NOTE ON THE BOREL METHOD OF MEASURE EXTENSION

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Introduction. This note concerns a countably additive measure on a Boolean ring of subsets of an abstract set, this measure being real-valued, admitting ∞ as a possible value. We are interested only in unique extensions, so we suppose the measure to be σ-finite. The following well known result will be referred to as the "extension theorem": "Every σ-finite measure on a ring extends uniquely to a σ-finite measure on the generated σ-ring." Besides the familiar proof using outer measure, there is a Borel-type proof using transfinite induction [4]. We attempt here to reduce the Borel-type proof to its ultimate simplicity, reducing the problem to the bounded case.

Proof for a bounded measure. In this section μ is a measure [1] on a ring $\mathcal{R}$ (of subsets of a fixed abstract set X) which is bounded above by a fixed real constant M : $0 \leq \mu(R) \leq M < \infty$, all $R \in \mathcal{R}$. The boundedness of μ will be applied in the transfinite induction.

If a sequence R_n, $n = 1, 2, \ldots$ ($R_n \in \mathcal{R}$) converges set-theoretically to $R \in \mathcal{R}$, then $\mu(R_n)$, $n = 1, 2, \ldots$ converges to $\mu(R)$. This is expressed symbolically:

$$(1) \qquad R_n \to R \Rightarrow \mu(R_n) \to \mu(R) \qquad (R, R_n \in \mathcal{R}) .$$

(1) The term "measure" will imply countable additivity.

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Assuming the extension theorem, (1) is a consequence of the countable additivity; but we will prove it independently, then use it as basis for proof of the extension theorem. It will suffice to prove the particular case:

(2) $$R_n \to 0 \Rightarrow \mu(R_n) \to 0 \qquad (R_n \in \mathcal{R}) .$$

In fact, the hypothesis of (1) asserts[2] that $R_n \Delta R \to 0$, and applying (2),

$$\mu(R_n) - \mu(R) \leq \mu(R_n - R) \leq \mu(R_n \Delta R) \to 0 .$$

But this implies $\limsup \mu(R_n) \leq \mu(R)$, and we show similarly that $\liminf \mu(R_n) \geq \mu(R)$.

To prove (2), assume its hypothesis, and set $S_n = \bigcup_{i=n}^{\infty} R_i$, so that[3] $R_n \subseteq S_n \downarrow 0$. The S_n need not belong to $\mathcal{R}$, but it will be shown in the next paragraph that they may be approximated by members of $\mathcal{R}$ as follows: To arbitrary $\epsilon > 0$ there corresponds a sequence T_n, $n = 1, 2, \ldots$ such that

(i) $T_n \in \mathcal{R}$, $\quad S_n \supseteq T_n \downarrow 0$

(ii) $S_n \supseteq R \in \mathcal{R} \Rightarrow \mu(R) - \mu(T_n) \leq \epsilon \sum_{i=1}^{n} \frac{1}{2^i}$.

Assuming this, $\mu(T_n) \downarrow 0$ by (i), then $\limsup \mu(R_n) \leq \epsilon$ by (ii), and since ϵ is arbitrary (2) will be proved, hence also (1).

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(2) The symmetric difference $E \Delta F = (E-F) \cup (F-E)$ is the ring addition, intersection being the ring multiplication.

(3) The symbols $\uparrow$, $\downarrow$ indicate monotone increasing, decreasing convergence.

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To establish the sequence T_n, set $E_n = \bigcup_{i=n}^{n'} R_i$ where n' is a fixed index $\geq n$ of sufficiently high rank that

$$\mu\left(\bigcup_{i=n}^{m} R_i\right) - \mu\left(\bigcup_{i=n}^{n'} R_i\right) < \frac{\epsilon}{2^n} \quad \text{for all } m \geq n' .$$

