SPHERICAL MODIFICATIONS AND THE STRONG CATEGORY OF MANIFOLDS

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Using the notion of spherical modification and results from Morse theory a general technique is described for constructing manifolds whose strong category is small ($\leqq 3$) but whose homological structure is complex.

Unless stated otherwise an n-manifold is a compact, differentiable n dimensional manifold without boundary.

Let V_1 be an n-manifold and suppose S^i is an i-sphere homeomorphically and smoothly imbedded in V_1 with a trivial normal bundle. Then S^i has a neighborhood of the form $S^i \times D^{n-i}$ (D^{n-i} is an $(n-i)$-disc). Clearly the boundary of $S^i \times D^{n-i} = S^i \times S^{n-i-1} =$ the boundary of $D^{i+1} \times S^{n-i-1}$. Smoothly identifying the boundary of $D^{i+1} \times S^{n-i-1}$ with the boundary of (V_1-interior $(S^i \times D^{n-i})$) results in a new manifold V_2. V_2 is said to be obtained from V_1 by a spherical modification of type $(i, n-i-1)$. (Cf. [8] page 504). The manifold V_2 has a sphere S^{n-i-1} (the associated sphere to S^i) imbedded in it with trivial normal bundle; namely,

$$\{0\} \times S^{n-i-1} \subset D^{i+1} \times S^{n-i-1} \subset V_2.$$

Clearly by reversing the procedure V_1 can be obtained from V_2 by a spherical modification of type $(n-i-1, i)$ determined by the associated sphere to S^i. Such a modification will be called an inverse to the given one.

Let V_2 be obtained from V_1 by performing a finite sequence S of spherical modifications on V_1. Associated to S is a $n+1$-manifold W called the trace of S with boundary of $W = V_1 \cup V_2$. The triple (W, V_1, V_2) is a manifold triad in the sense of [6] page 2. A rearrangement theorem says that the modifications S can be rearranged so that all modifications of type $(i, n-i-1)$ are performed before any of type $(i+1, n-i-2)$ and all modifications of type $(i, n-i-1)$ can be assumed to be carried out on the same manifold. Further the trace of the rearranged sequence is the same as the trace of S. (Cf. [8] page 514 and [6] page 44).

Assuming that the sequence S of modifications leading from V_1 to V_2 is already 'rearranged' as above one has a sequence of manifolds $V_1 = M_0$, $M_1, \cdots M_i$, $M_{i+1}, \cdots, M_k = V_2$ where M_{i+1} is obtained from M_i by modifications of type $(i, n-i-1)$ only. Suppose $b_{i+1} =$ the number of

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$(i, n-i-1)$ type modifications in S. Let $\{S_j^i\}_{j=1,\cdots,b_{i+1}}$ be the spheres in M_i determining the b_{i+1} $(i, n-i-1)$ type modifications and let $\{S_j^{n-i-1}\}_{j=1,2,\cdots,b_{i+1}}$ be the associated spheres in M_{i+1}. For any integer $b \geqq 0$ let $F(b)$ be the free abelian group on b generators ($F(0) = 0$). Define $C_i = F(b_i)$ where the $\{S_j^{i-1}\}_{j=1,\cdots,b_i}$ can be taken as representatives for the generators. Then $C_{i-1} = F(b_{i-1})$ is generated by $\{S_k^{i-2}\}_{k=1,\cdots,b_{i-1}}$. Define $d_i : C_i \to C_{i-1}$ by

$$d_i(S_j^{i-1}) = \sum_{k=1}^{b_{i-1}} A_{jk} S_k^{i-2}$$

where $A_{jk} = S_j^{i-1}.\ S_k^{n-i+1}$ = intersection number of S_j^{i-1} and S_k^{n-i+1}, where S_k^{n-i+1} = associated sphere to S_k^{i-2} (note S_j^{i-1} and S_k^{n-i+1} are both spheres in M_{i-1}). It is not hard to see that $(C_*, d) = (C_i, d_i)$ is a chain complex.

THEOREM 1. *Let W = trace of S (performed on V_1) then*

$$H_i(W, V_1) \cong H_i(C_*)$$

all i (homology with integer coefficients).

PROOF. [6] page 90.

