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ON SINGULAR POINTS OF NORMAL ARCS OF CYCLIC ORDER FOUR

BY

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1. **Introduction.** In [5] N. D. Lane and P. Scherk discuss arcs in the conformal (inversive) plane which are met by every circle at not more than three points; i.e., arcs of cyclic order three. This paper is concerned with the analysis of normal arcs of cyclic order four in the conformal plane.

An arc $\mathscr{A}$ is *normal* if each circle C can be oriented so that the points of $C \cap \mathscr{A}$ lie in the same order on C as they do on $\mathscr{A}$; cf. O. Haupt [2]. We note that a curve $\mathscr{C}_4$ of cyclic order four is always normal; cf. [2], 5.

It is well known that a normal arc $\mathscr{A}_4$ of cyclic order four is the union of a finite number of arcs of order three; cf. O. Haupt [3] and 4.1.3. of [1]. To derive this result Haupt used the contraction theorem (first attributed to S. Mukhopadhyaya) and the expansion theorem; cf. [1], 2.4.4; [6]; [7]; [1], 2.4.5. These results generally deal with the specific movement of intersection points of arcs with members of classes or order characteristics with a fundamental number k, this number being such that k distinct points uniquely determine one member of the class, cf. [1], 1.1. The proofs of his results are generally by induction on the fundamental number k.

It is of interest to derive results analogous to those of Haupt and Künneth for the analysis of arcs and curves of cyclic order four in the conformal plane. We shall use conformal methods for the derivation of several main results; cf. *2.2*, *2.4* and *3.2*.

In [5], certain differentiability properties are obtained for arcs of cyclic order three. We utilize the conformal analogues of the contraction and expansion theorems for normal arcs of cyclic order four to prove that an end-point of such an arc is ordinary and hence strongly differentiable; cf. *3.2* and *3.3*.

The definitions and notation which will be used in this paper are the same as in [4] and [5]. In particular, the same letters are used for the points of an arc as are used for their respective parameter values on the parameter interval. A *singular point* of an arc $\mathscr{A}_4$ is a point of cyclic order four.

2. Lemmas on normal arcs $\mathscr{A}_4$ of cyclic order four.

2.1. *Suppose the circle C_0 meets $\mathscr{A}_4$ at four distinct points a, b, p_1, p_2. Then if $t \in \mathscr{A}_4$ is sufficiently close to a, there is one and only one point $u=u(t) \in C(t, p_1, p_2) \cap \mathscr{A}_4$ near b. It depends continuously and monotonically on t. The points t and u move in opposite directions.*

Proof. Since $\mathscr{A}_4$ is of order four, $C_0=C(a, p_1, p_2)$ intersects $\mathscr{A}_4$ at a, b, p_1, p_2 and meets $\mathscr{A}_4$ nowhere else. If t is sufficiently close to a, then $C(t, p_1, p_2)$ will be close to C_0 and will intersect $\mathscr{A}_4$ at t, p_1, p_2 and a point $u(t)$ close to b. Also $C(t, p_1, p_2)$ meets $\mathscr{A}_4$ nowhere else. Thus $u(t)$ depends continuously on t.

The number n of the points p_i on $C(t, p_1, p_2)$ between t and $u(t)$ is constant. Since C_0 and $C(t, p_1, p_2)$ intersect at p_1 and p_2, and meet nowhere else, C_0 separates t and $u(t)$ if and only if n is odd; i.e., $n=1$.

Suppose $u(t)>b$ when $t>a$. Since C_0 intersects $\mathscr{A}_4$ at a, b, p_1, p_2 and does not meet $\mathscr{A}_4$ elsewhere, the subarc $[t, u(t)]$ of $\mathscr{A}_4$ will intersect C_0 at either a or b and at each of the n points of C_0 which lie between a and b. Hence t and $u(t)$ will lie on opposite sides of C_0 if and only if n is even; a contradiction. Hence $u(t)<b$ when $t>a$ and, symmetrically, $u(t)>b$ when $t<a$.

REMARKS. (i) The movement of t and $u(t)$ in 2.1 can continue as long as t and u do not move into p_1, p_2 or an endpoint of $\mathscr{A}_4$.

