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SHARPNESS IN YOUNG'S INEQUALITY FOR CONVOLUTION PRODUCTS

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ABSTRACT. Suppose that G is a locally compact group with modular function Δ and that p, q, r are three numbers in the interval $(1,\infty)$ satisfying $1/p + 1/q = 1 + 1/r$. If $c_{p,q}(G)$ is the smallest constant c such that $\|f * \Delta^{1/p'}g\|_r \leq c\|f\|_p\|g\|_q$ for all functions $f, g \in C_c(G)$ (here the convolution product is with respect to left Haar measure and p' is the exponent which is conjugate to p) then Young's inequality asserts that $c_{p,q}(G) \leq 1$. This paper contains three results about these constants. Firstly, if G contains a compact open subgroup then $c_{p,q}(G) = 1$ and, as an extension of an earlier result of J. J. F. Fournier, it is shown that there is a constant $c_{p,q} < 1$ such that if G does not contain a compact open subgroup then $c_{p,q}(G) \leq c_{p,q}$. Secondly, Beckner's calculation of $c_{p,q}(\mathbb{R})$ is used to obtain the value of $c_{p,q}(G)$ for all simply-connected solvable Lie groups and all nilpotent Lie groups. And thirdly, it is shown that for a nilpotent Lie group the set $L^p(G) * \Delta^{1/p'}L^q(G)$ is not contained in the union of the spaces $L^s(G)$, $s \in [1,r) \cup (r,\infty)$.

Consider a locally compact group G with modular function Δ. Young's inequality is the assertion that if p, q, r are three real numbers in the interval $(1,\infty)$ satisfying

$$p^{-1} + q^{-1} = 1 + r^{-1} \tag{1}$$

and if $f \in L^p(G)$ and $g \in L^q(G)$ then the convolution product $f * \Delta^{1/p'}g$ is finite a.e. and satisfies

$$\|f * \Delta^{1/p'}g\|_r \leq \|f\|_p\|g\|_q. \tag{2}$$

[Here, and throughout this paper, convolution products, Lebesgue spaces, and "a.e." are with respect to a left Haar measure on G and, for each number p in the interval $(1,\infty)$, p' denotes the index which is conjugate to p.] There is a constant implicit in (2), viz., the smallest number $c_{p,q}(G)$ for which the inequality

$$\|f * \Delta^{1/p'}g\|_r \leq c_{p,q}(G)\|f\|_p\|g\|_q \tag{3}$$

holds for all functions $f \in L^p(G)$ and $g \in L^q(G)$. The questions of sharpness in Young's inequality include, for given values of p and q and for a given group G, the problem of calculating $c_{p,q}(G)$, the problem of characterizing those functions f and g for which (3) is an equality, and the problem of characterizing the linear span of the set $L^p(G)*\Delta^{1/p'}L^q(G)$ of convolution products. All of these questions have been studied by a number of authors (see [1–3, 5, 7–9], for example).

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This paper is divided into two logically independent sections. The first section relates sharpness to the existence of compact open subgroups and contains two results which extend to non-unimodular groups some theorems due to J. F. F. Fournier [5] in the unimodular case. The second section deals with sharpness for semi-direct products and, for a class of semi-direct products, contains the proof of a conjecture of A. Klein and B. Russo [7] as well as some information about the linear span of the set of convolution products. Consequences of these results are, for certain solvable Lie groups, the calculation of the value of $c_{p,q}(G)$ for all numbers p, q, r in the interval $(1,\infty)$ satisfying (1) and, for certain nilpotent Lie groups, a proof of the fact that $L^p(G) * \Delta^{1/p'}L^q(G)$ is not contained in the union of the spaces $L^s(G)$, $s \in [1,r) \cup (r,\infty)$. This last result is an extension of the corresponding result of T. S. Quek and L. Y. H. Yap [8] for abelian groups and is in contrast to the Kunze-Stein phenomenon for semi-simple groups [3].

1. **Sharpness and compact open subgroups.** It is easy to see that equality holds in (2) if $f = g = 1_H$ for some compact open subgroup H of G, and hence that $c_{p,q}(G) = 1$ for all p and q if G contains a compact open subgroup. The following two theorems are essentially converses of this observation.

THEOREM 1. *Let p, q, r be three numbers in the interval $(1,\infty)$ satisfying (1). Then there is a constant $c_{p,q} < 1$ such that if G is any locally compact group which does not contain a compact open subgroup then $c_{p,q}(G) \leq c_{p,q}$.*

THEOREM 2. *Let p, q, r be as in Theorem 1, let G be a locally compact group with modular function Δ, and suppose that f and g are functions in $L^p(G)$ and $L^q(G)$, resp., such that*

$$0 < \|f * \Delta^{1/p'}g\|_r = \|f\|_p \|g\|_q.$$

Then there are complex numbers a and b, elements r and s of G, a compact open subgroup H of G, and a continuous function φ from G to $\mathbb{C}$ such that $\varphi = 0$ on $G - H$, $\varphi|_H$ is a homomorphism from H to $\{z \in \mathbb{C} : |z| = 1\}$, and $f = a\varphi(r\cdot)$ a.e. and $g = b\varphi(\cdot s)$ a.e.

As was stated in the preceding section, these two theorems were proven by Fournier [5] under the assumption that G is unimodular. The proofs of Theorem 1 and of [5, Theorem 1] both consist of two parts, with the first part being the reduction of the general case to a special case and the second part being the analysis of this special case. The proofs of these first parts as well as the proofs of Theorem 2 and [5; Theorem 3] are similar, with the major difference in each case being the introduction of various powers of the modular function as factors in a number of expressions. These similarities are such that after comparing the proofs of the first parts of Theorem 1 and [5, Theorem 1] the reader will have no difficulty in modifying the proof of [5, Theorem 3] to give a proof of Theorem 2, and the proof of Theorem 2 is therefore omitted.

PROOF OF THEOREM 1. Let G be a locally compact group with left Haar measure μ and modular function Δ. The first step in the proof is to show that if the theorem is true for some choice of numbers p, q, r satisfying (1) then it is true for all such choices. To see this, let U denote the interior of the triangle in the plane with vertices $(1, 0)$, $(0, 1)$, and $(1, 1)$, let V denote the subset of $(0, \infty)^3$ consisting of those points (p, q, r) satisfying (1) and for which the theorem is true, and assume that V is not empty. Notice that putting $\varphi(p, q, r) = (1/p, 1/q)$ for $(p, q, r) \in V$ defines a one-one function φ from V into U, that proving the theorem is equivalent to showing that φ is onto U and that, in turn, showing that φ is onto U is equivalent to showing that for each point a in the range of φ the horizontal and vertical lines in U through a belong to the range of φ.

Suppose that (p, q, r) is a point in V. Fix two functions f_0 and g_0 in $C_c(G)$ and consider the operator S defined by $S(g) = f_0 * \Delta^{1/p'} g$ for $g \in C_c(G)$ and, for each complex number z with $0 \leq \mathrm{Re}(z) \leq 1$, the operator T_z defined by $T_z(f) = f * \Delta^{z/q} g_0$ for $f \in C_c(G)$. Then

$$\|S(g)\|_p \leq \|f_0\|_p \|g\|_1 \quad \text{and} \quad \|S(g)\|_\infty \leq \|f_0\|_p \|g\|_{p'}$$

and

$$\|T_{iy}(f)\|_q \leq \|f\|_1 \|g_0\|_q \quad \text{and} \quad \|T_{1+iy}(f)\|_\infty \leq \|f\|_{q'} \|g_0\|_q$$

for $y \in \mathbb{R}$ and $f, g \in C_c(G)$ by (2) and it is easy to see that (T_z) is an analytic family of operators of admissible growth in the sense of [10]. Since $(1/p, 1/q)$ belongs to the range of φ it now follows easily from the Riesz-Thorin theorem and Stein's analytic interpolation theorem [10; Theorem 2] that the vertical and the horizontal lines in U through $(1/p, 1/q)$ belongs to the range of φ.

Put $c = 1 - 10^{-10}$. The preceding argument shows that it is sufficient to prove that if f and g are two non-negative-valued functions in $L^{4/3}(G)$ with $\|f\|_{4/3} = \|g\|_{4/3} = 1$ and with $\|f * \Delta^{1/4} g\|_2 > c$ then G contains a compact open subgroup. Accordingly, let f and g be two such functions and put $h = (f^{4/3} * g^{4/3})^{1/2}$. Then $\|h\|_2 = 1$ and

$$\begin{aligned}(f * \Delta^{1/4} g)(x) &= \int_G [f(y) g(y^{-1}x)]^{2/3} [f(y)]^{1/3} [(\Delta^{3/4} g)(y^{-1}x)]^{1/3} \, d\mu(y) \\ &\leq h(x) \|f\|_{4/3}^{1/3} \Big(\int_G \Delta(y^{-1}x) \, g^{4/3}(y^{-1}x) \, d\mu(y) \Big)^{1/4} \\ &= h(x)\end{aligned}$$

by Hölder's inequality with indices 2, 4, 4, and thus $0 \leq f * \Delta^{1/4} g \leq h$. Using the renormalization argument in [5, pp. 389–390], one may assume that $h(e) = 1$ and that $(f * \Delta^{1/4} g)(e) > c$.

Now put $k = (f \Delta^{-3/4} \check{g})^{1/3}$ and $\alpha = \|k\|_2$. Then $\|\Delta^{1/2} k^2\|_2 = 1$,

$$c < (f * \Delta^{1/4} g)(e) = \langle \Delta^{1/2} k^2, k \rangle \leq \|\Delta^{1/2} k^2\|_2 \|k\|_2 = \alpha \leq 1$$

by Hölder's inequality,

$$\begin{aligned}\|\Delta^{1/2} k^2 - \alpha^{-1} k\|_2^2 &= \|\Delta^{1/2} k^2\|_2^2 - 2\alpha^{-1} \langle \Delta^{1/2} k^2, k \rangle + 1 \\ &< 2(1 - c),\end{aligned}$$

and

$$\begin{aligned}\|f^{2/3} - (\Delta^{-3/4} \check{g})^{2/3}\|_2^2 &= \|f^{2/3}\|_2^2 - 2\langle f^{2/3}, (\Delta^{-3/4} \check{g})^{2/3} \rangle + \|(\Delta^{-3/4} \check{g})^{2/3}\|_2^2 \\ &< 2(1 - \alpha^2) \\ &\leq 4(1 - c).\end{aligned}$$

Since the graph of k lies between those of $f^{2/3}$ and $(\Delta^{-3/4}\check{g})^{2/3}$ one must then have

$$\|k - f^{2/3}\|_2^2 \leq 4(1-c) \quad \text{and} \quad \|k - (\Delta^{-3/4}\check{g})^{2/3}\|_2^2 \leq 4(1-c).$$

Now put

$$K = \{x \in G : (\Delta^{1/2}k)(x) > (2\alpha)^{-1}\}$$

and $\gamma = \langle k, \Delta^{-1/2}1_K\rangle \|\Delta^{-1/2}1_K\|_2^{-2}$. Then $\|\gamma\Delta^{-1/2}1_K\|_2 \leq \alpha$ and

$$\begin{aligned}\|k - \gamma\Delta^{-1/2}1_K\|_2^2 &\leq \|k - \alpha^{-1}\Delta^{-1/2}1_K\|_2^2 \\ &= \int_K \Delta^{-1}(\Delta^{1/2}k - \alpha^{-1})^2\, d\mu + \int_{G-K} k^2\, d\mu \\ &\leq 4\alpha^2\|\Delta^{1/2}k^2 - \alpha^{-1}k\|_2^2 \\ &< 8(1-c).\end{aligned}$$

These inequalities imply that

$$\|f^{2/3} - \gamma\Delta^{-1/2}1_K\|_2 \leq 5(1-c)^{1/2}$$

and that

$$\|g^{2/3} - \gamma(1_K)\check{}\|_2 \leq 5(1-c)^{1/2}.$$

Now, just as in [5, p. 390], it follows that

$$\begin{aligned}\|f - \gamma^{3/2}\Delta^{-3/4}1_K\|_{4/3} &\leq \|f^{2/3} - \gamma\Delta^{-1/2}1_K\|_2\|f^{1/3} + \gamma^{1/2}\Delta^{-1/4}1_K\|_4 \\ &\leq 10(1-c)^{1/2}\end{aligned}$$

and, similarly, that

$$\|g - \gamma^{3/2}(1_K)\check{}\|_{4/3} \leq 10(1-c)^{1/2}.$$

Now put $f_1 = \gamma^{3/2}\Delta^{-3/4}1_K$ and $g_1 = \gamma^{3/2}(1_K)\check{}$. Then $\|f_1\|_{4/3} \leq 1$ and $\|g_1\|_{4/3} \leq 1$, hence

$$\|f_1 * \Delta^{1/4}g_1 - f * \Delta^{1/4}g\|_2 \leq 20(1-c)^{1/2}$$

by Young's inequality, and therefore

$$\|f_1 * \Delta^{1/4}g_1\|_2 > c - 20(1-c)^{1/2}.$$

If one now carries out the renormalization of μ described in [5, bottom of p. 390] and replaces K by K^{-1} one may assume that $f = (\Delta^{3/4}1_K)\check{}$ and $g = 1_K$ and that

$$\|f\|_{4/3} = \|g\|_{4/3} = 1 \quad \text{and} \quad \|f * \Delta^{1/4}g\|_2 > d,$$

where $d = c - 20(1-c)^{1/2}$.

Notice that $\mu(K) = 1$. If one puts $h = (f^{4/3} * g^{4/3})^{1/2}$ and $\omega = \Delta^{1/4}(f * \Delta^{1/4}g)$ then $\|h\|_2 = 1$, $0 \leq f * \Delta^{1/4}g \leq h$, $h^2(x) = \mu(K \cap Kx^{-1})$ for $x \in G$, $f * \Delta^{1/4}g = \Delta^{1/4}h^2$, $(f * \Delta^{1/4}g)^2 = \omega h^2$, and

$$\check{\omega} = \omega = \Delta^{1/2}h^2 \leq \min\{\Delta^{1/2}, \Delta^{-1/2}\} \leq 1.$$

Now put $\alpha = .05$ and

$$H_a = \{x \in G : \omega(x) > a \text{ and } a^{2\alpha} < \Delta(x) < a^{-2\alpha}\}$$

for $0 < a < 1$. Then $e \in H_a = (H_a)^{-1}$ and H_a is an open subset of G with $\mu(H_a) < \infty$. So to prove the theorem it is sufficient to find a number a with $0 < a < 1$ and $(H_a)^2 \subseteq H_a$.

Consider two numbers a and b satisfying $0 < a < b < 1$. One has $\omega \le b^\alpha$ on the set where $\omega \le b$ as well as on the set where $\Delta \notin (b^{2\alpha}, b^{-2\alpha})$, hence

$$d^2 < \int_G \omega h^2 \, d\mu \le \int_{H_b} h^2 \, d\mu + b^\alpha \int_{G-H_b} h^2 \, d\mu,$$

and therefore

$$\mu(H_b) \ge \int_{H_b} h^2 \, d\mu > \frac{d^2 - b^\alpha}{1 - b^\alpha}.$$

On the other hand, one has $a < \omega < a^{-\alpha} h^2$ on the set H_a, and hence

$$\begin{aligned}\mu(H_a - H_b) &\le a^{-1-\alpha} \int_{H_a - H_b} h^2 \, d\mu \\ &\le a^{-1-\alpha}\Big[1 - \frac{d^2 - b^\alpha}{1 - b^\alpha}\Big] \\ &= \frac{1 - d^2}{(1 - b^\alpha) a^{1+a}}.\end{aligned}$$

It now follows easily from these two inequalities that if $xH_b \subseteq H_a - H_b$ for some element $x \in G$ then

$$d^2 \le \frac{1 + a^{1+\alpha} b^\alpha}{1 + a^{1+\alpha}}.$$

In view of the values of c and α, this inequality cannot hold for $a = .1$ and $b = .9$, and therefore no element x of G can satisfy $xH_{.9} \subseteq H_{.1} - H_{.9}$. So if one could show that

$$\big((H_{.7})^2 - H_{.7}\big) H_{.9} \subseteq H_{.1} - H_{.9} \tag{4}$$

then it would follow that $H_{.7}$ is the required compact open subgroup of G.

The proof of (4) will depend on two inequalities satisfied by the function ω. In fact, one has

$$\omega(xy) \ge \Delta^{1/2}(y)\, \omega(x) + \Delta^{-1/2}(x)\, \omega(y) - \Delta^{1/2}(x^{-1} y) \tag{5}$$

and

$$\omega(xy) \le \Delta^{1/2}(y)\, \omega(x) - \Delta^{-1/2}(x)\, \omega(y) + \Delta^{-1/2}(xy) \tag{6}$$

for all elements x and y in G, and a moderately lengthy by straightforward calculation using (5) and (6) and the multiplicativity of Δ will show that must (4) hold. Now (6) is easily seen to be a consequence of (5), and the proof of (5) goes as follows:

$$\begin{aligned}\omega(xy) &= \Delta^{1/2}(xy)\mu(K \cap K y^{-1} x^{-1}) \\ &= \Delta^{1/2}(xy)[1 - \mu(K - K y^{-1} x^{-1})] \\ &\ge \Delta^{1/2}(xy)[1 - \mu(K - K x^{-1}) - \mu(K x^{-1} - K y^{-1} x^{-1})] \\ &= \Delta^{1/2}(xy)\big[\mu(K \cap K x^{-1}) - \Delta^{-1}(x)[1 - \mu(K \cap K y^{-1})]\big] \\ &= \Delta^{1/2}(y)\, \omega(x) + \Delta^{-1/2}(x)\, \omega(y) - \Delta^{1/2}(x^{-1} y).\end{aligned}$$

2. **Sharpness for semi-direct products.** Let p, q, r be three numbers in the interval $(1, \infty)$ satisfying (1) and consider a locally group G which is the semi-direct product of a closed normal subgroup N and a closed subgroup H. Then Klein and Russo have shown that

$$c_{p,q}(G) \leq c_{p,q}(N)\, c_{p,q}(H) \tag{7}$$

[7; Lemma 2.4]. It is not hard to see that (7) is actually an equality if G is the product of N and H, and Klein and Russo have conjectured that (7) is always an equality. To the best of the author's knowledge, this conjecture has only been verified under the assumption that G is a Heisenberg group and that p' is an even integer [7; p. 185].

Quek and Yap have shown that if G is an abelian group which is neither compact nor discrete then the set of convolution products $L^p(G) * L^q(G)$ is not contained in the union of the spaces $L^s(G)$, $s \in [1, r) \cup (r, \infty)$ [8; Theorem 1.1] (see also [9; Corollary 2.5]). The corollary to the next theorem extends this result to connected and simply connected nilpotent Lie groups and contains a weaker result for solvable Lie groups. This result of Quek and Yap cannot be extended to arbitrary groups since it is know that a large class of semi-simple Lie groups satisfy the Kunze-Stein phenomena: $L^2(G) * L^p(G) \subseteq L^2(G)$ for $1 \leq p \leq 2$. An interesting open question is that of characterizing those groups for which $L^p(G) * \Delta^{1/p'} L^q(G)$ fails to be contained in the union of the spaces $L^s(G)$, $s \in [1, r) \cup (r, \infty)$.

THEOREM 3. *Let p, q, r be three numbers in the interval $(1, \infty)$ satisfying (1) and let G be a locally compact group which is the semi-direct product of a closed normal subgroup N and a closed group H which is isomorphic to $\mathbb{R}^m \times \mathbb{T}^n$ for some integers m and n. Then*

(a) *if $n = 0$ then $c_{p,q}(G) = c_{p,q}(N)\, c_{p,q}(H)$,*
(b) *$L^p(G) * \Delta^{1/p'} L^q(G) \not\subseteq \bigcup_{r<s<\infty} L^s(G)$, and*
(c) *if $m \geq 1$ and if there is a symmetric measurable set A in N, a $b > 0$, and a $c \in \mathbb{R}^m$ such that $\lambda(A) > 0$ and $\lambda\big(\alpha_{(u,v)}(A) A\big) \leq b e^{|\langle c, u \rangle|}$ for all $(u, v) \in \mathbb{R}^m \times \mathbb{T}^n$, where λ is a Haar measure on N and α_w is, for each $w \in H$, the restriction to N of conjugation by w, then $L^p(G) * \Delta^{1/p'} L^q(G) \not\subseteq \bigcup_{\substack{1 \leq s < \infty \\ s \neq r}} L^s(G)$.*

Put $A_p = \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}$ for each number $p \in (1, \infty)$ and recall that Beckner has proven that if p, q, r are three numbers in the interval $(1, \infty)$ satisfying (1) then $c_{p,q}(\mathbb{R}^m) = (A_p A_q A_{r'})^m$ for $m \geq 1$ [1; Theorem 3] (see also [2; Section 5]).

COROLLARY. *If p, q, r are as in Theorem 3 and if G is a Lie group then*

(a) *$c_{p,q}(G) = (A_p A_q A_{r'})^{\dim(G)}$ if G is simply-connected and solvable,*
(b) *$c_{p,q}(G) = (A_p A_q A_{r'})^{\dim(G) - rk(\Gamma)}$ if G is nilpotent, where Γ is the discrete central subgroup of the universal covering group $\tilde{G}$ of the connected component G_0 of G for which $\tilde{G}/\Gamma$ is isomorphic and homeomorphic to G_0 and $rk(\Gamma)$ is the rank of Γ,*
(c) *$L^p(G) * \Delta^{1/p'} L^q(G) \not\subseteq \bigcup_{r<s<\infty} L^s(G)$ if G is either nilpotent or else simply connected and solvable, and*

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(d) $L^p(G) * \Delta^{1/p'} L^q(G) \not\subseteq \bigcup_{\substack{1 \le s < \infty \\ s \ne r}} L^s(G)$ *if G is nilpotent and not compact.*

PROOF. Notice that if the connected component G_0 of G satisfies the conditions of the corollary then so does G itself, and hence one may assume that G is connected. Notice also that part (a) and the second case of part (c) follow easily from the structure of simply-connected solvable Lie groups [11; Theorem 3.18.6], from the part (a) of the theorem, and from the theorem of Beckner just quoted. Suppose that G is abelian. Then one may take G to be $\mathbb{R}^m \times \mathbb{T}^n$ for some non-negative integers m and n, and hence G satisfies (b) by Beckner's theorem and the above remarks about product groups and satisfies (d) and the first case of (c) by [8; Theorem 1.1].

Now suppose that G is nilpotent and not abelian. Let $\tilde{G}$ and Γ be as in the statement of (b), let π denote the mapping of $\tilde{G}$ onto G obtained by composing the quotient mapping of $\tilde{G}$ onto $\tilde{G}/\Gamma$ with the isomorphism of $\tilde{G}/\Gamma$ onto G, and let $\mathfrak{z}$ denote the center of the Lie algebra $\mathfrak{g}$ of G. Then $[\mathfrak{g},\mathfrak{g}]+\mathfrak{z} \neq \mathfrak{g}$, for otherwise one would have $\big[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]\big] = [\mathfrak{g},\mathfrak{g}]$, a contradiction. This means that there is an ideal $\mathfrak{n}$ in $\mathfrak{g}$ which is of co-dimension 1 and contains $\mathfrak{z}$. Put $k = \mathrm{rk}(\Gamma)$, $m = \dim(\mathfrak{z})$ and $n = \dim(\mathfrak{n})$, let T be an element of $\mathfrak{g}$ which is not in $\mathfrak{n}$, let $\tilde{Z}$ be the center of $\tilde{G}$, and let $\tilde{N}$ and $\tilde{H}$ be the connected subgroups of $\tilde{G}$ corresponding to $\mathfrak{n}$ and $\mathbb{R}T$, respectively. Finally, let $X_1, \ldots, X_n$, T be a Jordan-Hölder basis for $\mathfrak{g}$ (meaning that $\mathfrak{n}_k = \mathbb{R}X_1 + \cdots + \mathbb{R}X_k$ is an ideal in $\mathfrak{g}$ for $1 \le k \le n$) such that $X_1, \ldots, X_k$ a $\mathbb{Z}$-basis for Γ and $\mathfrak{z} = \mathfrak{n}_m$. Then exp is a diffeomorphism of $\mathfrak{g}$ onto $\tilde{G}$ which carries $\mathfrak{z}$, $\mathfrak{n}$, and $\mathbb{R}T$ onto $\tilde{Z}$, $\tilde{N}$, and $\tilde{H}$, resptively. Thus $\tilde{N}$, $\tilde{H}$, and $\tilde{Z}$ are all closed in $\tilde{G}$, $\tilde{Z}$ is the connected subgroup of $\tilde{G}$ corresponding to $\mathfrak{z}$, $\tilde{H}$ is isomorphic to $\mathbb{R}$, and $\tilde{G}$ is the semi-direct product of $\tilde{N}$ and $\tilde{H}$ [11; Corollary 3.6.4 and Lemma 3.18.4]. