar{F}, \bar{P}]\bar{R}/\bar{R} = [\bar{F}_0, \bar{P}]\bar{R}/\bar{R},$$

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whence

$$P = [F_0, P]R. \tag{6.31}$$

By (6.26), S has nilpotence class at most 3 and $\Omega_1(S)$ has exponent p. Then, from (d), (6.30), (6.31) and the method of proof of part (d), $P = \Omega_1(P)R \leqslant \Omega_1(S)$. So P has exponent p, as desired. □

Proof of Theorem C. Now we prove Theorem C. By Remark 5.6, we need to check only (5.15), (5.16) and parts (a), (c) and (d) of the theorem when $G = G^*$. Recall that we assumed $G = G^*$ before Lemma 6.1, and that we defined

$$\hat{G} = \mathrm{O}^p(G), \quad \hat{S} = S \cap \hat{G} \quad \text{and} \quad \hat{T} = \mathrm{O}_p(\hat{G})$$

in Theorem C. Moreover, by Proposition 4.5, $T^* = \mathrm{O}_p(G^*) = \mathrm{O}_p(G) = T$.

As $[\mathrm{O}_{p'}(G), T] \leqslant \mathrm{O}_{p'}(G) \cap \mathrm{O}_p(G) = 1$, we have $\mathrm{O}_{p'}(G) \leqslant C_G(T) \leqslant T$. Therefore, $\mathrm{O}_{p'}(G) = 1$.

Since $\hat{G} \lhd G$ and S is a Sylow p-subgroup of G,

$$\hat{S} \text{ is a Sylow } p\text{-subgroup of } \hat{G} \text{ and } \mathrm{O}_{p'}(Z(\hat{G})) \leqslant \mathrm{O}_{p'}(\hat{G}) \leqslant \mathrm{O}_{p'}(G) = 1.$$

So

$$Z(\hat{G}) = \mathrm{O}_{p'}(Z(\hat{G})) \times \mathrm{O}_p(Z(\hat{G})) = \mathrm{O}_p(Z(\hat{G})) \leqslant \mathrm{O}_p(\hat{G}) = \hat{T}.$$

Hence,

$$Z(\hat{G}) \leqslant Z(\hat{T}). \tag{6.32}$$

By Corollary 6.11, S has nilpotence class at most 3. As $\hat{S}$ is a subgroup of S, we obtain part (a) of the theorem.

Recall that $Z = Z(T)$. As before, let $Y = Z(G)$. In the proof of part (e) of Proposition 6.6, we obtained $Z = [Z, H]Y$, i.e. $Z = [Z, \hat{G}]Y$. Clearly,

$$[Z, \hat{G}] \leqslant Z \cap T \cap \hat{G} = Z(T) \cap \hat{T} \leqslant Z(\hat{T}).$$

Hence, $Z \leqslant Z(\hat{T})Z(G)$ and $[Z, \hat{G}] \leqslant [Z(\hat{T})Z(G), \hat{G}] = [Z(\hat{T}), \hat{G}]$. By Proposition 6.12, Z/Y is a standard module for G/L, and thus is not centralized by $\mathrm{O}^p(G)$, i.e. $\hat{G}$. So $1 < [Z, \hat{G}] \leqslant [Z(\hat{T}), \hat{G}]$, and $Z(\hat{T})$ is not contained in $Z(\hat{G})$. Therefore, by (6.32),

$$Z(\hat{G}) < Z(\hat{T}). \tag{6.33}$$

By Proposition 6.10 and 6.13, $T/Z(G)$ is abelian, T/Z is elementary abelian, and $[\hat{T}, \hat{G}] = \hat{T}$. Since $\hat{T} \leqslant T$, we obtain

$$\hat{T}' \leqslant T' \cap \hat{T} \leqslant Z(G) \cap \hat{T} \leqslant Z(\hat{G}).$$

By (6.33), $Z(\hat{G}) < Z(\hat{T}) \leqslant \hat{T}$. This proves part (c) of the theorem.

Parts (d) and (e) of Proposition 6.13 give part (d) of the theorem.

Now recall statements (5.15) and (5.16) in Remark 5.6. Since $G = G^*$, we may restate them as follows.

(5.15′) If U/V is a chief factor of G and $Z(\hat{G}) \leqslant V < U \leqslant \hat{T}$, then U/V is not a central chief factor.

(5.16′) If $q > 2$, then $L = T$, and every chief factor U/V of G as in (5.15′) is a standard module for G/T.

Take a chief factor U/V of G as in (5.15′). Then $Z(\hat{G}) \leqslant V < U \leqslant \hat{T}$ and

$$V \leqslant U \cap VY = V(U \cap Y) = V(U \cap Z(G)) \leqslant VZ(\hat{G}) = V.$$

Thus, $V = U \cap VY$. We obtain an isomorphism of G-modules

$$UY/VY = U(VY)/VY \cong U/(U \cap VY) = U/V.$$

Therefore, UY/VY is a chief factor of G isomorphic to U/V.

Consider a chief series of G that contains the series

$$1 \leqslant Y < Z \leqslant T \leqslant G.$$

Since $Y \leqslant VY < UY \leqslant T$, the proof of the Jordan–Hölder Theorem for chief series [**16**, pp. 125–127] shows that some chief factor W/X from this chief series satisfies $Y \leqslant X < W \leqslant T$ and is isomorphic to UY/VY, and hence to U/V.

Since Z/Y is a standard module for G/L (by Proposition 6.12), it is a non-central chief factor of G, and we have

$$W/X = Z/Y \quad \text{or} \quad Z \leqslant X < W \leqslant T.$$

However, in the latter case, W/X is not central, by Proposition 6.6. Thus, in all cases, W/X, and hence U/V, are not central. This proves (5.15′).

To prove (5.16′), assume that $q > 2$ and take a chief factor U/V as above. By Proposition 6.12, $L = T$. Therefore, L centralizes U/V. By Proposition 6.9, U/V is a standard module for G/T, as desired.

This completes the proof of Theorem C. □

7. Examples

As mentioned in § 1, the group S_{MCL} in Theorem C has an advantage over the group S_2 in the exceptional case of [**12**] in being defined more explicitly and having (like $J(S)$) the property that no other subgroup of S is isomorphic to it. But Theorem C has the disadvantage of allowing a wider family of exceptions to specifying a characteristic subgroup of S that is normal in G. We illustrate this in Examples 7.1–7.3, where S is 'large' enough that one of the groups S_1 or S_2 in the exceptional case of [**12**] is normal in G, but 'small' enough that conditions (a)–(i) in Theorem C are satisfied and neither $Z(S)$ nor S_{MCL} is normal. Examples 7.2 and 7.3 also show that some of the restrictions on p and q in Theorem C are necessary.

In Theorem C, $\tilde{J}(S)$ is not normal in G, while S_{MCL} may be normal. In contrast, in Examples 7.4 and 7.5, $Z(J(S))$ is normal, while $Z(S_{\mathrm{MCL}})$ is not. In Examples 7.6 and 7.7, (E_0) is satisfied, but no non-identity characteristic subgroup of S is normal in G.

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Example 7.1. Let Q be a quaternion group of order 8 if $p = 2$ and a non-abelian group of order p^3 and exponent p if p is odd. It is well known that the automorphism group of Q contains a subgroup H isomorphic to $\mathrm{SL}(2, p)$ that centralizes $Z(Q)$. (For $p = 2$, take H as in Example 7.2.) Let E be a standard module for H.

Let m be a natural number and $Q_1, \dots, Q_m$ be isomorphic copies of Q. We embed E, $Q_1, \dots, Q_m$ in their direct product $T = E \times Q_1 \times \cdots \times Q_m$ and let H act on T by acting on each component according to the action above. Let G be the semi-direct product of T by H.

Let S be the product of T with a Sylow p-subgroup $\langle \sigma \rangle$ of H, and let K be the product of T with the centre of H. It is easy to verify that $T = \mathrm{O}_p(G)$ and that G satisfies (E_0) for $p^n = p$. To verify the hypothesis of Theorem C, we must show that $S = \tilde{J}(S)$.

Clearly,

$$Z(G) = Z(Q_1) \times \cdots \times Z(Q_m), \quad Z(S) = C_E(\sigma) \times Z(G), \quad \mho^1(Z(S)) = 1 \tag{7.1}$$

and $Z(T) = E \times Z(G)$. Then $T/Z(S)$ is abelian and $Z_2(S)/Z(S) = Z(S/Z(S)) \leqslant T/Z(S)$. So

$$Z_2(S) \leqslant T < S. \tag{7.2}$$

Consider first the case in which p is odd. Here, T has exponent p. It is well known that σ centralizes a subgroup B of order p^2 in Q. Let $B_1, \dots, B_m$ be the corresponding subgroups of $Q_1, \dots, Q_m$. Let

$$\tilde{B} = B_1 \times \cdots \times B_m, \quad A^* = E \times \tilde{B} \quad \text{and} \quad A = C_E(\sigma) \times \tilde{B} \times \langle \sigma \rangle.$$

It is easy to see that A and A^* are large abelian subgroups of S and that

$$d(S) = d(T) = p^{2m+2}, \quad J(T) = T, \quad J(S) = S \quad \text{and} \quad S' = \Phi(S) = C_E(\sigma) \times \tilde{B}. \tag{7.3}$$

Next, consider the case in which $p = 2$. Then (see Example 7.2) Q contains elements i, j, k such that

$$i^\sigma = j, \quad j^\sigma = i, \quad k = ij \quad \text{and} \quad k^\sigma = k^{-1}.$$

Let $i_1, \dots, i_m$ and $j_1, \dots, j_m$ and $k_1, \dots, k_m$ be elements of $Q_1 \times \cdots \times Q_m$ corresponding to i, j and k, and let $\sigma' = i_1 i_2 \cdots i_m \sigma$ and

$$\tilde{B} = \langle k_1, \dots, k_m \rangle, \quad A^* = E \times \tilde{B} \quad \text{and} \quad A = C_E(\sigma) \times \tilde{B}\langle \sigma' \rangle.$$

Then

$$\sigma'^2 = (i_1 i_2 \cdots i_m)\sigma^{-1}(i_1 i_2 \cdots i_m)\sigma = (i_1 i_2 \cdots i_m)(j_1 j_2 \cdots j_m) = k_1 k_2 \cdots k_m.$$

Since σ' centralizes σ'^2, σ' centralizes $\tilde{B}$. It is easy to see that A and A^* are large abelian subgroups of S, and (7.3) is still valid in this case.

Thus, (7.1)–(7.3) hold for all choices of p. Note that $|S| = p|T| = p \cdot p^2 \cdot (p^3)^m = p^{3m+3}$ and, by (7.1), $|Z(S)| = p^{m+1}$. Therefore,

$$|S|\,|Z(S)| = p^{3m+3} \cdot p^{m+1} = p^{4m+4} = (p^{2m+2})^2 = d(S)^2.$$

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By Lemma 2.12, the minimal CL-subgroups of S are the large abelian subgroups of S, and $S = S_{\mathrm{CL}} = S_{\mathrm{MCL}} = \tilde{J}(S)$.

By (7.1) and (7.3), $Z(S) \neq Z(G)$ and $\tilde{J}(S) = S$. Since $S = S_{\mathrm{MCL}}$, it follows from Lemma 2.19 that neither of the two subgroups $Z(S)$ and S_{MCL} of Theorem C is normal in G, and G satisfies conditions (a)–(i) of Theorem C.

In contrast, (7.1)–(7.3) yield that $\tilde{J}(S) = S$, $\mho^1(Z(S)) = 1$ and S' is not contained in $Z(S)$. Hence, S has nilpotence class at least 3 (in fact, precisely 3). Therefore, if $p \neq 3$, then S satisfies the hypothesis of the exceptional case of [**12**] discussed in § 1 (i.e. case (c) of Theorem D of [**12**]), and one of the pair of subgroups S_1, S_2 given in that case is normal in G.

Actually, the proof of Theorem D of [**12**] (on p. 450 of [**12**], where $Z_2(G)$ in (7.1) should be corrected to $Z_2(S)$) shows a little more for $p \neq 3$: $S_2 \lhd G$ because we have the conditions

$$\tilde{J}(S) = S, \quad \mho^1(Z(S)) = 1, \quad Z(S) \neq Z(G) \quad \text{and} \quad \Omega_1(Z_2(S)) \leqslant \mathrm{O}_p(G).$$

As $S = S_{\mathrm{MCL}}$, our suspicion (in § 1) that $S_2 \geqslant S_{\mathrm{MCL}}$ is false. (Note that here we obtained $S_2 \lhd G$ without assuming that S_1 is not normal in G. Indeed, one may calculate that $S_1 = Z(G) \lhd G$ here.)

Again, assume $p \neq 3$. Since S_2 is an intersection of subgroups $\mathrm{O}_p(G^*)$ for groups G^* that satisfy (E_0), $S_2 \geqslant \Phi(S) = C_E(\sigma) \times \tilde{B}$ by (7.3). It is easy to see that the normal closure of $\Phi(S)$ in G is equal to T. Since $S_2 \lhd G$, we have $S_2 = T$.

This example illustrates another difference between Theorem C and the results of [**12**]. If $p \neq 3$ and S is 'too small' to satisfy the hypothesis of [**12**], then, by Remark 1.2 of [**12**], a group G satisfying (E_0) will have a unique non-central chief factor within $\mathrm{O}_p(G)$ (and this chief factor lies within $Z(\mathrm{O}_p(G))$). But for G in this example, G has precisely $m+1$ non-central chief factors within $\mathrm{O}_p(G)$, since one occurs for each of E, $Q_1/Z(Q_1), \dots, Q_m/Z(Q_m)$.

Now assume that $p \geqslant 5$ and $m = 1$. Then $S_2 = T = E \times Q_1$ and T has exponent p. Let $x_1 = \sigma$. Take x_2 in $B_1 \setminus Z(G)$, x_5 in $E \setminus C_E(\sigma)$, and x_6 in $Q_1 \setminus B_1$, and take $x_3 = [x_1, x_5]$ and $x_4 = [x_2, x_6]$. Then

$$E = \langle x_3, x_5 \rangle, \quad Q_1 = \langle x_2, x_4, x_6 \rangle, \quad Z(Q) = \langle x_4 \rangle, \quad T = \langle x_2, x_3, \dots, x_6 \rangle$$

and $[x_i, x_j] = 1$ whenever $1 \leqslant i, j \leqslant 6$ and $|j - i| \leqslant 3$. Since $\langle x_1, x_5 \rangle$ is a non-abelian group of order p^3 generated by elements of order p, it has exponent p. Now

$$\langle x_1, \dots, x_5 \rangle = \langle x_1, x_3, x_5 \rangle \times \langle x_2, x_4 \rangle$$

and there exists an isomorphism ϕ of $\langle x_1, \dots, x_5 \rangle$ onto T given by $\phi(x_i) = x_{i+1}$ for $i = 1, 2, \dots, 5$. (This example comes from Example 8.2 of [**12**] and § 9 of [**10**].)

We saw above that T does not contain S_{MCL}. The isomorphism ϕ shows more generally that T does not contain any non-identity subgroup S^* satisfying the condition that every subgroup of S isomorphic to S^* is equal to S^*.

Example 7.2. In Theorem C, part (d) yields that if $\hat{T}/Z(\hat{G})$ is not elementary abelian and $p \neq 2$, then $Z(S)$ or S_{MCL} is normal in G. Here, we show that the assumption that $p \neq 2$ is necessary.

Let H be a group isomorphic to the symmetric group of order 3. Let U be the direct product of two cyclic groups of order 4 with a quaternion group of order 8. Then

$$H = \langle \sigma, \tau \rangle \quad \text{and} \quad U = \langle a \rangle \times \langle b \rangle \times \langle i, j \rangle,$$

where $\sigma^2 = \tau^3 = 1$, $a^4 = b^4 = i^4 = j^4 = 1$ and $i^2 = j^2 = [i, j]$. Let $ij = k$, as usual.

We let H act faithfully on U by defining

$$a^\sigma = b, \quad b^\sigma = a, \quad i^\sigma = j, \quad j^\sigma = i, \quad a^\tau = b, \quad b^\tau = a^{-1}b^{-1}, \quad i^\tau = j^{-1}, \quad j^\tau = k^{-1}.$$

Inside U, let $c = ai$, $d = bj$ and $z = [i, j]$. Note that $\Phi(U) = \langle a^2, b^2, z \rangle = \langle c^2, d^2, z \rangle$.

Let $T = \langle c, d, \Phi(U) \rangle$. Then $z = [c, d]$ and $\Phi(T) = \Phi(U) = Z(T)$. Since

$$c^\sigma = d, \quad d^\sigma = c, \quad c^\tau = dz \quad \text{and} \quad d^\tau = c^{-1}d^{-1}z,$$

T is invariant under H. Let G be the semi-direct product of T by H.

Let $S = \langle T, \sigma \rangle$. Then S is a Sylow 2-subgroup of G,

$$|T| = 2^5, \quad |S| = 2^6, \quad Z(S) = C_{Z(T)}(\sigma) = \langle c^2d^2, z \rangle \quad \text{and} \quad Z(G) = \langle z \rangle. \tag{7.4}$$

Moreover, $T = \mathrm{O}_p(G)$ and G satisfies (E_0) for $p^n = 2$. Since $\langle c, Z(T) \rangle$ is an abelian subgroup of T of order 2^4 and T is not abelian,

$$d(S) \geqslant d(T) = 2^4.$$

We claim that $d(S) = 2^4$. Suppose A is an abelian subgroup of S. Then $|A| \leqslant 2^4$ if $A \leqslant T$. So assume that A is not contained in T. Then

$$A \cap Z(T) \leqslant C_{Z(T)}(\sigma) = Z(S) = \langle c^2d^2, z \rangle < Z(T)$$

and $(A \cap T)Z(T)$ is an abelian subgroup of T. Therefore,

$$T > (A \cap T)Z(T) > A \cap T, \quad |A \cap T| \leqslant |T|/2^2 = 2^3 \quad \text{and} \quad |A| = 2|A \cap T| \leqslant 2^4,$$

as desired. Thus, $d(S) = 2^4$.

Let $A^* = \langle \sigma d, z \rangle$. Since $\langle c, Z(T) \rangle$ and $\langle d, Z(T) \rangle$ are abelian subgroups of order 2^4 in T that generate T, we have $T = J(T)$. Moreover,

$$\begin{aligned}
(\sigma d)^2 &= \sigma^{-1} d \sigma d = cd, \\
(\sigma d)^4 &= (cd)^2 = (aibj)^2 = (ab)^2k^2 \\
&= a^2b^2z = c^2zd^2zz = c^2d^2z.
\end{aligned}$$

So σd has order 8, $A^* = \langle \sigma d \rangle \times \langle z \rangle$ and A^* is abelian of order 16. Therefore, $J(S) \geqslant \langle J(T), A^* \rangle = S$ and $S = J(S)$.

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Here,

$$|S|\,|Z(S)| = 2^6 \cdot 2^2 = 2^8 = d(S)^2.$$

By Lemma 2.12, the minimal CL-subgroups of S are the large abelian subgroups of S, and $S = S_{\mathrm{CL}} = S_{\mathrm{MCL}} = \tilde{J}(S)$. By (7.4), $Z(S) \neq Z(G)$. Now, as in Example 7.1, neither of the subgroups $Z(S)$ and S_{MCL} of Theorem C is normal in G, but one of the subgroups S_1, S_2 for this case of [**12**] is normal in G. (In fact, $S_2 = T \triangleleft G$, as in Example 7.1.) So G satisfies conditions (a)–(i) of Theorem C. However, it is easy to see that

$$\hat{G} = \mathrm{O}^p(G) = T\langle z\rangle, \quad \hat{T} = \mathrm{O}_p(\hat{G}) = T, \quad Z(\hat{G}) = C_{Z(T)}(z) = \langle z\rangle = Z(G)$$

and $\hat{T}/Z(\hat{G})$ is not elementary abelian, unlike the case when p is odd.

Further calculation shows that, for every large abelian subgroup A of S, $|\Omega_1(Z(A))| = |\Omega_1(A)| \leqslant 2^3 < d(S)$ because A is not elementary abelian. Since $|\Omega_1(A)| = 2^3$ for $A = \langle c, Z(T)\rangle$, the parameter $mz(S)$ in Theorem B is equal to 2^3 and we have

$$1 < S_\Phi = \langle \Phi(A) \mid A \text{ is a large abelian subgroup of } S \text{ and } |\Omega_1(A)| = 2^3\rangle.$$

Since $Z(S) \neq Z(G)$, Lemma 2.19 and Theorem B yield that S_Φ is a normal subgroup of G. (In fact, $S_\Phi = \Phi(T) = Z(T) > 1$.)

Example 7.3. In Theorem C, part (h) yields that if $L > T$ and $q > 2$, then $Z(S)$ or S_{MCL} is normal in G. Here, we show that the assumption that $q > 2$ is necessary.

Let F be the Galois field of order 2^6. Then the multiplicative group $F^\times$ contains a unique subgroup M of order 9, and the Galois group of F contains a unique element σ of order 2, given by $x \mapsto x^8$. We may regard σ and the elements of M as permutations of F. Then σ normalizes M.

Let $H = M\langle\sigma\rangle$. Then H is a dihedral group of order 18. Therefore, $H/\Omega_1(M)$ is isomorphic to the symmetric group of degree 3, so that H acts on a Klein 4-group E with kernel $\Omega_1(M)$.

Let R be the set of all triples (x, y, z) for $x, y \in F$ and $z \in \mathrm{GF}(2)$. Define a bilinear mapping of $F \times F$ into $\mathrm{GF}(2)$ by $f(x, y) = T(xy^8)$, where T denotes the trace function from F to $\mathrm{GF}(2)$. Note that $f(x^\alpha, y^\alpha) = f(x, y)$ whenever $\alpha \in M$ or $\alpha = \sigma$, and hence whenever $\alpha \in H$.

We define multiplication on R by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + f(x', y)),$$

and we let $(x, y, z)^\alpha = (x^\alpha, y^\alpha, z)$ for $(x, y, z) \in R$ and $\alpha \in H$. Straightforward calculation shows that R is a group and that

$$[(x, y, z), (x', y', z')] = (0, 0, f(x', y) + f(x, y')).$$

Moreover, H acts faithfully on R by automorphisms. Finally, we embed E and R in their direct product T, and we embed T and H in their semi-direct product G.

Let $S = T\langle\sigma\rangle$. Then S is a Sylow 2-subgroup of G and $T = \mathrm{O}_2(G)$, and G satisfies (E_0) for $p^n = 2$. It is easy to see that R is an extra-special group of order 2^{13} and

$$|S| = 2^{16}, \quad Z(T) = E \times Z(R), \quad Z(S) = C_E(\sigma) \times Z(R) \quad \text{and} \quad |Z(S)| = 4.$$

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Let

$$R_1 = \{(x, y, z) \mid x, y \in \mathrm{GF}(8) \text{ and } z \in \mathrm{GF}(2)\}$$

and

$$A_1 = E \times R_1.$$

Then R_1 is an elementary abelian subgroup of R of order 2^7 that is centralized by σ. Let $A = C_E(\sigma) \times R_1 \times \langle\sigma\rangle$. Easy calculation shows that

$$A_1 \text{ and } A \text{ are elementary abelian subgroups of order } 2^9 \text{ in } S, \quad d(T) = d(S) = 2^9,$$
$$T = J(T) \quad \text{and} \quad S = J(S).$$

Therefore, $|S|\,|Z(S)| = 2^{16}{\cdot}2^2 = 2^{18} = d(S)^2$. By Lemma 2.12, the minimal CL-subgroups of S are the large abelian subgroups of S, and $S = S_{\mathrm{CL}} = S_{\mathrm{MCL}} = \tilde{J}(S)$.

As in Examples 7.1 and 7.2, neither of the subgroups $Z(S)$ and S_{MCL} of Theorem C is normal in G, but one of the subgroups S_1, S_2 for this case of [**12**] is normal in G. (As in Examples 7.1 and 7.2, $S_2 = T \lhd G$.) Since

$$L = C_G(Z(T)) = C_G(EZ(R)) = T\Omega_1(M) > T,$$

we have $L > T$, unlike the case when $q > 2$.

Example 7.4. Here we verify a case of Thompson's conjecture in § 1 when $S = J(S)$ and show that neither S_Φ nor S_{MCL} is normal in this case.

Assume $p \geqslant 5$. For convenience, we take $q = p$. Let G be the group denoted by G_{-a} in Example 8.1 of [**12**]. Then

$$G = \langle x \in P \mid x \text{ is a } p\text{-element}\rangle$$

for a rank-1 parabolic subgroup P of the simple group $G_2(p)$, P/G is a cyclic p'-group, and S is a Sylow p-subgroup of G, P and $G_2(p)$.

Let F be the field $\boldsymbol{F}_p$. In the usual notation for simple groups of Lie type [**4**], $S = U$ and $G = \langle x_{-a}(F), S\rangle$ for the short root a in a fundamental root system $\{a, b\}$ of type G_2. As usual, let $T = \mathrm{O}_p(G)$. Then

$$|S| = p^6, \quad G/T \cong \mathrm{SL}(2, p), \quad G \text{ satisfies } (E_0), \quad d(S) = p^3,$$
$$S = J(S) = \tilde{J}(S), \quad |Z(S)| = p \quad \text{and} \quad Z(S) = Z(T) \lhd G.$$

Moreover, T is an extra-special group of order p^5 and exponent p, and $T/Z(T)$ is a chief factor of order p^4 in G, and thus not a standard module for G/T.

In the usual notation, the Chevalley commutator formulae [**4**] give

$$Z(T) = x_{3a+2b}(F), \quad T = \langle x_b(F), x_{b+a}(F), x_{b+2a}(F), x_{b+3a}(F)\rangle,$$
$$S' = \langle x_{b+a}(F), x_{b+2a}(F), x_{b+3a}(F), Z(T)\rangle \quad (\text{of order } p^4),$$
$$[S', S] = \langle x_{b+2a}(F), x_{b+3a}(F), Z(T)\rangle \quad (\text{of order } p^3),$$
$$[S', S, S] = \langle x_{b+3a}(F), Z(T)\rangle = Z_2(S) \quad (\text{of order } p^2).$$

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Moreover, $S = \langle x_a(F), T\rangle$ and $Z_2(S) = C_T(x_a(F))$. Thus, S has nilpotence class 5, and it is a p-group of maximal class.

By Proposition 2.8 and Theorem 2.9, $S_{\mathrm{CL}} \geqslant \tilde{J}(S) = S$ and S_{CL} is a CL-subgroup of S. So $S = S_{\mathrm{CL}}$ and $f(S) = |S|\,|Z(S)| = p^6 \cdot p = p^7$. Let $S^* = C_S(Z_2(S))$. Then calculation shows that

$$S^* = \langle S', x_a(F)\rangle, \quad Z(S^*) = Z_2(S), \quad |S^*| = p^5, \quad |S^*|\,|Z(S^*)| = p^5 \cdot p^2 = p^7 = f(S)$$

and S^* is the unique minimal CL-subgroup of S. Therefore,

$$S_{\mathrm{MCL}} = S^* \quad \text{and} \quad S_{\Phi} = \Phi(S^*) = (S^*)' = [S', S].$$

Hence, none of S_{Φ}, $Z(S_{\mathrm{MCL}})$ or S_{MCL} is normal in G.

Here, $Z(J(S)) = Z(S) = Z(T) \lhd G$, in accordance with Thompson's conjecture in § 1.

Example 7.5. Assume p is odd. Let T be an extra-special group of order p^7 and exponent p, let H be $\mathrm{PSL}(2,p)$ and let σ be an element of order p in H. Let F be the prime field $\boldsymbol{F}_p$.

In Example 10.4 of [**8**] (where T, H and σ are denoted by H, L and x, respectively), it is shown that there exists a semi-direct product, G, of T by H satisfying the following conditions.

(a) $H/Z(H)$ is the direct sum of two copies, V_1 and V_2, of a three-dimensional vector space V over F on which H acts irreducibly as an orthogonal group.

(b) σ acts with cubic minimal polynomial on V_1 and V_2.

(c) For $S = T\langle\sigma\rangle$, S is a Sylow p-subgroup of G and $d(S) = d(T) = p^4$ and $J(S) = S$.

(d) $C_S(\sigma)$ is an elementary abelian subgroup of G of order p^4.

Clearly, $T = \mathrm{O}_p(G)$, $Z(S) = Z(T)$ and G satisfies (E_0) for $p^n = p$. Since $S = J(S)$, Proposition 2.8 and Theorem 2.9 yield that $S = S_{\mathrm{CL}} = \tilde{J}(S)$ and $f(S) = |S|\,|Z(S)| = p^8 \cdot p = p^9$. Let $S^* = C_S(Z_2(S))$.

This example is similar to Example 7.4. By similar methods, one sees that

$$|Z_2(S)| = p^3, \quad |S^*| = p^6 \quad \text{and} \quad Z(S^*) = Z_2(S);$$

S^* is the unique minimal CL-subgroup of S; and $S_{\mathrm{MCL}} = S^*$ and $S_{\Phi} = \Phi(S^*) = Z_2(S)$. Thus, none of S_{MCL}, $Z(S_{\mathrm{MCL}})$ or S_{Φ} is normal in G.

Since $\mathrm{SL}(2,p)$ is not involved in G, G is p-stable, by [**13**, Theorem 8.12].

Example 7.6. (Here, p is arbitrary.) Let H be $\mathrm{SL}(2,p)$, let V be a standard module for H, and embed V and H in their semi-direct product G.

There exist elements u, v of V and w of H such that

$$V = \langle u, v\rangle, \quad u^w = uv \quad \text{and} \quad v^w = v.$$

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Let $S = \langle V, w\rangle$, so that S is a Sylow p-subgroup of G. Then

$$u^p = v^p = w^p = 1, \quad [u,w] = v, \quad V = \mathrm{O}_p(G) \quad \text{and} \quad G \text{ satisfies } (E_0) \text{ for } p^n = p.$$

It is easy to see that V is the unique non-identity normal p-subgroup of G (because H permutes the non-identity elements of V transitively) and that there exists a unique automorphism α of S such that

$$u^\alpha = w, \quad w^\alpha = u^{-1} \quad \text{and} \quad v^\alpha = v.$$

Thus, V is not characteristic in S, and no non-identity characteristic subgroup of S is normal in G.

For an arbitrary power q of p, we may take H to be $\mathrm{SL}(2,q)$ instead of $\mathrm{SL}(2,p)$ and then generalize the proof above to show that no non-identity characteristic subgroup of S is normal in G. Alternatively, one may embed G in a rank-1 parabolic subgroup of $\mathrm{PSL}(3,q)$ and use [**4**, pp. 200–202] and the method of Example 7.7.

Example 7.7. In Theorem A and several related results, S has nilpotence class 2 if $p \neq 3$. We show here that the assumption that $p \neq 3$ is necessary.

Assume that $p = 3$. Let $q = 3^n$ for some natural number n. Take G and S to be the subgroups of $G_2(q)$ analogous to the subgroups G and S of $G_2(p)$ for p as in Example 7.4. (A different construction of G and S for $q = 3$ is given below.) Thus,

$$G = \langle x \in P \mid x \text{ is a 3-element}\rangle$$

for a rank-1 parabolic subgroup P of the simple group $G_2(q)$, P/G is a cyclic $3'$-group, and S is a Sylow 3-subgroup of G, P and $G_2(q)$. As usual, let $T = \mathrm{O}_3(G)$.

It is easy to see that G satisfies (E_0). By [**15**, pp. 358–359], S has nilpotence class 3 if $q = 3$. Since $G_2(q)$ contains $G_2(3)$, S has nilpotence class at least 3 in general. We will show that no non-identity characteristic subgroup of S is normal in G. Therefore, S satisfies conditions (a)–(f) of Theorem A. In particular, S has nilpotence class precisely 3.

Suppose W is a characteristic subgroup of S that is normal in G. Then $W \lhd N_P(S)$. By the Frattini argument (Lemma 2.1), $P = GN_P(S)$. Hence, $W \lhd P$. We must show that $W = 1$.

Since q is a power of 3, there exists an automorphism α of $G_2(q)$ that preserves S and takes P to the other rank-1 parabolic subgroup P^* of $G_2(q)$ that contains S [**4**, p. 206]. Then α preserves W, and $W = W^\alpha \lhd P^\alpha = P^*$. Hence, $W \lhd \langle P, P^*\rangle = G_2(q)$. As $G_2(q)$ is simple, $W = 1$, as desired.

Let $F = \boldsymbol{F}_q$. The main reason that this example is very different from Example 7.4 (where $Z(S) \lhd G$) is that here [**4**, pp. 206–210]

$$[x_a(F), x_{2a+b}(F)] = [x_{a+b}(F), x_{2a+b}(F)] = 1,$$

because F has characteristic 3. Indeed,

$$Z(S) = \langle x_{2a+b}(F), x_{3a+2b}(F)\rangle, \quad |Z(S)| = q^2 \quad \text{and} \quad d(S) = q^4.$$

For the case when $q = 3$, one can also construct G without using the group $G_2(3)$. One takes T to be a direct product

$$T = \langle x_2, x_6\rangle \times \langle x_3, x_5\rangle,$$

where $\langle x_2, x_6\rangle$ is a non-abelian group of order 3^3 and exponent 3, and $\langle x_3, x_5\rangle$ is an elementary abelian group of order 9. Let $x_4 = [x_2, x_6]$, and define automorphisms x_1 and x_7 of T by

$$\begin{gathered} x_2^{x_1} = x_2, \quad x_3^{x_1} = x_3, \quad x_5^{x_1} = x_3^{-1}x_5, \quad x_6^{x_1} = x_2x_3x_4x_5x_6, \\ x_2^{x_7} = x_2x_3^{-1}x_4^{-1}x_5x_6^{-1}, \quad x_3^{x_7} = x_3x_5, \quad x_5^{x_7} = x_5, \quad x_6^{x_7} = x_6. \end{gathered}$$

Then $x_i^3 = 1$ for $i = 1, \dots, 7$. Let G be the semi-direct product of T by $\langle x_1, x_7\rangle$. Then $T = \mathrm{O}_p(G)$.

By § 9 of [**10**], $\langle x_1, x_7\rangle$ is isomorphic to $\mathrm{SL}(2,3)$ and, for $S = \langle x_1, T\rangle$, there exists an isomorphism ϕ of S onto $\langle x_7, T\rangle$ determined by

$$\phi(x_i) = x_{i+1} \quad \text{for } i = 1, \dots, 6.$$

Clearly, $\langle x_1\rangle$ and S are Sylow 3-subgroups of $\langle x_1, x_7\rangle$ and of G, and G satisfies (E_0). Let g be an element of $\mathrm{SL}(2,3)$ such that $\langle x_7\rangle^g = \langle x_1\rangle$. Then the mapping given by $x \mapsto \phi(x)^g$ is an automorphism of S.

Suppose W is a characteristic subgroup of S that is normal in G. Then

$$W = \phi(W)^g \quad \text{and} \quad \phi(W) = W^{g^{-1}} = W.$$

From the definition of ϕ, we see that $W = 1$, as desired.

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