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ASYMPTOTIC SHAPE OF FINITE PACKINGS

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ABSTRACT. Let K be a convex body in $\mathbf{E}^d$ and denote by C_n the set of centroids of n non-overlapping translates of K. For $\varrho > 0$, assume that the parallel body $\operatorname{conv} C_n + \varrho K$ of $\operatorname{conv} C_n$ has minimal volume. The notion of parametric density (see [21]) provides a bridge between finite and infinite packings (see [4] or [14]). It is known that there exists a maximal $\varrho_s(K) \geq 1/(32d^2)$ such that $\operatorname{conv} C_n$ is a segment for $\varrho < \varrho_s$ (see [5]). We prove the existence of a minimal $\varrho_c(K) \leq d + 1$ such that if $\varrho > \varrho_c$ and n is large then the shape of $\operatorname{conv} C_n$ can not be too far from the shape of K. For $d = 2$, we verify that $\varrho_s = \varrho_c$. For $d \geq 3$, we present the first example of a convex body with known ϱ_s and ϱ_c; namely, we have $\varrho_s = \varrho_c = 1$ for the parallelotope.

1**. Introduction.** Finite packings of circles have been investigated already at the beginning of the century (see [18]). The attention turned towards packings in euclidean d-space $\mathbf{E}^d$, $d \geq 3$, after the Sausage Conjecture of László Fejes Tóth in 1975 (see [12]). The conjecture states that for $d \geq 5$, the volume of the convex hull of n non-overlapping balls in $\mathbf{E}^d$ is minimal when the centers are aligned (and the convex hull of the balls is a "sausage").

In this paper, we consider packings of copies of a convex body K by *translates* of K, and we assume it without mentioning. The symbol C_n always denotes a set of centroids of n non-overlapping translates of K. If $\dim(\operatorname{conv} C_n) = 1$ the arrangement C_n is called a sausage. The sausage with minimal $V(\operatorname{conv} C_n + \varrho K)$ is denoted by S_n. An extremely fruitful notion concerning finite packings is the notion of parametric density (see [21]); namely, the maximum of $nV(K)/V(\operatorname{conv} C_n + \varrho K)$ for given n and $\varrho > 0$. Note that the case $\varrho = 1$ is the classical problem. The solution of the Sausage Conjecture by U. Betke, M. Henk and J. M. Wills (see [4] and [14]) is based on this notion. In addition, now we have a tool to connect the specific properties of finite and infinite packings:

There exists a maximal $\varrho_s(K) \geq 1/(32d^2)$ such that $\operatorname{conv} C_n$ is a segment for $\varrho < \varrho_s$ (see [5]). On the other hand, for $\varrho > d + 1$, the optimal density is the same as the infinite packing density (see [4] or [14]). This paper concentrates on the shape of the optimal packing for large ϱ.

Denote by $\Delta(K)$ the average part of the space taken up by a copy of K in the densest infinite translative packing of K, and so $V(K)/\Delta(K)$ is the packing density $\delta(K)$ of K. Assume that $V(\operatorname{conv} C_n + \varrho K)$ is minimal, and hence the parametric density is maximal.

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Cluster like packings show that asymptotically $V(\operatorname{conv} C_n + \varrho K)$ is always at most $n\Delta(K)$, and the *critical radius* ϱ_c is defined as the infimum of ϱ such that $V(\operatorname{conv} C_n + \varrho K) \sim n\Delta(K)$ for large n. It satisfies $\frac{1}{2} \leq \varrho_c \leq d+1$ (and even $\varrho_c \leq 2$ if K is centrally symmetric) (see Section 3).

Theorem A below gives a more exact formulation what the shape of the optimal arrangement is for large ϱ (see Propositions 3.1, 3.2 and 3.3 for proofs and more precise statements).

For a convex body C, set $r_K(C) = \max\{\lambda \mid \exists x,\ x + \lambda K \subset C\}$ (the *inradius*) and $R_K(C) = \min\{\lambda \mid \exists x,\ C \subset x + \lambda K\}$ (the *circumradius* with respect to K). Note that maximizing the parametric density is equivalent with minimizing $V(\operatorname{conv} C_n + \varrho K)$.

THEOREM A. *Let $d \geq 2$ and $\varrho > \varrho_c(K)$ for some convex body K.*

(i) $r_K(\operatorname{conv} C_n)$ tends to infinity as $n \longrightarrow \infty$;

(ii) if $\varrho > d+1$ then there exists $\omega(\varrho)$ with $\lim_{\varrho\longrightarrow\infty} \omega(\varrho) = 1$ such that if $V(\operatorname{conv} C_n + \varrho K)$ is minimal then $R_K(\operatorname{conv} C_n)/r_K(\operatorname{conv} C_n) < \omega(\varrho)$ for large n.

REMARK. If K is centrally symmetric then $\omega(\varrho)$ can be defined even for $\varrho > 2$.

Let $\varrho > 0$. Then for large n, in the arrangement minimizing the surface-area of $\operatorname{conv} C_n + \varrho K$ (or any mean-projection of $\operatorname{conv} C_n + \varrho K$), the shape of $\operatorname{conv} C_n$ is asymptotically a ball (see [6] or [22] if $\varrho = 1$, but the same proof works in fact for all positive ϱ). Here we can consider for $\varrho > \varrho_c(K)$ and the optimal $C_{n,\varrho}$ the *normalized shape* $K_{n,\varrho} = n^{-1/d} \operatorname{conv} C_{n,\varrho}$. By Theorem A and a Blaschke argument it follows that there are convergent subsequences of $K_{n,\varrho}$. In general the shape of the limit body as well as its uniqueness remains unknown. It can not be expected that for general convex body K the asymptotical shape of the optimal packing is homothetic to K. It definitely does not hold when $\varrho = 1$ and K is a circle in the plane (see [19]). Here we can give the limit of all this limit bodies for $\varrho \longrightarrow \infty$. Theorem A together with $V(\operatorname{conv} C_n + \varrho K) \sim n\Delta(K)$ states that

$$\lim_{n,\varrho\longrightarrow\infty} K_{n,\varrho} = \big(\delta(K)\big)^{-1/d} \cdot K.$$

In the paper [4], the authors state the so-called Strong Sausage Conjecture; namely, if K is a ball and $\varrho < \varrho_c$ then the sausage minimizes $V(\operatorname{conv} C_n + \varrho K)$. If $d \geq 3$ then one can not expect a similar statement even for general centrally symmetric convex bodies (see [1]).

At the end of Section 3, we consider the example of a parallelotope P. Until now, this is the only known example of a convex body in E^d, $d \geq 3$, satisfying the Strong Sausage Conjecture for general packings. In this case, $\varrho_s(P) = \varrho_c(P) = 1$, and if $\varrho > 1$ then the optimal shape is asymptotically homothetic to P. Note that for $d \geq 3$ and $\varrho > \varrho_c$, to determine even the asymptotic behavior of the optimal packing of a convex body K different from the parallelotope seems to be out of reach at the moment.

We verify the Strong Sausage Conjecture for a planar convex domain K (see Sections 4 and 5). Denote by $A(K)$ the area of K and by $P(K)$ the area of the smallest parallelogram containing K which satisfy $A(K) \leq \Delta(K) \leq P(K)$.

THEOREM B. *Assume that $A(\operatorname{conv} C_n + \varrho K)$ is minimal for $\varrho > 0$.*

(i) $\operatorname{conv} C_n$ *is a segment if either $\varrho \le 3/7$ or $\varrho < \Delta(K)/P(K)$ and n is large.*

(ii) if $\varrho > \Delta(K)/P(K)$ then $R_K(\operatorname{conv} C_n)/r_K(\operatorname{conv} C_n) < \omega(\varrho)$ for large n where $\omega(\varrho) = 1 + 550/\sqrt{\varrho}$ for $\varrho \ge 3$.

REMARK. For centrally symmetric domains the sausage is always optimal if $\varrho < \Delta(K)/P(K)$ (see [4]). The case of the circle is implicitly contained already in the paper [18] of Thue.

Note that $3/4 \le \Delta(K)/P(K) \le 1$. For non-centrally symmetric convex bodies, the situation can be actually more complicated than for the symmetric ones. Assume that K is a triangle, and hence $\Delta(K)/P(K) = 3/4$. Then for $n = 3$ and $\varrho > 1/2$, the sausage is not optimal.

If the n translates of K are chosen from some lattice packing of K then we also call the finite packing of K as lattice packing. The results above equally apply to finite lattice packings. It is especially transparent in the planar case when the densest infinite lattice packing is also the densest translative packing (see [11]). In Theorem A, the only changes needed are that the lower bound for ϱ is $1.5(d+1)$ (and 3 if K is centrally symmetric). The manuscript [1] considers asymptotic shapes of lattice packings in E^3 and moreover the asymptotic shape for lattice packings, $\varrho > \varrho_c$, is described explicitely in [3] and [20]. The result is the Wulff-shape in crystallography.

2. **Some basic inequalities.** We prove various formulae involving $r_K(C)$ and $R_K(C)$. These inequalities will be used in the subsequent sections. The following statement is a version of Steinhagen's theorem for general convex bodies:

LEMMA 2.1. *Any convex body C is contained in a strip bounded by two parallel hyperplanes which support a copy of $d\, r_K(C)K$.*

REMARK. The constant d is attained when K is a simplex and $C = -K$.

PROOF. We prove the statement by induction on d, where the case $d = 1$ readily holds. We may assume that $r_K(C) = 1$ and $K \subset C$. For some $1 \le m \le d$, there exist $x_0, \ldots, x_m \in \operatorname{bd} K \cap \operatorname{bd} C$ such that the relative interior of the convex hull of the common outer normal vectors at $x_0, \ldots, x_m$ contains the origin. Define L as the linear m-space spanned by the $m+1$ normal vectors, denote by C_0 the projection of the region enclosed by the supporting hyperplanes onto L and by K_0 the projection of $\operatorname{conv}\{x_0, \ldots, x_m\}$ onto L.

If $m < d$ then applying the induction hypothesis to the m-simplices C_0 and K_0 yields the lemma.

So assume that $m = d$ and F_j is the facet of C_0 containing x_j, $j = 0, \ldots, d$. Denote by H_j the hyperplane through the centroids of the facets F_k of C_0 different from F_j, and let H_j^+ (H_j^-) be the halfspace containing (not containing) F_j. Since $\bigcap_{j=0}^d \operatorname{int} H_j^+$ is contained in $\operatorname{int} C_0$, we may assume that a vertex of K_0 lies in H_0^-. Now the statement follows as the distance of F_0 from the opposite vertex of C_0 is d times the distance of H_0 and F_0. ■

A similar argument yields that

$$R_{-K}(C) \le d\, R_K(C), \tag{1}$$

with equality if and only if K and C are homothetic simplices.

For any vector $u \neq 0$, denote by $\|u\|_K$ twice the ratio of the length of u and the length of the longest segment in K parallel to u, and set $D_K(C)$ to be the maximum of $\|x-y\|_K$ for $x, y \in C$. If K is the unit ball then $D_K(C)$ is the diameter of C. A simple application of Helly's theorem (reducing to the case when C is a simplex) and similar arguments as for Lemma 2.1 yield that

$$R_K(C) \le \frac{d}{2} D_K(C). \tag{2}$$

Here one has again equality when K is a simplex and $C = -K$.

The mixed volume $V_i(C, K)$, $i = 0, \ldots, d$ of H. Minkowski is defined by the formula

$$V(\alpha C + \beta K) = \sum_{i=0}^{d} \binom{d}{i} V_i(C, K) \alpha^{d-i} \beta^i$$

for $\alpha, \beta \ge 0$ (see *e.g.* [7] or [17]). Here $V_i(C, K) = V_{d-i}(K, C)$ and $V_0(C, K) = V(C)$. The mixed volumes are continous, linear and monotonic in both variables, and they are invariant under simultaneous volume preserving affine transformations of C and K.

Assume that C is a d-polytope and U denotes the set of outer unit normal vectors to the facets of C. If $|F_u|$ is the $(d-1)$-area of the facet of C with normal vector u and $h_K(u)$ is the value of the support function of K then

$$V_1(C, K) = \frac{1}{d} \sum_{u \in U} h_K(u) |F_u|. \tag{3}$$

In addition, the mixed volumes satisfy the celebrated Alexandrov-Fenchel inequality. The case we need is that for $1 \le i < j \le d$,

$$V_i(C, K)^j \ge V_j(C, K)^i V(C)^{j-i}.$$

In some particular cases, also the stability of the Alexandrov-Fenchel inequality is known. Assume that $V(C) = V(K) = 1$ and the centroids C and K coincide. If the diameter of C and K are at most D then their Hausdorff distance $\delta(C, K)$ satisfies (see [13])

$$\delta(C, K) < 12d\, D \Big(V_{d-1}(C, K) - 1 \Big)^{\frac{1}{d+1}}. \tag{4}$$

In the planar case, Bonnesen's inequality is more convenient for our purposes. Setting $A(C, K) = V_1(C, K)$, we have (see [13])

$$A(C, K)^2 \ge A(C)A(K) + \tfrac{1}{4} A(K)^2 \Big(R_K(C) - r_K(C) \Big)^2. \tag{5}$$

For $u \in \mathbf{S}^{d-1}$, denote by $\pi_K(u)$ the $(d-1)$-area of the orthogonal projection of K onto the hyperplane normal to u. If C is a segment with length $|C|$ parallel to u then $V_{d-1}(C, K) = \frac{1}{d} |C| \pi_K(u)$. We deduce that

$$D_K(C) \le 2d\, V_{d-1}(C, K) / V(K), \tag{6}$$

which in turn yields by (2) that

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LEMMA 2.2. *For any convex body C,*

$$V(K)R_K(C) \le d^2 V_{d-1}(C, K).$$

REMARK. If K is the unit ball B^d then the optimal constant is known; namely, instead of d^2, $2\kappa_{d-1}/(d\kappa_d)$ where $\kappa_d = V(B^d)$.

Using $R_K(C)r_C(K) = 1$ and interchanging the role of K and C, we also deduce

COROLLARY 2.3. *For any convex body C,*

$$d^2 V_1(C, K) r_K(C) \ge V(C).$$

We also need an estimate for $V(C)$ in terms of both of the in- and circumradius.

LEMMA 2.4. *For any convex body C,*

$$\frac{1}{d^2} V(K) r_K^{d-1}(C) R_K(C) \ \le \ V(C) \ \le \ d^2 V(K) r_K(C) R_K^{d-1}(C).$$

PROOF. Let $u \in \mathbf{S}^{d-1}$. As our problem is affine invariant, we may assume that the width $w_K(u)$ of K parallel to u is equal the length of the longest segment contained in K parallel to u, and hence

$$V(K) \ge \frac{1}{d} w_K(u) \pi_K(u). \tag{7}$$

This estimate yields the upper bound for $V(C)$ by Lemma 2.1. We deduce the lower bound by the relation $R_K(C)r_C(K) = 1$ and interchanging the role of K and C. ∎

By John's theorem (see [15]), any convex body K contains a unique ellipsiod E of maximum volume (the so-called *Löwner ellipsoid*), and assuming that the origin is the center of E, we have $K \subset dE$. If K is centrally symmetric then even $K \subset \sqrt{d}E$, and the extremal bodies are the simplex and the parallelotope, respectively. K. Ball proved (see [2]) that these bodies also minimize the ratio $V(E)/V(K)$ among convex bodies or centrally symmetric convex bodies, respectively. These properties of the Löwner ellipsoid yield by Stirling's formula

LEMMA 2.5. *Assume that $V(K) = 1$ and the Löwner ellipsoid of K is a ball centered at the origin. Then*

$$\frac{1}{e} B^d \subset K \subset d\sqrt{d} B^d.$$

The left hand side is asymptotically tight but probably $d\sqrt{d}$ can be replaced by $O(d)$. More precisely,

CONJECTURE 2.6. *Assume that $V(K) = 1$ and the Löwner ellipsoid of K is a ball. Then the diameter of K is maximal if K is the regular simplex.*

Finally, we show that given the area of the Löwner ellips, the triangle and the square have some extremal properties also with respect to packings.

PROPOSITION 2.7. *Given the area of the Löwner ellips of K, the domain maximizing $\Delta(K)$ and $P(K)$ is the triangle, and among centrally symmetric domains that is the parallelogram.*

REMARK. Assume that K is centrally symmetric. Then some lattice packing of K is densest among any packings of K (allowing rotation), and hence the parallelogram maximizes $\Delta(K)$ even in the case of general packings.

PROOF. Assume that B^2 is the Löwner ellips of K. Then $A(K)$ is maximal if K is the regular triangle by the result of K. Ball above. Now Fáry's theorem (see [9]) states that $\Delta(K)/A(K) \leq 3/2$ with equality if and only if K is a triangle. On the other hand, $P(K)/A(K) \leq 2$ (see [7]), with equality if and only if K is a triangle.

Assume that K is centrally symmetric. There exist some x_1, x_2, x_3 such that $\pm x_j \in \mathrm{bd} K \cap \mathrm{bd} B^2$, $j = 1, 2, 3$, and any two consecutive of the six points has acute or right angle (allowing $x_1 = x_2$). Denote by H the hexagon (possibly square) whose sides are tangent to B^2 in $\pm x_j$. We conclude that $\Delta(K) \leq A(H)$ as H tiles the plane, and $A(H)$ is readily maximal when H is the square. The statement for $P(K)$ can be similarly proved. ■

3. **Packings with $\varrho > \varrho_c$.** Observe that the body $P_n = \big(n\Delta(K)/V(K)\big)^{1/d} K$ contains the centroids of n non-overlapping translates of K (see [6]) and for $\varrho > 0$,

$$V(P_n + \varrho K) = n\Delta(K) + \sum_{i=1}^{d} \binom{d}{i} \big(n\Delta(K)\big)^{\frac{d-i}{d}} V(K)^{\frac{i}{d}} \varrho^i. \tag{8}$$

Let $\operatorname{conv} C_n$ minimize $V(\operatorname{conv} C_n + \varrho K)$. We deduce by (8) that asymptotically $V(\operatorname{conv} C_n + \varrho K)$ is at most $n\Delta(K)$.

Denote by $Z(K)$ the minimal volume of a cylinder containing K. It is well known that $V(K) \geq \frac{1}{d} Z(K)$ (see [7]). Then the sausage arrangement corresponding to $Z(K)$ shows that $V(\operatorname{conv} C_n + \varrho K) \leq n\varrho^{d-1} Z(K)$, which in turn yields that $\varrho_c \geq (1/d)^{1/(d-1)}$. On the other hand, we have (see [4])

$$V(\operatorname{conv} C_n + \varrho K) \geq n\Delta(K) \quad \text{for } \varrho \geq d+1, \tag{9}$$

and even for $\varrho \geq 2$ if K is centrally symmetric. These estimates yield that $\varrho_c \leq d+1$ (and $\varrho_c \leq 2$ if C is centrally symmetric). For $d = 2$, we determine the exact value of ϱ_c in the next section.

Assume that $\varrho > \varrho_c$, and set $\varrho_0 = \frac{1}{2}(\varrho + \varrho_c)$ and $Q_n = \operatorname{conv} C_n + \varrho_0 K$. Then there exists some positive function $\phi(n)$ with $\lim_{n\to\infty} \phi(n) = 0$ and $V(Q_n) \geq \big(1 - \phi(n)\big) n\Delta(K)$. It follows by Minkowski's formula that for $\varepsilon = \varrho - \varrho_0$,

$$V(\operatorname{conv} C_n + \varrho K) = V(Q_n + \varepsilon K) > \big(1 - \phi(n)\big) n\Delta(K) + d\, V_1(Q_n, K)\varepsilon.$$

Comparing this to (8) yields the existence of some constants c_1 and c_2 independent of n satisfying

$$V_1(Q_n, K) \leq c_1 n^{\frac{d-1}{d}} + c_2 \phi(n) n.$$

Since for large n, $V(Q_n) > \frac{1}{2}\Delta(K)\, n$, Corollary 2.3 yields

PROPOSITION 3.1. *If $\varrho > \varrho_c$ then $r_K(\operatorname{conv} C_n)$ tends to infinity.*

Now we give more precise information about the shape of $\operatorname{conv} C_n$ if $\varrho > d+1$ and n is large. Let $0 < \varepsilon < \varrho - d - 1$. Minkowski's formula, (9) and the Alexandrov-Fenchel inequality yield that

$$V\big(\operatorname{conv} C_n + (d+1)K + \varepsilon K\big) > n\Delta(K) + d\,\Delta(K)^{\frac{d-1}{d}} V(K)^{1/d}\, n^{\frac{d-1}{d}}\, \varepsilon.$$

Therefore $Q_n = \operatorname{conv} C_n + (d+1+\varepsilon)K$ satisfies that if

$$n > \frac{1}{\varepsilon^{\frac{d}{d-1}}}\,\varrho^{\frac{d^2}{d-1}} \quad \text{then} \quad V(Q_n) \geq n\Delta(K) + \varrho^d V(K). \tag{10}$$

In particular, we deduce by (8) and $V\big(Q_n + (\varrho - d - 1 - \varepsilon)K\big) \leq V(P_n + \varrho K)$ that

$$\sum_{i=1}^{d-1} \binom{d}{i} V_i(Q_n, K)(\varrho - d - 1 - \varepsilon)^i < \sum_{i=1}^{d-1} \binom{d}{i} V(Q_n)^{\frac{d-i}{d}} V(K)^{\frac{i}{d}} \varrho^i. \tag{11}$$

The Alexandrov-Fenchel inequality yields that if

$$V_{d-1}(Q_n, K) \geq \left(\frac{\varrho}{\varrho - d - 1 - \varepsilon}\right)^{d-1} V(Q_n)^{\frac{1}{d}} V(K)^{\frac{d-1}{d}}$$

then for any $i \leq d-2$,

$$V_i(Q_n, K) \geq \left(\frac{\varrho}{\varrho - d - 1 - \varepsilon}\right)^{i} V(Q_n)^{\frac{d-i}{d}} V(K)^{\frac{i}{d}}$$

holds. It follows by (11) that

$$V_{d-1}(Q_n, K) < \left(\frac{\varrho}{\varrho - d - 1 - \varepsilon}\right)^{d-1} V(Q_n)^{\frac{1}{d}} V(K)^{\frac{d-1}{d}}. \tag{12}$$

PROPOSITION 3.2. *Let $\varrho > d+1$ and n be large. If $V(\operatorname{conv} C_n + \varrho K)$ is minimal then*

$$\frac{R_K(\operatorname{conv} C_n)}{r_K(\operatorname{conv} C_n)} < d^{2d+3} \left(\frac{2\varrho}{\varrho - d - 1}\right)^{d^2}.$$

REMARK. Closer look at the proof shows that $n > \varrho^d \big(d\varrho/(\varrho - d - 1)\big)^{d^3}$ is sufficient. If K is centrally symmetric then $d+1$ can be replaced with 2.

PROOF. We may assume $V(Q_n) \leq 2n\Delta(K)$ by (8). Set $\varepsilon = \min\{\frac{1}{2}(\varrho - d - 1), 1\}$, and hence Lemma 2.2 and (12) yield that

$$R_K(Q_n) < d^2 \left(\frac{2\varrho}{\varrho - d - 1}\right)^{d-1} \left(\frac{2\Delta(K)}{V(K)}\right)^{1/d} n^{1/d}.$$

Since by Lemma 2.4, we have

$$r_K(Q_n) > \frac{1}{2\,d^{2d}} \left(\frac{2\varrho}{\varrho - d - 1}\right)^{-(d-1)^2} \left(\frac{\Delta(K)}{V(K)}\right)^{1/d} n^{1/d},$$

the inequality $R_K(\operatorname{conv} C_n)/r_K(\operatorname{conv} C_n) < 2\,R_K(Q_n)/r_K(Q_n)$ yields the proposition. ■

Finally we show that if ϱ is large then the asymptotic shape of the optimal $\operatorname{conv} C_n$ as n tends to infinity is close to being homothetic to K.

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PROPOSITION 3.3. *Let $\varrho > d^{7d}$ and $n > \varrho^{2d}$. If $V(\operatorname{conv} C_n + \varrho K)$ is minimal then*

$$\frac{R_K(\operatorname{conv} C_n)}{r_K(\operatorname{conv} C_n)} < 1 + \frac{100\, d^{3.5}}{\varrho^{1/(d+1)}}.$$

PROOF. As the problem is affine invariant, we may assume that $V(K) = 1$ and the Löwner ellipsoid of K is a ball centered at the origin. In addition, set $\varepsilon = 1$ and let $\tilde{Q}_n$ be the body homothetic to Q_n with $V(\tilde{Q}_n) = 1$ and sharing a common centroid with K. With these normalizations, we deduce by (12) that

$$V_{d-1}(\tilde{Q}_n, K) < \left(\frac{\varrho}{\varrho - d - 2}\right)^{d-1}, \tag{13}$$

which in turn yields by Lemma 2.5 and (6) that

$$D_{B^d}(\tilde{Q}_n) < d^{5/2}\left(\frac{\varrho}{\varrho - d - 2}\right)^{d-1}.$$

Substituting these estimates into (4) shows that

$$\delta(\tilde{Q}_n, K) < 12 d^{7/2}\left(\frac{\varrho}{\varrho - d - 2}\right)^{d-1}\left(\left(\frac{\varrho}{\varrho - d - 2}\right)^{d-1} - 1\right)^{1/(d+1)}.$$

The desired inequality finally follows as $R_K(\tilde{Q}_n) \leq 1 + e\,\delta(\tilde{Q}_n, K)$ and $r_K(\tilde{Q}_n) \geq 1 - e\,\delta(\tilde{Q}_n, K)$. ■

The proofs were based on the inequality $V(\operatorname{conv} C_n + \varrho K) \geq n\Delta(K)$ for $\varrho \geq d + 1$ (and for $\varrho \geq 2$ if K is centrally symmetric). Denote by $\Delta_Z(K)$ the minimal determinant of a packing lattice of K and assume that C_n is a set of n points of such a lattice. Then $V(\operatorname{conv} C_n + \varrho K) \geq n\Delta_Z(K)$ for $\varrho \geq 1.5(d + 1)$ and even for $\varrho \geq 3$ if K is centrally symmetric (see [6]), which in turn yields the corresponding statements for lattice packings. If K is a ball then even the condition $\varrho \geq \sqrt{21}/2 = 2.2913$ is sufficient (see [14]).

EXAMPLE. Let $\operatorname{conv} C_n$ be the convex hull of the centers of n non-overlapping translates of the parallelotope P such that $V(\operatorname{conv} C_n + \varrho P)$ is minimal for some $\varrho > 0$. Then Theorem 1.1 in [4] and $V(\operatorname{conv} C_n + P) \geq V(S_n + P)$ show that $\operatorname{conv} C_n = S_n$ if $\varrho < 1$. If $\varrho > 1$ then

$$\left(1 - O(n^{-1/(4d^2)})\right) \cdot P \subset \frac{\operatorname{conv} C_n}{n^{1/d}} \subset \left(1 + O(n^{-1/(4d)})\right) \cdot P,$$

and actually $\operatorname{conv} C_n = (m - 1)P$ assuming that $n = m^d$. The proof is based on some inequalities for mixed volumes involving P, like the Alexandrov–Fenchel inequality.

4. **Planar packings for small ϱ.** We prove the statements of Theorem B in several steps. In this section, K is a convex domain and $\varrho < \Delta(K)/P(K)$.

The perimeter of a convex polytope Q with respect to the norm $\|\cdot\|_K$ is denoted by $U_K(Q)$. If $Q = \operatorname{conv}\{x, y\}$ then $U_K(Q) = 2\|x - y\|_K$. Set $K^* = \frac{1}{2}(K - K)$. Then

$P(K) = P(K^*)$ and $\|\cdot\|_K = \|\cdot\|_{K^*}$. Since for any discrete set Λ, $\Lambda + K$ is a packing if and only if $\Lambda + K^*$ is a packing, we also have $\Delta(K) = \Delta(K^*)$. The celebrated inequality of Oler states (see [16])

$$\frac{A(\operatorname{conv} C_n)}{\Delta(K)} + \frac{U_K(\operatorname{conv} C_n)}{4} + 1 \geq n. \tag{14}$$

For the sausage S_n one has

$$A(\operatorname{conv} S_n + \varrho K) = (n-1)P(K)\varrho + A(K)\varrho^2. \tag{15}$$

We deduce by (14) and Minkowski's formula that $A(\operatorname{conv} C_n + \varrho K) \leq A(\operatorname{conv} S_n + \varrho K)$ yields the inequality

$$4\left(\frac{1}{\varrho} - \frac{P(K)}{\Delta(K)}\right) A(\operatorname{conv} C_n) \leq P(K)U_K(\operatorname{conv} C_n) - 8A(\operatorname{conv} C_n, K). \tag{16}$$

LEMMA 4.1.

$$A(\operatorname{conv} C_n) \geq A(\operatorname{conv} C_n, -K) - A(\operatorname{conv} C_n, K),$$

with equality if and only if $\operatorname{conv} C_n$ *is a segment or* $n = 3$ *and* $\operatorname{conv} C_n = K$.

PROOF. The core of the proof is the claim that if the vertices of the triangle T induce a packing of three translates of K then

$$A(T) \geq A(T, -K) - A(T, K). \tag{17}$$

First we consider the case that K is a triangle. Let $u \in E^2$ be with $K \cap (u + K) = \emptyset$. We prove that the convex hull of K and $K + u$ contains a parallelogram P which contains a translate of K. We can assume that K and $K + u$ have a common point x (a vertex of K). Let f be the opposite face of x. Then $P = f + u$ yields a parallelogram with the above properties. Since P is symmetric it also contains a translate of $-K$ and it follows $A(T, -K) \leq A(T, T + K) = A(T) + A(T, K)$.

Applying this to the three face vectors of T we obtain three translates of $-K$ in $T + K$. If two of them are not identical then $T + K$ even contains the sum $l + (-K)$ where l is a line segment and it follows $A(T, T + K) \geq A\big(T, l + (-K)\big) > A(T, -K)$. Hence equality can only occur if these three translates coincide. So each side of $-K$ has to be a side of one of the translates of K and this is only the case if $K = T$.

For general K we define x by $-K \subset x + R_T(-K)T$, and hence each side of $x + R_T(-K)T$ contains a point of $-K$. Denoting by $-H$ the convex hull of these points, (3) yields that $A(T, -K) = A(T, -H) = A\big(T, R_T(-K)T\big) = R_T(-K)A(T)$. Since the vertices of T yield a packing for the triangle H it follows $A(T, -K) = A(T, -H) \leq A(T) + A(T, H) \leq A(T) + A(T, K)$. In the equality case we have $H = T$ and from $A(T, T+K) = A(T, -K) = A(T, -H)$ it follows $K = H = T$.

Now triangulate $\operatorname{conv} C_n$ using only the centroids of the corresponding n translates of K. Adding up the corresponding inequalities for each triangle in the triangulation yields the required inequality. Equality can occur only if $\operatorname{conv} C_n$ is a segment or each triangle in the triangulation is congruent to K; namely, if $n = 3$ and $K = \operatorname{conv} C_n$. ∎

PROPOSITION 4.2. *If* $\varrho \leq 3/7$ *and* $A(\operatorname{conv} C_n + \varrho K)$ *is minimal then* $C_n = S_n$.

PROOF. Assume that $A(\operatorname{conv} C_n + \varrho K)$ is minimal. Since by (3) (see also [4]), for any convex, compact set C the inequality

$$P(K^*)U_{K^*}(C) \le 8A(C, K^*) \tag{18}$$

holds, we deduce by (16) and some simple considerations that

$$\left(\frac{1}{\varrho} - \frac{P(K)}{\Delta(K)}\right) A(\operatorname{conv} C_n) \le A(\operatorname{conv} C_n, -K) - A(\operatorname{conv} C_n, K).$$

Since $P(K)/\Delta(K) \le 4/3$ (see [4]), $\varrho \le 3/7$ and Lemma 4.1 yield that either $\operatorname{conv} C_n$ is a segment or $\varrho = 3/7$, $n = 3$ and $K = \operatorname{conv} C_n$. In the later case $A(\operatorname{conv} C_n + \varrho K) > A(\operatorname{conv} S_n + \varrho K)$, which in turn implies the proposition. ■

If K is a triangle and $C_3 = K$ then $V(C_3 + \varrho K) < V(S_3 + \varrho K)$ for $\varrho > 1/2$. It follows that one can not improve too much on the bound $3/7$ of Proposition 4.2. In addition, the next Proposition shows that if $\varrho < 3/4$ and n is large then the sausage arrangement is optimal. Thus non-symmetric convex bodies may behave more irregularly than the centrally symmetric ones.

PROPOSITION 4.3. *Let $\varrho < \Delta(K)/P(K)$ and $n > 2000\big(\Delta(K)/P(K) - \varrho\big)^{-2}$. With these conditions, $A(\operatorname{conv} C_n + \varrho K)$ is minimal if and only if $C_n = S_n$.*

PROOF. Since our problem is affine invariant, we may assume that $B = B^2$ is the Löwner ellips of K^*. Assume that $A(\operatorname{conv} C_n + \varrho K)$ is minimal. There exist two parallel supporting lines l_1 and l_2 of $\operatorname{conv} C_n$ with distance at most $3r_B(\operatorname{conv} C_n)$ (see [8]), and let $s \subset \operatorname{conv} C_n$ be the a segment whose orthogonal projection to l_1 is the same as the projection of $\operatorname{conv} C_n$. Then $\operatorname{conv} C_n \subset s + 3r_B(\operatorname{conv} C_n)B$ and

$$P(K)U_K(s) \le 8A(s, K), \tag{19}$$

which in turn yields by (16) that

$$4\left(\frac{1}{\varrho} - \frac{P(K)}{\Delta(K)}\right) A(\operatorname{conv} C_n) \le 3P(K)U_K(B)r_B(\operatorname{conv} C_n).$$

Here $P(K) = P(K^*) \le 4$ by Lemma 2.7 and $U_K(B) = U_{K^*}(B) \le 2\pi$, and hence

$$\left(\frac{\Delta(K)}{P(K)} - \varrho\right) A(\operatorname{conv} C_n) < 6\pi\, r_B(\operatorname{conv} C_n). \tag{20}$$

Assume that $\operatorname{conv} C_n$ is two dimensional and let $\sigma_n \subset \operatorname{conv} C_n$ be the set of centroids of the corresponding n translates of K. Since $\sigma_n + B$ is also a packing, any segment of length $\sqrt{2}$ along l_1 contains the projection of at most $3r_B(\operatorname{conv} C_n)/\sqrt{2} + 1$ points of σ_n. We deduce that $(s/\sqrt{2})\big(3r_B(\operatorname{conv} C_n)/\sqrt{2} + 1\big) \ge n$, which in turn yields the estimate

$$A(\operatorname{conv} C_n) > s\, r_B(\operatorname{conv} C_n) \ge \frac{2n\, r_B(\operatorname{conv} C_n)}{3r_B(\operatorname{conv} C_n) + \sqrt{2}}.$$

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Substituting this into (20) results in

$$2\left(\frac{\Delta(K)}{P(K)}-\varrho\right)n<6\pi\big(3r_B(\operatorname{conv}C_n)+\sqrt{2}\big). \tag{21}$$

On the other hand, $A(\operatorname{conv}C_n)\leq n\Delta(K^*)$ (see [6]) and $A(K^*)/\Delta(K^*)\geq 0.8926$ (see [10]) yield that

$$r_B(\operatorname{conv}C_n)\leq\sqrt{2}r_{K^*}(\operatorname{conv}C_n)\leq\sqrt{2}\sqrt{\frac{\Delta(K^*)}{A(K^*)}}\sqrt{n}<1.5\sqrt{n},$$

and hence

$$2\left(\frac{\Delta(K)}{P(K)}-\varrho\right)n<6\pi\left(\frac{9}{2}\sqrt{n}+\sqrt{2}\right).$$

Now we deduce by some elementary calculations that if $n>2000\big(\Delta(K)/P(K)-\varrho\big)^{-2}$ then $\operatorname{conv}C_n$ must be a segment. ∎

5. **Planar packings for $\varrho>\varrho_c$.** The results of this section correspond to the ones from Section 2. The difference is that having Bonnesen's and Oler's inequality at hand allows much more precise statements.

PROPOSITION 5.1. *Let $\varrho>P(K)/\Delta(K)$ and $A(\operatorname{conv}C_n+\varrho K)$ be minimal. Then for $n\geq 10^4\varrho^2/\big(\varrho-P(K)/\Delta(K)\big)^2$, we have the estimate*

$$\frac{R_K(\operatorname{conv}C_n)}{r_K(\operatorname{conv}C_n)}\leq\frac{400\varrho^2}{\left(\varrho-\frac{\Delta(K)}{P(K)}\right)^2}.$$

PROOF. It follows by Lemma 2.1 that there exists a segment $s\subset\operatorname{conv}C_n$ such that $\operatorname{conv}C_n\subset s+4r_K(\operatorname{conv}C_n)K$. We deduce by (8) that

$$A(\operatorname{conv}C_n)+2\varrho A(\operatorname{conv}C_n,K)\leq\Delta(K)n+2\varrho\sqrt{\Delta(K)A(K)}\sqrt{n},$$

which in turn yields by (14) and (19) that

$$\begin{aligned}2\left(\varrho-\frac{\Delta(K)}{P(K)}\right)A(\operatorname{conv}C_n,K)\;&\leq\;2\varrho\sqrt{\Delta(K)A(K)}\sqrt{n}\\&\quad+\Delta(K)U_K(K)r_K(\operatorname{conv}C_n)+\Delta(K).\end{aligned}$$

Note that $r_K(\operatorname{conv}C_n)\leq\sqrt{\Delta(K)/A(K)}\sqrt{n}$ as $A(\operatorname{conv}C_n)\leq n\Delta(K)$ (see [6]), $U_K(K)=U_{K^*}(K^*)\leq 8$ (see [11]) and $\frac{2}{3}\leq A(K)/\Delta(K)\leq 1$ by Fáry's theorem. We deduce for $Q_n=\operatorname{conv}C_n+\frac{3}{2}K$ after some elementary calculations that

$$2\left(\varrho-\frac{\Delta(K)}{P(K)}\right)\frac{A(Q_n,K)}{\Delta(K)}\leq 2\varrho\sqrt{n}+8\sqrt{\frac{3}{2}}\sqrt{n}+3\varrho. \tag{22}$$

Assume that the origin is the centroid of K. Then $K^*\subset\frac{3}{2}K$, and hence

$$\frac{1}{4}P(K)U_K(\operatorname{conv}C_n)\leq 3A(\operatorname{conv}C_n,K)$$

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by (18). We deduce using Oler's inequality that

$$A(Q_n) = A\left(\operatorname{conv} C_n + \frac{3}{2}K\right) \geq n\Delta(K). \tag{23}$$

On the other hand,

$$A(Q_n, K) \geq \frac{A(Q_n)}{4r_K(Q_n)} \geq \frac{n\Delta(K)}{4r_K(Q_n)}$$

holds by Corollary 2.3, and combining this with (22) yields that

$$r_K(Q_n) \geq \frac{\varrho - \frac{\Delta(K)}{P(K)}}{32\varrho}\sqrt{n}. \tag{24}$$

In particular, if $n \geq 10^4\varrho^2/\big(\varrho - P(K)/\Delta(K)\big)^2$ then $r_K(\operatorname{conv} C_n) \geq \frac{1}{2}r_K(Q_n)$. Finally, Lemma 2.4 yields the proposition as $V(\operatorname{conv} C_n) \leq n\Delta(K)$. ■

The proof of the next proposition is basically the same as for Proposition 3.3, only rather applies the stability formula of Bonnesen.

PROPOSITION 5.2. *Let $\varrho \geq 3$ and $A(\operatorname{conv} C_n + \varrho K)$ be minimal. Then for $n \geq 2500\varrho^2$, we have the estimate*

$$\frac{R_K(\operatorname{conv} C_n)}{r_K(\operatorname{conv} C_n)} \leq 1 + \frac{550}{\sqrt{\varrho}}.$$

PROOF. Set $Q_n = \operatorname{conv} C_n + \frac{3}{2}K$. Since $A\big(Q_n + (\varrho - \frac{3}{2})K\big) \leq A(P_n + \varrho K)$ and $A(Q_n) \geq n\Delta(K)$, we deduce by (8) that

$$2A(Q_n, K)\cdot\left(\varrho - \frac{3}{2}\right) \leq 2\varrho\sqrt{A(K)\Delta(K)}\cdot\sqrt{n} + \left(3\varrho - \frac{9}{4}\right)A(K).$$

Using Bonnesen's inequality (5), and the estimates $A(Q_n) \geq n\Delta(K)$ and $\frac{2}{3} \leq A(K)/\Delta(K) \leq 1$, results in

$$1 + \frac{1}{6n}\big(R_K(Q_n) - r_K(Q_n)\big)^2 \leq \left(\frac{\varrho}{\varrho - \frac{3}{2}}\right)^2\left(1 + \frac{3}{2\sqrt{n}}\right)^2.$$

For $n \geq 2500\varrho^2$ and $\varrho \geq 3$, the right hand side is at most $1 + 5/\varrho$. On the other hand, $\sqrt{n} \leq 96r_K(\operatorname{conv} C_n)$ by (24), and hence $R_K(Q_n) - r_K(Q_n) = R_K(\operatorname{conv} C_n) - r_K(\operatorname{conv} C_n)$ yields the proposition by some simple calculations. ■

The results of Sections 4 and 5 yield that $\varrho_c = P(K)/\Delta(K) = \varrho_s$, and

COROLLARY 5.3. *The Strong Sausage Conjecture holds for any planar convex domain.*

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