The T_n are defined inductively:

$$T_1 = E_1 , \quad T_n = E_n \cap T_{n-1} \quad \text{for } n > 1 .$$

Since $S_n \supseteq E_n \supseteq T_n$ (i) is satisfied, and (ii) is satisfied for $n = 1$. In fact, if $S_1 \supseteq R \in \mathcal{R}$ then $R \cap \bigcup_{i=1}^{m} R_i$, $m = 1, 2, \ldots$ increases monotonely to R, so that

$$\mu(R) - \mu(T_1) = \lim_{m\to\infty} \mu\left(R \cap \bigcup_{i=1}^{m} R_i\right) - \mu(E_1) \leq \lim_{m\to\infty} \mu\left(\bigcup_{i=1}^{m} R_i\right) - \mu(E_1) \leq \frac{\epsilon}{2} .$$

It remains to show that (ii) is satisfied for n, assuming it for $n-1$. Assuming the hypothesis of (ii),

$$\mu(R) - \mu(E_n) = \lim_{m\to\infty} \mu\left(R \cap \bigcup_{i=n}^{m} R_i\right) - \mu(E_n) \leq \lim_{m\to\infty} \mu\left(\bigcup_{i=n}^{m} R_i\right) - \mu(E_n) \leq \frac{\epsilon}{2^n}$$

$$\mu(E_n) - \mu(T_n) = \mu(E_n - T_n) = \mu(E_n - T_{n-1}) = \mu(E_n \cup T_{n-1}) - \mu(T_{n-1}) .$$

Since $E_n \subseteq S_n \subseteq S_{n-1}$ and therefore $S_{n-1} \supseteq E_n \cup T_{n-1} \in \mathcal{R}$, by the induction hypothesis,

$$\mu(E_n \cup T_{n-1}) - \mu(T_{n-1}) \leq \epsilon \sum_{i=1}^{n-1} \frac{1}{2^i} .$$

The combined inequalities give the right member of (ii), thus completing the induction and so establishing (1).

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The definition of set-theoretical convergence carries over, without essential modification, to Moore-Smith sequences of sets. For double-index sequences we have the following analogue of (2):

$$(3) \qquad R_{mn} \to 0 \Longrightarrow \mu(R_{mn}) \to 0 \qquad (R_{mn} \in \mathcal{R})$$

In fact, if the right member of (3) were false there would exist $\epsilon > 0$ and two subsequences of the positive integers: $m_1 < m_2 < \ldots$, $n_1 < n_2 < \ldots$ such that $\mu(R_{m_k n_k}) \geq \epsilon$ for $k = 1, 2, \ldots$. But because of (2) this would contradict the convergence $\lim_{k \to \infty} R_{m_k n_k} = 0$ implied by the left member of (3).

We now show that if $R_n \to E \ (R_n \in \mathcal{R})$, then $\mu(R_n)$, $n = 1, 2, \ldots$ converges to a limit which depends only on the limit set E. Applying (3), we establish $\mu(R_n)$, $n = 1, 2, \ldots$ as a Cauchy sequence:

$$\begin{aligned} |\mu(R_m) - \mu(R_n)| &= \max\{\mu(R_m) - \mu(R_n), \mu(R_n) - \mu(R_m)\} \\ &\leq \max\{\mu(R_m - R_n), \ \mu(R_n - R_m)\} \\ &\leq \mu(R_m - R_n) + \mu(R_n - R_m) \ = \ \mu(R_m \Delta R_n) \to 0 \ . \end{aligned}$$

Given a second sequence converging to the same limit: R $R'_n \to E \ (R'_n \in \mathcal{R})$, we have $R'_n \Delta R_n \to 0$, so that

$$\mu(R'_n) - \mu(R_n) \leq \mu(R'_n - R_n) \leq \mu(R'_n \Delta R_n) \to 0 \ .$$

Therefore $\lim \mu(R'_n) \leq \lim \mu(R_n)$ and, by symmetry, we have the inverse inequality, completing the proof of the assertion.

