J. Appl. Prob. **38,** 280–283 (2001)
Printed in Israel

Dear Editor,

A note on D-policy bulk queueing systems

We offer new studies of the queueing process of D-policy models and correct results of [2].

1. Preliminaries

In [2], the D-policy did not apply to the queueing process in the general case. The results of [2] are corrected by applying [1] and [3]. Throughout, we use the notation of [2].

Customers enter the system in accordance with a bulk Poisson input of intensity λ and with $a(z)$ as the probability generating function of arriving batches. They are serviced singly in accordance with the i.i.d. random variables $\Sigma_1, \Sigma_2, \dots$ with the joint probability density function $B(x)$ and Laplace–Stieltjes transform $\sigma(\theta)$. When the system is exhausted, service is suspended with the server staying idle in the system, leaving the system for multiple vacation trips or a single vacation trip. Each suspension mode lasts until the system's workload becomes D or greater at one of the 'observation epochs'. Let $\tau = (\tau_0, \tau_1, \dots)$ be the sequence of such observation epochs, $X_0, X_1, \dots$ be the increments of units' replenishment over τ, and $Y_0, Y_1, \dots$ be the respective increments of the workload. With $\nu = \inf\{k : B_k = Y_0 + \cdots + Y_k > D\}$, τ_ν is the first passage time. The observed value B_ν of $\{B_k\}$ at τ_ν is the workload, and A_ν is the queue length (where $A_k = X_0 + \cdots + X_k$) at τ_ν (the *first excess level projection* of B_ν onto $\{A_k\}$ [1], [2]).

Now, at the beginning of the first busy period after suspension, the server starts servicing *one virtual customer* whose service time is B_ν, so that $Q_n = Q(T_n)$, $n = 0, 1, \dots,$ where $T_1 = \Sigma_1$, if $Q_0 > 0$, and $T_1 = \tau_\nu + B_\nu$, if $Q_0 = 0$. The embedded chain is that of an $M/G/1$ queue with bulk input and start-up time. We need [2]:

$$L(z, \vartheta, \theta) := \mathbb{E}[z^{A_\nu} \, \mathrm{e}^{-\vartheta B_\nu} \, \mathrm{e}^{-\theta \tau_\nu}]$$
$$= \gamma_0(z, \vartheta, \theta) - [1 - \gamma(z, \vartheta, \theta)] \mathcal{L}_s \left\{ \frac{\gamma_0(z, \vartheta + s, \theta)}{1 - \gamma(z, \vartheta + s, \theta)} \right\} (D), \qquad (1.1)$$

where

$$\gamma_0(z, \vartheta, \theta) = \mathbb{E}[z^{X_0} \, \mathrm{e}^{-\vartheta Y_0} \, \mathrm{e}^{-\theta \tau_0}], \qquad \gamma(z, \vartheta, \theta) = \mathbb{E}[z^{X_1} \, \mathrm{e}^{-\vartheta Y_1} \, \mathrm{e}^{-\theta \chi_1}],$$
$$\chi_n = \tau_n - \tau_{n-1}, \quad n = 1, 2, \dots, \quad \text{and} \quad \mathcal{L}_s F(x) = \mathrm{Lapl}^{-1} \left(\frac{1}{s} F(s) \right)(x), \quad x \geq 0.$$

Multiple vacations.

$$\tau_0 = X_0 = Y_0 = 0, \qquad \gamma_0 = 1$$

and

$$\gamma(z, \vartheta, \theta) = \mathbb{E}[z^{X_1} \, \mathrm{e}^{-\vartheta Y_1} \, \mathrm{e}^{-\theta \chi_1}] = \gamma(\lambda(z\sigma(\vartheta), \theta)),$$
$$\sigma(\vartheta) = \mathbb{E}[\mathrm{e}^{-\vartheta \Sigma_1}], \qquad \lambda(z, \theta) = \lambda - \lambda a(z) + \theta,$$
$$a(z) = \mathbb{E}[z^{U_1}], \qquad \gamma(\theta) = \mathbb{E}[\mathrm{e}^{-\theta \chi_1}]. \qquad (1.2)$$

Received 23 November 1999; revision received 27 November 2000.

(The latter is the Laplace–Stieltjes transform of a one vacation trip.) Consequently,

$$L(z,\vartheta,\theta) = 1 - [1-\gamma(\lambda(z\sigma(\vartheta),\theta))]\mathcal{L}_s\left\{\frac{1}{1-\gamma(\lambda(z\sigma(\vartheta+s),\theta))}\right\}(D). \tag{1.3}$$

Dormant server.

$$\gamma(z,\vartheta,\theta) = \frac{\lambda}{\lambda+\theta}a(z\sigma(\vartheta)),$$

yielding

$$L(z,\vartheta,\theta) = 1 - \left[1 - \frac{\lambda}{\lambda+\theta}a(z\sigma(\vartheta))\right]\mathcal{L}_s\left\{\frac{1}{1-(\lambda/(\lambda+\theta))a(z\sigma(\vartheta+s))}\right\}(D). \tag{1.4}$$

Single vacation.

$$\gamma_0(z,\vartheta,\theta) = \gamma(\lambda(z\sigma(\vartheta),\theta)),$$
$$\gamma(z,\vartheta,\theta) = \frac{\lambda}{\lambda+\theta}a(z\sigma(\vartheta)),$$

$$L(z,\vartheta,\theta) = \gamma(\lambda(z\sigma(\vartheta),\theta)) - \left[1 - \frac{\lambda}{\lambda+\theta}a(z\sigma(\vartheta))\right]\mathcal{L}_s\left\{\frac{\gamma(\lambda(z\sigma(\vartheta+s),\theta))}{1-(\lambda/(\lambda+\theta))a(z\sigma(\vartheta+s))}\right\}(D). \tag{1.5}$$

The corresponding formulas for the marginal functional $\mathcal{B}(\vartheta) = \mathbb{E}[\mathrm{e}^{-\vartheta B_\nu}]$ will read as follows.

Multiple vacations.

$$\mathcal{B}(\vartheta) = 1 - [1-\gamma(\lambda(\sigma(\vartheta)))]\mathcal{L}_s\left\{\frac{1}{1-\gamma(\lambda(\sigma(\vartheta+s)))}\right\}(D),$$

where $\lambda(z) = \lambda(z,0)$.

Dormant server.

$$\mathcal{B}(\vartheta) = 1 - [1-a(\sigma(\vartheta))]\mathcal{L}_s\left\{\frac{1}{1-a(\sigma(\vartheta+s))}\right\}(D).$$

Single vacation.

$$\mathcal{B}(\vartheta) = \gamma(\lambda(\sigma(\vartheta))) - [1-a(\sigma(\vartheta))]\mathcal{L}_s\left\{\frac{\gamma(\lambda(\sigma(\vartheta+s)))}{1-a(\sigma(\vartheta+s))}\right\}(D).$$

Now, the probability generating function $p(z)$ of the invariant probability measure p of the embedded queueing process is:

$$p(z) = p_0\frac{z\mathcal{B}(\lambda(z)) - \sigma(\lambda(z))}{z-\sigma(\lambda(z))}, \qquad p_0 = \frac{1+\bar{\mathcal{B}}\lambda a - \rho}{1-\rho},$$

where $\rho = \lambda a S$, $a = \mathbb{E}[U_1]$, $S = \mathbb{E}[\Sigma_1]$, and $\bar{\mathcal{B}} = \rho\bar{\tau}$, with $\bar{\tau} = \mathbb{E}[\tau_\nu]$.

https://doi.org/10.1239/jap/996986664 Published online by Cambridge University Press

2. Continuous time parameter process

The limiting distribution $\pi = (\pi_0, \pi_1, \dots)$ of $Q(t)$ exists given $\rho < 1$ and is sought in the form of its probability generating function $\pi(z) = (1/C)ph(z)$, where C is given below, and $h(z) = (h_i(z); i = 0, 1, \dots)^\top$ with the probability generating functions of the respective rows of the integrated (over $\mathbb{R}_+$) semi-regenerative kernel

$$K(t) = \{K_{ik}(t) = \mathbb{P}^i\{Q(t) = k, T_1 > t\}; i, k = 0, 1, \dots\}$$

being given by

$$h_i(z) = \sum_{k=0}^{\infty} z^k \int_0^{\infty} K_{ik}(t)\,\mathrm{d}t = z^i \Delta(z), \qquad i > 0, \tag{2.1}$$

where

$$\Delta(z) = \frac{1 - \sigma(\lambda(z))}{\lambda(z)}. \tag{2.2}$$

Theorem. ([3].)

$$T(z, \theta) = \int_0^{\infty} \mathrm{e}^{-\theta t} \mathbb{E}[z^{N_t} \mathbf{1}_{\{\tau_\nu + B_\nu > t\}}]\,\mathrm{d}t = \frac{1 - L(z, \lambda(z, \theta), \theta)}{\lambda(z, \theta)},$$

where L satisfies (1.1), (1.3), (1.4) or (1.5) and $\lambda(z, \theta)$ is defined in (1.2).

In our case,

$$h_0(z) = T(z, \theta) = \int_0^{\infty} \mathbb{E}[z^{N_t} \mathbf{1}_{\{\tau_\nu + \Sigma > t\}}]\,\mathrm{d}t = \frac{1 - L(z, \lambda(z), 0)}{\lambda(z)}, \tag{2.3}$$

with the following variants of L.

Multiple vacations.

$$L(z, \lambda(z), 0) = 1 - [1 - \gamma\{\lambda[z\sigma(\lambda(z))]\}]\mathcal{L}_s \left\{ \frac{1}{1 - \gamma\{\lambda(z\sigma(\lambda(z) + s))\}} \right\}(D).$$

Dormant server.

$$L(z, \vartheta, 0) = 1 - [1 - a\{z\sigma(\lambda(z))\}]\mathcal{L}_s \left\{ \frac{1}{1 - a\{z\sigma(\lambda(z) + s)\}} \right\}(D).$$

Single vacation.

$$L(z, \vartheta, 0) = \gamma\{\lambda[z\sigma(\lambda(z))]\} - [1 - a\{z\sigma(\lambda(z))\}]\mathcal{L}_s \left\{ \frac{\gamma\{\lambda[z\sigma(\lambda(z) + s)]\}}{1 - a\{z\sigma(\lambda(z) + s)\}} \right\}(D).$$

From (2.1)–(2.3) we get

$$ph(z) = \Delta(z)p(z) + p_0 \frac{\sigma(\lambda(z)) - L(z, \lambda(z), 0)}{\lambda(z)}.$$

The mean stationary value of the service cycle is equal to $C = pc$, where

$$c = (c_i = \mathbb{E}^i[T_1]; i = 0, 1, \dots)^\top, \qquad c_0 = \bar{\tau}(1 + \rho), \qquad c_i = S, \quad i > 0,$$

and $\bar{\tau} = \mathbb{E}[\tau_\nu]$. This yields

$$C = p_0(\bar{\tau}(1 + \rho) - S) + S.$$

Finally,

$$\pi(z) = \frac{1}{C} ph(z).$$

https://doi.org/10.1239/jap/996986664 Published online by Cambridge University Press

References

[1] Dshalalow, J. H. (1997). On the level crossing of multi-dimensional delayed renewal processes. *J. Appl. Math. Stoch. Anal.* **10,** 355–361.
[2] Dshalalow, J. H. (1998). Queueing processes in bulk systems under the *D*-policy. *J. Appl. Prob.* **35,** 976–989.
[3] Dshalalow, J. H. (2001). Time dependent analysis of multivariate marked renewal processes. Submitted.

Yours sincerely,

Jewgeni H. Dshalalow

Department of Applied Mathematics,
Florida Institute of Technology,
Melbourne, FL 32901,
USA.
Email address: eugene_d@bellsouth.net

https://doi.org/10.1239/jap/996986664 Published online by Cambridge University Press