THEOREM 2. *Let M be an n dimensional, compact, connected manifold of the form $M = W \cup D^n$ where W = trace of a finite sequence S of spherical modifications on V_1 = boundary of $D^n = S^{n-1}$ and D^n is attached to W by smoothly identifying (boundary D^n) to ((boundary W) $= V_1$). Then $H_0(M) = Z$ (integers) and for $i > 0$ $H_i(M) \cong H_i(C_*)$ where C_* is obtained from S as described above.*

PROOF. Consider the sequence $H_i(M) \xrightarrow{f} H_i(M, D^n) \xrightarrow{g} H_i(W, V_1)$ where f is from the homology sequence of the pair (M, D^n) and is thus an isomorphism for $i > 0$ and g is induced by excision and homotopy and is thus an isomorphism for all i. Since M is connected $H_0(M) = Z$ and for $i > 0$ the theorem follows from theorem 1.

COROLLARY. *If $d_i = 0$ all $i > 0$ then $H_i(M) = C_i = F(b_i)$ all $i > 0$ where b_i = number of $(i-1, n-i-1)$type modifications in S.*

PROOF. Follows directly from theorem 2 and the definition of $H_i(C_*)$.

Before getting to the main result one further definition is needed. Let M be an n-manifold. Define $C(M)$ to be the minimum number of contractable in themselves, open sets needed to cover M. (See strong category [2] page 360.)

Theorem 3 is a statement of the main result although the technique of proof is of more interest than the theorem. (See remark following the proof of theorem 3.)

Note that if, for example, $b_i = 0$ for $0 < i \leqq 33$ and $n = 100$ then the '$C(M) \leqq 3$' portion of the theorem falls under the case mentioned in [1] page 201.

THEOREM 3. *Let* $\{b_i\}_{i=0,1,\cdots,n}$ *be a sequence of non-negative integers satisfying the following conditions*: $b_0 = 1$, $b_i = b_{n-i}$ *and if* $n = 2m$ *then* $b_m = 2t$. *Under these conditions there exists an orientable, connected n-manifold* M *with* $H_i(M) = F(b_i)$ *and* $C(M) \leqq 3$.

PROOF. Let $N = \sum_{i=1}^{m} b_i$ if $n = 2m+1$ and let $N = \sum_{i=1}^{m-1} b_i + t$ if $n = 2m$ and $b_m = 2t$. Let D_1^n be an n-disc and in the boundary of $D_1^n = S^{n-1} = V_1$ pick out N mutually disjoint $(n-1)$ discs. Call them $D(b_i, j)$ where $1 \leqq i \leqq m$ and $1 \leqq j \leqq b_i$ if $n = 2m+1$ if $n = 2m$, then for $1 \leqq i \leqq m-1$, $1 \leqq j \leqq b_i$ and for $i = m$, $1 \leqq j \leqq t$ (note that no discs are picked if $b_i = 0$). In each disc $D(b_i, j)$ imbed an $(i-1)$-sphere S_j^{i-1} with trivial normal bundle. (e.g. $S^{i-1} =$ Boundary

$$D^i \subset D^i \subset D^i \times \{0\} \subset D^i \times D^{n-i-1} = D^{n-1}).$$

Performing on V_1 spherical modifications determined by these N different spheres gives a manifold V with N mutually disjoint spheres $\{S_j^{n-i-1}\}$ (S_j^{n-i-1} is associated to S_j^{i-1}) imbedded in it. Performing on V, N spherical modifications inverse to those performed on V_1 gives a manifold V_2 which is again S^{n-1}. Finally, perform on V_2 an $(n-1, -1)$ spherical modification determined by V_2 itself. Let $W =$ trace of these $2N+1$ modifications and let $M = D_1^n \cup W$ (M is clearly a compact, connected n-manifold).

For $n = 2m+1$ if $1 \leqq i \leqq m$ then $m+1 \leqq n-i \leqq n-1$. Hence the number of $(i-1, n-i-1)$ modifications $=$ number of $(n-i-1, i-1)$ modifications $= b_i = b_{n-i}$ $(1 \leqq i \leqq n-1)$. For $n = 2m$ a slight change occurs; namely, the number of $(m-1, m-1)$ modifications performed on $V_1 = t$, and the number of $(m-1, m-1)$ modifications performed on $V = t$, so the total number of $(m-1, m-1)$ modifications $= 2t = b_m$. In both cases there is only one $(n-1, -1)$ modification performed thus $C_i = F(b_i) 1 \leqq i \leqq n$ for any n.

Consider now $d_i : C_i \to C_{i-1}$. To compute d_i it is necessary to find the intersection numbers $S_j^{i-1} \cdot S_k^{n-i}$ where S_j^{i-1} is a generator of C_i and S_k^{n-i} is associated to a generator S_k^{i-2} of C_{i-1}. If $n = 2m$ then $i-1 \neq n-i$ and thus by the construction of W, $S_j^{i-1} \cap S_k^{n-i} = \emptyset$ for $2 \leqq i \leqq n-1$. However if $n = 2m+1$ then $i-1 = n-i$ for $i = m+1$. In this case, then, the associated spheres to generators of C_m are the generators of C_{m+1}.

Let S_k^{m-1} be a generator of C_m and let S_k^m (a generator of C_{m+1}) be associated to S_k^{m-1}. Now S_k^m is identified with $\{0\} \times S^m \subset D^m \times S^m$ introduced when $S^{m-1} \times D^{m+1}$ is replaced by $D^m \times S^m$ under the modification. Let $P \neq 0$ be a point in D^m and let $\bar{S}_k^m$ be the sphere $\{P\} \times S^m \subset D^m \times S^m$. Then $S_k^m \cap \bar{S}_k^m = \emptyset$ and performing the modification with respect to $\bar{S}_k^m$ gives the

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same result as using S^m_k since $\bar{S}^m_k$ and S^m_k are isotopic. (Cf. [9] 776). Thus for n odd or even its clear that if $2 \leqq i \leqq n-1$ then $d_i = 0$.

Further suppose all of the $(0, n-2)$ modifications performed on V_1 to be orientable (i.e. V is orientable). This corresponds to identifying (boundary $S^0 \times D^{n-1}$) to (boundary $D^1 \times S^{n-2}$) in such a way that the orientations on one component of the boundary are the same while those on the other component are opposite. Hence when the $(n-1, -1)$ modification is performed on V_2 the intersection number is $S^{n-1} \cdot S^0_j = 1-1 = 0$ where S^0_j in V_2 is associated to S^{n-2}_j in V and S^{n-2}_j is associated to the o-sphere determined by $D(b_1, j)$. Also for W its clear that $C_0 = 0$ and thus $d_i = 0$ for $i = 1, \cdots, n$ and by the corollary to theorem 2 $H_i(M) = H_i(C_*) = F(b_i)$. This proves the first part of theorem 3.

To see that $C(M) \leqq 3$ consider the following: By [7] page 14 (the trace of the modification corresponding to $D(b_i, j)$ on $V_1) \cup D^n_1 = D^n_1$ with an n-disc $C(b_i, j)$ attached to the boundary of $D^n_1 = V_1$. Actually $C(b_i, j) \cap V_1 = S^{i-1}_j \times D^{n-i}$ (= tubular neighborhood of S^{i-1}_j used to determine the spherical modification.) Thus $D^n_1 \cup$ (trace of the N different spherical modifications on $V_1) = D^n_1 \cup C_1$ where C_1 is a set of N mutually disjoint n-discs $C(b_i, j)$. Repeating the above argument on V using the N inverse modifications to those done on V_1 one obtains that $D^n_1 \cup C_1 \cup$ (trace of the N inverse modifications) $= D^n_1 \cup C_1 \cup C_2$ where C_2 is a set of N mutually disjoint n-discs $C(b_{n-i}, j)$. Further, when performing the inverse modification to the one determined by $D(b_i, j)$ a tubular neighborhood of small enough 'radius' can be used so that

$$C(b_{n-i}, j) \cap C(b_i, j) = \bar{D}^i \times S^{n-i-1} \subset D^i \times S^{n-i-1} \subset V$$

where $D^i \times S^{n-i-1}$ is the set introduced into V by the modification and where $\bar{D}^i$ is an i-disc $\subset$ interior D^i. Changing the 'radius' does not effect the modification in any significant way (Cf. [9] page 776). Thus all the discs in C_2 can be taken disjoint from D^n_1. Finally, performing the $(n-1, -1)$ spherical modification determined by V_2 corresponds to attaching an n-disc D^n_2 to $D^n_1 \cup C_1 \cup C_2$ by identifying the boundary of D^n_2 to V_2. Thus $M = (D^n_1 \cup C_2) \cup C_1 \cup D^n_2$ where $(D^n_1 \cup C_2)$, C_1, and D^n_2 each consist of finitely many mutually disjoint n-discs.

If $(D_1, D_2, \cdots, D_k)$ is a set of mutually disjoint n-discs in a connected n-manifold (which M is) then D_1 can be joined to D_2 by a smooth arc α so that $\alpha \cap \bigcup_{i=1}^k D_i =$ two points, one in boundary D_1 and one in boundary D_2. A tubular neighborhood T of α can be picked so that $T \cap D_i =$ an $n-1$ disc in boundary D_i $(i = 1, 2)$ and T misses $D_3, \cdots, D_k$. Thus D_1 and D_2 can be joined to form a set E_2 which is contractable in itself. Repeat this construction on $(E_2, D_3, \cdots, D_k)$ starting with E_2 and D_3 to form E_3. Finally one obtains a set E_k which is contractable in itself.

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Hence M can be covered by 3 such contractable sets and if each of them is expanded slightly M can be covered by their interiors and it follows that $C(M) \leqq 3$. This completes the proof of theorem 3.

REMARK. The theorem only asserts the existence of a manifold of a certain type. A manifold satisfying theorem 3 can be constructed in a simple manner as indicated below. The more involved construction given in the proof of the theorem gives a general technique for constructing manifolds with $C(M) \leqq 3$ as there are few restrictions placed on the spherical modifications involved. For example, by changing the (0, 0) modification one can obtain the 2-torus, the klein bottle or the projective plane.

For $n = 2m+1$ one can obtain a manifold satisfying theorem 3 as follows: Let $N = \sum_{i=1}^{m} b_i$ and denote by M_i, b_i copies of $S^i \times S^{n-i}$. Define M to be the connected sum of $M' = \bigcup_{i=1}^{N} M_i$ (i.e. fix a component C of M' and connect all other components of M' to C by $(0, n-1)$-modifications. It is not difficult to prove directly that M satisfies the theorem and is in fact a special case of the construction given in the proof of theorem 3. A similar argument holds for n even.

In theorem 3 if $n = 2m$ then b_m is assumed to be even. This assumption can easily be removed in certain cases. Let $b_m = 2t+1$ and suppose there exists an $(m-1, n-m-1)$ spherical modification ϕ on S^{n-1} which again yields S^{n-1}. Then as in the proof of theorem 3 perform the $N(= \sum_{i=1}^{m-1} b_i+t\}$ spherical modifications on V_1 together with one more; namely ϕ, to obtain V. Then performing the N inverse modifications on V one obtains $V_2 = S^{n-1}$. (No inverse modification is needed to 'cancel' ϕ.) Thus the number of $(m-1, n-m-1)$ type modifications $= 2t+1 = b_m$. Its easy to see that the rest of the proof goes through as before.

If $n = 2, 4, 8, 16$ such spherical modifications as ϕ exist, the trace of ϕ being the real projective plane with two 2-discs removed if $n = 2$, the complex projective plane with two 4-discs removed if $n = 4$, the quaternionic projective plane with two 8-discs removed if $n = 8$, and the Cayley projective plane with two 16-discs removed if $n = 16$. (Cf. [4] page 708). However for $n = 2$ the (0, 0) modification is non-orientable but for $n = 4, 8$ and 16 one has:

COROLLARY 1. *If $n = 4, 8, 16$ the restriction that b_m be even in theorem 3 can be removed.*

COROLLARY 2. *If $b_1 \neq 0$ and $n \geqq 2$ in theorem 3 then $C(M) = 3$.*

PROOF. This follows from [2] page 258 theorem 29.3.

Let f be a Morse function on an n-manifold M ($f : M \to R$ (reals) with a finite number of critical points all of which are non-degenerate). To each critical point of f is attached an index i (an integer $0 \leqq i \leqq n$) (Cf. [7]

page 5). Define $\mu(M)$ to be the minimum number of different indices appearing in f as f ranges over all Morse functions on M. It is well known that $C(M) \leqq \mu(M) \leqq n+1$ (Cf. [3] or [5]). Further if the number of points with index i for a Morse function f on M is zero then $H_i(M) = 0$ (Cf. [7] page 20). Hence if $b_i > 0$, $i = 0, \cdots, n$ in theorem 3 then the manifold M constructed there has the property that $C(M) \leqq 3$ and $\mu(M) = n+1$. Thus $\mu(M)-C(M) \geqq n-2$ and, in view of corollary 2, $n-2$ is the maximum difference if $b_1 \neq 0$ and $n \geqq 2$.

COROLLARY 3. If $n \geqq 2$ then there exists an n-manifold M (actually there exists infinitely many non-diffeomorphic such n-manifolds) such that $\mu(M)-C(M) = n-2$ and if $n = 1$ then clearly $\mu(S^1)-C(S^1) = 0$ where S^1 is the 1-sphere.

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