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THE GENERALIZED SINGULAR DIRECT PRODUCT FOR QUASIGROUPS

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1. **Introduction.** In [2], A. Sade gives a construction for quasigroups which he calls the *singular direct product*. In this paper we generalize Sades' construction. As an application we obtain a recursive construction for quasigroups orthogonal to their transposes. All quasigroups considered in this paper will be finite.

2. **The generalized singular direct product for quasigroups.** Let $(V, \odot)$ be an idempotent quasigroup and $(Q, \circ)$ a quasigroup with subquasigroup P. Let $P' = Q \backslash P$ and for each (v, w), $v \neq w \in V$, let $\otimes(v, w)$ be a binary operation on P' so that $(P', \otimes(v, w))$ is a quasigroup. We remark that the $|V|^2 - |V|$ operations $\otimes(v, w)$ are not necessarily related to each other nor to the operation $\circ$. We define the *generalized singular direct product*, denoted by $V \times Q(P, P' \otimes(v, w))$, to be the quasigroup $\oplus$ defined on the set $P \cup (P' \times V)$ by the conditions:

(1) $p_1 \oplus p_2 = p_1 \circ p_2$ if $p_1, p_2 \in P$;
(2) $(p', v) \oplus p = (p' \circ p, v)$ if $p \in P$, $p' \in P'$, $v \in V$;
(3) $p \oplus (p', v) = (p \circ p', v)$ if $p \in P$, $p' \in P'$, $v \in V$;
(4) $(p', v) \oplus (p_2', v) = p_1' \circ p_2'$ if $p_1' \circ p_2' \in P$
$\qquad = (p_1' \circ p_2', v)$ if $p_1' \circ p_2' \in P'$;
(5) $(p_1', v) \oplus (p_2', w) = (p_1' \otimes (v, w) p_2', v \odot w)$ if $v \neq w$.

We note that if $\otimes(v, w)$ is the same operation for all $v \neq w \in V$, we have the singular direct product as defined by Sade [2], whereas if $P = \varnothing$ and $\otimes(v, w) = \circ$ for all $v \neq w \in V$ we have the ordinary direct product.

3. **Construction of quasigroups orthogonal to their transposes.** The transpose of a quasigroup is the quasigroup obtained by leaving the headline and sideline of its multiplication table fixed and replacing the associated latin square with its transpose. Clearly any latin square orthogonal to its transpose must be diagonal so that a suitable permutation gives an idempotent latin square which is orthogonal to its transpose. In what follows all quasigroups which are orthogonal to their transposes will be considered idempotent.

Now let $(V, \odot)$ and $(Q, \circ)$ be quasigroups each of which is orthogonal to its transpose, and let P be a subquasigroup of Q such that $|P'| \neq 6$, where $P' = Q \backslash P$. Let $(P', \circ_1)$ and $(P', \circ_2)$ be orthogonal quasigroups. Denote by $(V, T_\odot)$, $(Q, T_\circ)$,

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$(P', T_{\circ_1})$, and $(P', T_{\circ_2})$ the transposes of the quasigroups $(V, \odot)$, $(Q, \circ)$, $(P', \circ_1)$, and $(P', \circ_2)$ respectively. Now write

$$V^2 \backslash \{(v, v) \mid v \in V\} = S_1 \cup S_2,$$

where $(v, w) \in S_1$ if and only if $(w, v) \in S_2$. We now construct the generalized singular direct product

$$(\mathscr{V}, \oplus) = V \times Q(P, P' \otimes (v, w)),$$

where

$$\begin{aligned} (P', \otimes(v, w)) &= (P', \circ_1) \quad \text{if } (v, w) \in S_1 \\ &= (P', T_{\circ_2}) \quad \text{if } (v, w) \in S_2, \\ V &= (V, \odot), \end{aligned}$$

and

$$Q = (Q, \circ).$$

If we denote the transpose of $(\mathscr{V}, \oplus)$ by $(\mathscr{V}, T_{\oplus})$, then

$$(\mathscr{V}, T_{\oplus}) = V \times Q(P, P' \otimes (v, w)),$$

where

$$\begin{aligned} (P', \otimes(v, w)) &= (P', \circ_2) \quad \text{if } (v, w) \in S_1 \\ &= (P', T_{\circ_1}) \quad \text{if } (v, w) \in S_2, \\ V &= (V, T_{\odot}), \end{aligned}$$

and

$$Q = (Q, T_{\circ}).$$

The proof that $(\mathscr{V}, \oplus)$ and $(\mathscr{V}, T_{\oplus})$ are orthogonal is analogous to a similar result in Sade's paper and can be found there [2]. As a consequence of this result we have the following theorem.

THEOREM 1. *If V and Q are quasigroups orthogonal to their transposes, and if Q contains a subquasigroup P such that $|Q \backslash P| \neq 6$, then there is a quasigroup of order $v(q-p)+p$ orthogonal to its transpose, where v, q, and p denote the orders of V, Q, and P respectively.*

4. **Applications.** It is known that there is a quasigroup orthogonal to its transpose for every order n, such that $n \not\equiv 2 \pmod 4$, $n \not\equiv 3 \pmod 9$, or $n \not\equiv 6 \pmod 9$, (see, e.g., [1], [2]). The following theorems extend this class.

THEOREM 2. *There are an infinite number of quasigroups of order $n \equiv 2 \pmod 4$ orthogonal to their transposes.*

Proof. Since $34 = 11(4-1)+1$ and there are quasigroups of order 11 and 4 orthogonal to their transposes and there is a pair of orthogonal quasigroups of order 3, the generalized singular direct product gives a quasigroup of order 34 orthogonal to its transpose. Now let Q be a quasigroup of order $q=4t+2>10$, containing a subquasigroup of order 4 orthogonal to its transpose. (The above example shows that there is one.) Let V be a quasigroup of odd order $v=2k+1$ orthogonal to its transpose. Since $q-4>6$ the generalized singular direct product produces a quasigroup of order $v(q-4)+4$ orthogonal to its transpose and containing a subquasigroup of order 4. Now $v(q-4)+4=4(vt-k)+2$ and since $4t+2<4(vt-k)+2$ iteration of this procedure produces an infinite number of quasigroups of order $n\equiv 2 \pmod 4$, each of which is orthogonal to its transpose.

THEOREM 3. *There are an infinite number of quasigroups of order* $n\equiv 3 \pmod 9$ *orthogonal to their transposes.*

Proof. Since $57=7(9-1)+1$, remarks similar to those in Theorem 2 give a quasigroup of order 57 orthogonal to its transpose and containing a subquasigroup of order 9. Let Q be a quasigroup of order $q=9t+3>15$ orthogonal to its transpose containing a subquasigroup of order 9. Let V be a quasigroup of order $v=3s+1$ orthogonal to its transpose. Then the generalized singular direct product has order $v(q-9)+9$ and contain a subquasigroup of order 9. But $v(q-9)+9=9(vt-2s)+3$ so that $9t+3<9(vt-2s)+3$. Again, iteration produces an infinite number of quasigroups orthogonal to their transposes of order $n\equiv 3 \pmod 9$.

THEOREM 4. *There are an infinite number of quasigroups of order* $n\equiv 6 \pmod 9$ *orthogonal to their transposes.*

Proof. Since $33=4(9-1)+1$, the preceding remarks give a quasigroup of order 33 orthogonal to its transpose and containing a subquasigroup of order 9. Now let Q be a quasigroup of order $q=9t+6>15$ orthogonal to its transpose and containing a sub-quasigroup of order 9. Let V be a quasigroup of order $v=3s+1$ orthogonal to its transpose. Then the generalized singular direct product has order $v(q-9)+9=9(vt-s)+6>9t+6$ and contains a subquasigroup of order 9. As above this procedure produces an infinite number of quasigroups orthogonal to their transposes of order $n\equiv 6 \pmod 9$.

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