(ii) It can happen that $\mathscr{A}_4$ is only one-sidedly conformally differentiable at an interior point p_i. Hence $u(t)$ need not move continuously on $\mathscr{A}_4$ if t or $u(t)$ move through p_1 or p_2.

(iii) 2.1 and the above remarks (excluding the reference to endpoints in (i)) remain valid if the arc $\mathscr{A}_4$ is replaced by a curve $\mathscr{C}_4$ of cyclic order four.

(iv) The spiral $\mathscr{A}_4$ given by $r=e^\theta$; $0\leq\theta\leq 2\pi$, shows that the statement of 2.1 is false if the normality condition is omitted.

2.2. THEOREM 1. *Let C_0 be a circle which meets $\mathscr{A}_4$ at points $p_0<q_0<r_0<s_0$. If $\mathscr{B}$ is the closed subarc of $\mathscr{A}_4$ between p_0 and s_0, then there exists at least one singular point in the interior of $\mathscr{B}$.*

Proof. Consider the parameter interval $I_0=[p_0, s_0]$. We define a sequence of intervals and a corresponding sequence of circles by induction. Having defined

$$I_n = [p_n, s_n]$$

with $I_n \subset I_0$ and

$$C_n = C_n(p_n, q_n, r_n, s_n)$$

through $p_n<q_n<r_n<s_n$, we define $I_{n+1}\subset I_n$ and C_{n+1} as follows.

Divide $I_n=[p_n, s_n]$ into eight equal subintervals with the endpoints A_i; $i=-4, -3, \ldots, 0, 1, \ldots, 4$. Our goal is the construction of a circle which passes through four points of either the interval $[A_{-4}, A_3]$ or the interval $[A_{-3}, A_4]$ of $\mathscr{A}_4$.

In the following, put $i=1, 2, 3$ in turn. Suppose at the ith step only $i-1$ of the points of $C_n \cap I_n$ lie in the closed subinterval $[A_{-i}, A_i]$ of $\mathscr{A}_4$ and that there are points of $C_n \cap I_n$ on both sides of $[A_{-i}, A_i]$. Then we move the two points which lie outside, adjacent to, and on opposite sides of $[A_{-i}, A_i]$ toward this interval, using 2.1, while keeping the other two points of $C_n \cap I_n$ fixed. Eventually, at least one of these moving points reaches $[A_{-i}, A_i]$. If necessary, we proceed with the next step. In this manner we obtain a new interval

$$I_{n+1} = [p_{n+1}, s_{n+1}] \subset I_n$$

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and a new circle

$$C_{n+1} = C_{n+1}(p_{n+1}, q_{n+1}, r_{n+1}, s_{n+1})$$

through $p_{n+1} < q_{n+1} < r_{n+1} < s_{n+1}$.

Let $l(I_n)$ be the length of I_n. By our construction, we obtain for each n, $l(I_{n+1}) < 7l(I_n)/8$. Thus $\{I_n\}$ is a nested sequence of closed intervals such that $\lim l(I_n)=0$. Hence there exists $y \in I_0$ such that $\cap_n I_n = \{y\}$ The corresponding point $y \in \mathscr{B}$ is the required singular point.

If y is not an interior point of $\mathscr{B}$, then hold q_0 and r_0 fixed and let t move a small distance from p_0 toward q_0 before defining the sequence $\{I_n\}$. By 2.1, there is a point u which moves a small distance from s_0 toward r_0. Now use the interval $[t, u]$ as I_0 and $C(t, q_0, r_0, u)$ as C_0 and construct $\{I_n\}$ and $\{C_n\}$ as above. This will ensure that we obtain a singular point in $[t, u]$ and hence in the interior of $\mathscr{B}$, as required.

REMARKS. (i) In the above proof, the normality of $\mathscr{A}_4$ is used only in the application of 2.1.

(ii) Theorem 1 becomes false if the normality condition is omitted. The spiral $r=e^{\theta}$; $0 \leq \theta \leq 2\pi$, is an arc of cyclic order four, but it has no singular points.