Let $\mathcal{R}^*$ be the class of limits of convergent sequences R_n, $n = 1, 2, \ldots$ $(R_n \in \mathcal{R})$; this is a ring extension of $\mathcal{R}$, which we refer to as the "limit ring" of $\mathcal{R}$. The assertion of the preceding paragraph implies that a function μ^* is well defined on $\mathcal{R}^*$ by the formula:

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$$\mu^*(R^*) = \lim \mu(R_n), \quad \text{where } R_n \to R^* \quad (R_n \in \mathcal{R},\ R^* \in \mathcal{R}^*).$$

It follows from (1) that μ^* extends μ, and that any measure extension of μ to $\mathcal{R}^*$ must coincide with μ^*. To prove the finite additivity of μ^*, consider a disjoint union[4] of two members of $\mathcal{R}^*$: $R^* + S^*$. We have

$$R_n \to R^*, \quad S_n \to S^*, \quad R_n \cap S_n \to 0 \quad (R_n, S_n \in \mathcal{R}).$$

Applying (2),

$$\mu^*(R^*+S^*) = \lim \mu(R_n \cup S_n) = \lim(\mu(R_n)+\mu(S_n)+\mu(R_n \cap S_n)) = \mu^*(R^*)+\mu^*(S^*).$$

So μ^* will be a measure if we show, further, that

$$(4) \qquad R_n^* \uparrow R^* \Rightarrow \mu^*(R_n^*) \uparrow \mu^*(R^*) \quad (R_n^*, R^* \in \mathcal{R}^*).$$

In order to prove (4) we introduce the class $\mathcal{U}$ consisting of all finite or countable unions of members of $\mathcal{R}$: $\mathcal{R} \subseteq \mathcal{U} \subseteq \mathcal{R}^*$. We begin by proving that

$$(5) \qquad U_n \uparrow U \Rightarrow \mu^*(U_n) \uparrow \mu^*(U) \quad (U_n, U \in \mathcal{U})$$

We have: $U_n = \bigcup_m R_{nm}$ $(R_{nm} \in \mathcal{R})$, $U = \bigcup_n \bigcup_m R_{nm}$. We may re-order the R_{mn} as terms of a simple sequence (with indices $1, 2, \ldots$) then take the partial (finite) unions as the terms R_n of a sequence increasing to U:

$$R_n \uparrow U \ (R_n \in \mathcal{R}), \qquad \mu(R_n) \uparrow \mu^*(U).$$

Then (5) follows from the fact that each R_n is contained in some U_m. Now assume the hypothesis of (4), and let $\epsilon > 0$ be arbitrary. Considering R_n^* as the superior limit of a sequence converging to R_n^* whose terms belong to $\mathcal{R}$, we see

(4) We substitute $+$ or Σ for $\cup$ to indicate a disjoint union.

that there exist sets $U_n \in \mathcal{U}$ such that

$$U_n \supseteq R^*_n \quad \text{and} \quad \mu^*(U_n) - \mu^*(R^*_n) < \frac{\epsilon}{2^n} .$$

Then $U'_n = \bigcup_{i=1}^{n} U_i$, $n = 1, 2, \ldots$ is an increasing sequence of sets of $\mathcal{U}$ having a limit $U \in \mathcal{U}$, so by (5) ,

$$\mu^*(U'_n) \uparrow \mu^*(U) \geq \mu^*(R^*) .$$

But $\mu^*(U'_n) - \mu^*(R^*_n) = \mu^*\left(\bigcup_{i=1}^{n} U_i - R^*_n\right) \leq \sum_{i=1}^{n} \mu^*(U_i - R^*_i) < \epsilon$.

Then, since ϵ is arbitrary, $\lim \mu^*(R^*_n) \geq \mu^*(R^*)$, and the inverse inequality is obvious, proving (4). This establishes μ^* as the unique measure extension to $\mathcal{R}^*$ of the original measure μ on $\mathcal{R}$.

We define inductively the transfinite sequence of rings:

$$\mathcal{R} = \mathcal{R}_0 \subset \mathcal{R}_1 \subset \mathcal{R}_2 \subset \ldots \subset \mathcal{R}_\alpha \subset \ldots \subset \mathcal{R}_\sigma$$

For $\alpha > 0$, an ordinal of the first kind, $\mathcal{R}_\alpha = \mathcal{R}^*_{\alpha-1}$, the limit ring of $\mathcal{R}_{\alpha-1}$. For an ordinal α of the second kind, $\mathcal{R}_\alpha = \bigcup_{\beta<\alpha} \mathcal{R}_\beta$. The ordinal σ is the smallest such that $\mathcal{R}^*_\sigma = {}_\sigma$, so $\mathcal{R}_\sigma$ is the σ-ring generated by $\mathcal{R}$. (The symbol σ denotes this ordinal and also serves as the sign of the generated σ-ring.) In the case $\sigma = 0$, $\mathcal{R}$ is already an σ-ring and there is no measure extension problem. An ordinal $\alpha (0 \leq \alpha \leq \sigma)$ will be called "accessible" if

(i) The original measure μ on $\mathcal{R}$ extends uniquely to a measure μ_α on $\mathcal{R}_\alpha$, μ_α being bounded by M , the bound of μ.

(ii) For arbitrary $\epsilon > 0$ and $S \in \mathcal{R}_\alpha$, there exists $U \in \mathcal{U}$ (class of finite or countable unions of members of $\mathcal{R}$) such that

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$$U \supseteq S \quad \text{and} \quad \mu_\alpha(U) - \mu_\alpha(S) < \epsilon \quad .$$

To complete the proof of the extension theorem for the bounded case, it will suffice to prove inductively that every α $(0 \leq \alpha \leq \sigma)$ is accessible. Since 0 is accessible, it suffices to prove $\alpha > 0$ accessible, assuming every $\beta < \alpha$ accessible.

First, let α be of the first kind so that $\alpha-1$ is accessible. As has been shown, $\mu_{\alpha-1}$ extends uniquely to a measure μ_α on $\mathcal{R}^*_{\alpha-1} = \mathcal{R}_\alpha$ which is obviously bounded by M. But μ_α, the unique measure extension of $\mu_{\alpha-1}$, itself the unique measure extension of μ to $\mathcal{R}_{\alpha-1}$, is the unique measure extension of μ to $\mathcal{R}_\alpha$, so (i) is satisfied for α. The class $\mathcal{U}_{\alpha-1}$ of finite or countable unions of members of $\mathcal{R}_{\alpha-1}$, is related to $\mathcal{R}_\alpha$ as is $\mathcal{U}$ to $\mathcal{R}_1$. So, for arbitrary $\epsilon > 0$ and $S \in \mathcal{R}_\alpha$, there exists $V \in \mathcal{U}_{\alpha-1}$ such that

$$V \supseteq S \quad \text{and} \quad \mu_\alpha(V) - \mu_\alpha(S) < \frac{\epsilon}{2} \quad .$$

By the definition of $\mathcal{U}_{\alpha-1}$ and the fact that $\mathcal{R}_{\alpha-1}$ is a ring, we may express V as a disjoint finite or countable union of members of $\mathcal{R}_{\alpha-1}$:

$$V = \Sigma T_n \quad (T_n \in \mathcal{R}_{\alpha-1}) \quad .$$

Since $\alpha-1$ is accessible and μ_α extends $\mu_{\alpha-1}$, there exists $U_n \in \mathcal{U}$ such that

$$U_n \supseteq T_n \quad \text{and} \quad \mu_\alpha(U_n) - \mu_\alpha(T_n) < \frac{\epsilon}{2^{n+1}} \quad .$$

Setting $U = \bigcup_n U_n$, we have $S \subseteq V \subseteq U \in \mathcal{U}$

$$\mu_\alpha(U) - \mu_\alpha(S) = \mu_\alpha(U) - \mu_\alpha(V) + \mu_\alpha(V) - \mu_\alpha(S) < \mu_\alpha(U) - \mu_\alpha(V) + \frac{\epsilon}{2} \,.$$

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But since μ_α is a finite measure on $\mathcal{R}_\alpha$,

$$\mu_\alpha(U)-\mu_\alpha(V) = \mu_\alpha(U-V) \leq \mu_\alpha\{\bigcup_n U_n - T_n)\} \leq \Sigma\, \mu_\alpha(U_n - T_n) < \frac{\epsilon}{2} .$$

The combined inequalities establish (ii) for α.

Now let α be of the second kind. Because of the induction hypothesis, a function μ_α is well defined on $\mathcal{R}_\alpha$ by the formula $\mu_\alpha(S) = \mu_\beta(S)$, where $\beta < \alpha$ is such that $S \in \mathcal{R}_\beta$ ($S \in \mathcal{R}_\alpha$). It must be shown that, in virtue of this definition, α is accessible. Already, by the induction hypothesis, (ii) is satisfied, μ_α is finitely additive and bounded by M. Since any measure extension of μ to $\mathcal{R}_\alpha$ would satisfy the defining formula for μ_α, μ_α is the only possible such extension. It remains only to prove the countable additivity of μ_α. Suppose that $S = \sum_1^\infty S_n$ (S, $S_n \in \mathcal{R}_\alpha$). Because of the finite additivity,

$$\mu_\alpha(S) \geq \sum_1^\infty \mu_\alpha(S_n) .$$

Applying (ii) for α, for given $\epsilon > 0$ there exist sets $U_n \in \mathcal{U}$ such that

$$U_n \supseteq S_n \quad \text{and} \quad \mu_\alpha(U_n) - \mu_\alpha(S_n) < \frac{\epsilon}{2^n} .$$

Because μ_α, restricted to $\mathcal{R}_1$, is the measure μ_1,

$$\mu_\alpha(S) \leq \mu_\alpha\left(\bigcup_1^\infty U_n\right) \leq \sum_1^\infty \mu_\alpha(U_n) < \sum_1^\infty \mu_\alpha(S_n) + \epsilon .$$

Since ϵ is arbitrary the combined inequalities give

$$\mu_\alpha(S) = \sum_1^\infty \mu_\alpha(S_n) .$$

This completes the induction, and therefore the proof of the extension theorem for the bounded case.

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Proof for a σ-finite measure. Henceforth μ is a σ-finite measure on the ring $\mathcal{R}$. Applying the extension theorem for the bounded case, we show in this section that the σ-finite measure μ extends uniquely to a σ-finite measure on the generated σ-ring $\mathcal{R}_\sigma$. Let $\mathcal{R}_o$ be the subring of $\mathcal{R}$ consisting of the sets of $\mathcal{R}$ of finite measure. Because μ is σ-finite the σ-ring generated by $\mathcal{R}_o$ is $\mathcal{R}_\sigma$. Therefore to extend μ uniquely to a measure on $\mathcal{R}_\sigma$ it will suffice to do the same for its restriction to $\mathcal{R}_o$. So we assume henceforth, without loss of generality, that the original measure μ on $\mathcal{R}$ is finite.

Between $\mathcal{R}$ and $\mathcal{R}_\sigma$ we interpolate the ring $\mathcal{R}'$ consisting of all members of $\mathcal{R}_\sigma$ contained in some member of $\mathcal{R}$. In other words, $R' \in \mathcal{R}'$ if and only if $R' \in \mathcal{R}_\sigma$ and $R' \subseteq R$ for some $R \in \mathcal{R}$ (depending on R'). Let $\mathcal{R}_1$ be a given ring (of subsets of X) and let E be a given set (subset of X). The "trace" of $\mathcal{R}_1$ on E, denoted $\mathcal{R}_1 \cap E$, is the ring of sets of the form $R_1 \cap E$, where $R_1 \in \mathcal{R}_1$. Since a member of $\mathcal{R}_\sigma$ is contained in $R \in \mathcal{R}$ if and only if it belongs to the trace $\mathcal{R}_\sigma \cap R$, we have

$$\mathcal{R}' = \bigcup_{R \in \mathcal{R}} (\mathcal{R}_\sigma \cap R), \quad \mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{R}_\sigma$$

Our procedure will be to first extend μ uniquely to a finite measure μ' on $\mathcal{R}'$, then to extend μ' uniquely to a σ-finite measure on $\mathcal{R}_\sigma$.

First, we extend μ uniquely to a finite measure μ' on $\mathcal{R}'$. For arbitrary $R \in \mathcal{R}$, the restriction of μ to $\mathcal{R} \cap R$ is bounded, and the σ-ring generated by $\mathcal{R} \cap R$ is(5)

$$(\mathcal{R} \cap R)_\sigma = \mathcal{R}_\sigma \cap R .$$

(5) The left member is merely the notation for the generated σ-ring. Its identity with the right member is proved as theorem E, sec. 5 [3].

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The extension theorem for the bounded case asserts that this restriction extends uniquely to a finite measure μ_R on $\mathcal{R}_\sigma \cap R$. We will combine the measures μ_R, $R \in \mathcal{R}$ to form a single measure μ' on $\mathcal{R}'$. We observe that if $R_1 \subseteq R_2$ $(R_1, R_2 \in \mathcal{R})$ then μ_{R_1} and the restriction of μ_{R_2} to $\mathcal{R}_\sigma \cap R_1$ must coincide on their common domain, since they are both measure extensions of the restriction of μ to $\mathcal{R} \cap R_1$ (uniqueness of measure extension in the bounded case). Hence, for any two members R_1, R_2 of $\mathcal{R}$, μ_{R_1}, μ_{R_2} coincide on $\mathcal{R}_\sigma \cap (R_1 \cap R_2)$. Therefore a function μ' is well defined on $\mathcal{R}'$ by the formula

$$\mu'(R') = \mu_R(R'), \quad \text{where } R \text{ is such that } R' \subseteq R \in \mathcal{R} \ (R' \in \mathcal{R}').$$

It is already obvious that μ' is a finite measure on $\mathcal{R}'$ which extends every μ_R, and so also extends μ. But any measure on $\mathcal{R}'$ which extends μ must also extend every μ_R $(R \in \mathcal{R})$ (uniqueness of measure extension in the bounded case), so μ' is the only possible measure extension of μ to $\mathcal{R}'$.

We now extend μ' uniquely to a σ-finite measure on $\mathcal{R}_\sigma$. Every member of $\mathcal{R}_\sigma$ may be expressed as a disjoint, finite or countable, union of members of $\mathcal{R}'$. Consider two such representations of the same set $E \in \mathcal{R}_\sigma$:

$$E = \Sigma S_n = \Sigma T_m \quad (S_n, T_m \in \mathcal{R}').$$

Since μ' is a measure on $\mathcal{R}'$,

$$\sum_n \mu'(S_n) = \sum_n \mu'(S_n \cap \sum_m T_m) = \sum_n \sum_m \mu'(S_n \cap T_m) = \sum_m \mu'(T_m).$$

Therefore a function μ_σ is well defined on $\mathcal{R}_\sigma$ by the formula $\mu_\sigma(E) = \Sigma \mu'(S_n)$, where $E = \Sigma S_n$ $(S_n \in \mathcal{R}')$ is any representation

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The function μ_σ extends μ', it is finitely additive and, if it is shown to be a measure, it will be σ-finite. But any measure on $\mathcal{R}_\sigma$ extending μ' must satisfy the defining equation for μ_σ, so μ_σ is the only possible such extension. It remains only to prove the countable additivity fof μ_σ. Suppose that

$$E = \sum_1^\infty E_n \quad (E_n \in \mathcal{R}_\sigma) .$$

Each term of the union has a representation of the form(6)

$$E_n = \sum_{m=1}^\infty S_{nm} \quad (S_{nm} \in \mathcal{R}') .$$

In the special case where $E \in \mathcal{R}'$ we have

$$\mu_\sigma(E) = \mu'(E) = \mu'\left(\sum_{n=1}^\infty \sum_{m=1}^\infty S_{nm}\right) = \sum_{n=1}^\infty \sum_{m=1}^\infty \mu'(S_{nm}) = \sum_{n=1}^\infty \mu_\sigma(E_n) .$$

In the general case we may express E : $E = \sum_1^\infty S_n \quad (S_n \in \mathcal{R}')$.

We apply the special case, taking account of the definition μ_σ, and noting that $S_n \cap E_m \in \mathcal{R}'$:

$$\mu_\sigma(E) = \sum_{n=1}^\infty \mu'(S_n) = \sum_{n=1}^\infty \mu'\left(S_n \cap \sum_{m=1}^\infty E_m\right) = \sum_{n=1}^\infty \sum_{m=1}^\infty \mu'(S_n \cap E_m)$$

$$= \sum_{m=1}^\infty \sum_{n=1}^\infty \mu'(S_n \cap E_m) = \sum_{m=1}^\infty \mu_\sigma(E_m) .$$

This establishes the countable additivity of μ_σ, and so completes our proof of the extension theorem.

(6) By adjoining, if necessary, an infinity of terms equal to the empty set, we may suppose all representations to be countably infinite.

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