2.3. By an argument similar to that used in 2.2, one obtains the following.

Let $p_0 < q_0 < r_0$ be three points on $\mathscr{A}_4$ and let $\mathscr{B}$ be the closed subarc of $\mathscr{A}_4$ bounded by p_0 and r_0. Let $a \in \mathscr{A}_4 \backslash \mathscr{B}$. Suppose that there exists a circle through the points a, p_0, q_0, r_0. If Γ_a is the system of circles passing through the point a, then there exists at east one Γ_a-singular point y in the interior of $\mathscr{B}$; i.e., *for any neighborhood N of y on $\mathscr{B}$ there exists a circle of Γ_a that meets N at least three times.*

2.4. THEOREM 2. *Suppose the open subarc of $\mathscr{A}_4$ contains two singular points. Let $a \in \mathscr{A}_4 \backslash \mathscr{B}$. Then some circle of Γ_a will meet $\mathscr{B}$ at three distinct points.*

Proof. By our assumption, there are eight points

$$w_1 < w_2 < w_3 < w_4 < x_1 < x_2 < x_3 < x_4$$

on $\mathscr{B}$ such that $w_1 \in C(w_2, w_3, w_4)$ and $x_1 \in C(x_2, x_3, x_4)$. We may assume e.g. $a < w_1$.

Let t move on $\mathscr{B}$ monotonically from w_4 towards x_4. By 2.1, the circle $C(w_2, w_3, t)$ meets $\mathscr{A}_4$ at a fourth point $u(t)$ which moves from w_1 monotonically towards a. If $u(t)=a$ for some t with $w_4 < t \leq x_4$, then $C(w_2, w_3, t)$ has the required properties. By 2.1, Remark (i), we may therefore assume that $a < w_1' = u(x_4) < w_1$.

Now let t' move on $\mathscr{B}$ from w_3 monotonically towards x_3. Then $C(w_2, x_4, t')$ meets $\mathscr{A}_4$ at a fourth point $u'(t')$ which moves from w_1' monotonically towards a. Again we may assume that $a < w_1'' = u'(x_3) < w_1'$.

Finally let t'' move on $\mathscr{B}$ from w_2 monotonically towards x_2. Then $C(x_3, x_4, t'')$ meets $\mathscr{A}_4$ at some point $u''(t'')$ which moves from w monotonically towards a.

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As $x_1 \in C(x_3, x_4, x_2)$ this circle cannot meet $\mathscr{A}_4$ elsewhere. Hence $u''(t'')=a$. for some t'' such that $w_2<t''<x_2$. The circle $C(x_3, x_4, t'')$ has the required properties.

2.5. *Let $y<z$ be two singular points of $\mathscr{A}_4$; $a \notin [y, z]$. Then $[y, z]$ contains at least one Γ_a-singular point.*

Proof. Construct a monotonically shrinking sequence of open arcs $\mathscr{B}_\lambda \subset \mathscr{A}_4$ such that $[y, z] \subset \mathscr{B}_\lambda \subset \mathscr{A}_4$, $a \notin \mathscr{B}_\lambda$ and $\bigcap^\infty \mathscr{B}_\lambda = [y, z]$. By 2.4 and 2.3, each $\mathscr{B}_\lambda$ contains a Γ_a-singular point x_λ. The sequence $\{x_\lambda\}$ has accumulation points. Such a point is Γ_a-singular and lies in $[y, z]$.

Arguments which are analogous to those used in 2.3, 2.4 and 2.5 yield the following.

2.6. *Let a, b, y, z be distinct points on $\mathscr{A}_4$ such that $b \notin [y, z]$ and a does not lie between any two of b, y, z. Finally, suppose that y and z are Γ_a-singular. Then $[y, z]$ contains a $\Gamma_{a,b}$-singular point x;* i.e., *for any neighbourhood N of x there exists a circle passing through a and b which meets N at least twice.*

2.7. *Let a, b, c, y, z be distinct points on $\mathscr{A}_4$ such that $c \notin [y, z]$, b does not lie between any two of c, y, z, and a does not lie between any two of b, c, y, z. Finally, suppose that y and z are $\Gamma_{a,b}$-singular. Then $[y, z]$ contains a $\Gamma_{a,b,c}$-singular point x,* i.e., *for any neighbourhood N of x there exists a circle passing through a, b and c, which meets N at least once.*

3. Singular and ordinary points on arcs of order four.

3.1. We use the results in Section 2 to derive the following.

THEOREM 3. *The number of singular points of a normal arc of cyclic order four is bounded.*

Proof. Suppose $\mathscr{A}_4$ has at least k interior singular points, say $x_1<x_2<\cdots<x_k$. Choose $a<x_1$ in the interior of $\mathscr{A}_4$. By 2.5, each interval $[x_\lambda, x_{\lambda+1}]$ contains a Γ_a-singular point. This yields (at least) $h=[k/2]$ Γ_a-singular points $y_1<y_2<\cdots<y_h$ with $x_1 \le y_1 < y_h \le x_k$.

Next choose $b \in (a, x_1)$. Then 2.6 yields $m=[h/2]\Gamma_{a,b}$-singular points $z_1<z_2<\cdots<z_m$ such that $y_1 \le z_1 < z_m \le y_h$. Finally, let $c \in (b, x_1)$ such that none of $z_1, \ldots, z_m$ lies on $C(a, b, c)$. Then there are at least $m-1$ $\Gamma_{a,b,c}$-singular points in (z_1, z_m), and hence in the interior of $\mathscr{A}_4$. As there is not more than one such point, we have $m-1 \le 1$, i.e. $m \le 2$, $h \le 5$, and $k \le 11$.

3.2. Since an arc bounded by two singular points or by a singular point and an endpoint has order three if it has no singular points in its interior, 3.1 yields the following.

Every normal arc $\mathscr{A}_4$ is the union of a bounded number of arcs of cyclic order three.

REMARK. The above statement remains valid if the normality condition is removed, but the proof requires other methods; cf. [3].

3.3. *An end-point of $\mathscr{A}_4$ is ordinary.*

Proof. Assume the end-point p of $\mathscr{A}_4$ is singular. Then for each neighbourhood $N^{(1)}$ of p there exists a circle which meets $N^{(1)}$ four times, say at $p_1<q_1<r_1<s_1$. By 2.2, there exists a singular point $y^{(1)}$ in (p_1, s_1). Now take a new smaller neighbourhood $N^{(2)}$ of p with $y^{(1)} \notin N^{(2)}$. By 2.2, there exists another singular point $y^{(2)}$ different from $y^{(1)}$. Repeating this process and using 2.2, we obtain an infinite number of singular points. This is impossible, by Theorem 3.

3.4. In 3.3, it was shown that an end-point p of $\mathscr{A}_4$ is ordinary. Thus there exists a neighbourhood N_3 of p on $\mathscr{A}_4$ which is of order three. But is is known that $N_3 \cup \{p\}$ is strongly differentiable at p; cf. ([4], 3.5.). This proves the following result.

THEOREM 4. *An end-point of $\mathscr{A}_4$ is strongly differentiable.*

3.5. In [5], it is shown that an interior point of an arc of order three is strongly once differentiable, (necessarily satisfies Condition I'). The authors also proved that if such a point is differentiable, then it is strongly differentiable.

It is easily seen that corresponding results for interior points of arcs of order four are not generally valid. An elementary differentiable cusp point with characteristic $(2, 1, 1)_0$ of an arc of order four does not satisfy Condition I' and hence is not strongly differentiable.

3.6. If $\mathscr{A}_4$ is strongly conformally differentiable, we can choose the point a in the proof of Theorem 3 such that a does not lie on the osculating circle of any of the singular points $x_1, \ldots, x_k$. Then the Γ_a-singular point in each interval $[x_\lambda, x_{\lambda+1}]$ cannot coincide with either x_λ or $x_{\lambda+1}$. Similarly, by choosing the point b such that b does not lie on the tangent circle through a of any of the Γ_a-singular points $y_1, \ldots, y_h$, we can ensure that the $\Gamma_{a,b}$-singular point in each $[y_\mu, y_{\mu+1}]$ does not coincide with y_μ or $y_{\mu+1}$. Finally, we choose c as in the proof of Theorem 3. Then the proof of Theorem 3 yields the following.

A strongly differentiable normal arc of cyclic order four contains at most four singular points. Thus every such arc is the union of at most five arcs of cyclic order three.

3.7. A closed curve $\mathscr{C}_4$ of cyclic order four is automatically normal. From 3.6, one obtains the following well-known result.

THEOREM 5. *A strongly differentiable closed curve of cyclic order four contains at most four singular points.*

REMARK. The authors have proved, using different methods, that the statement of Theorem 5 remains valid if "strongly" is ommitted.

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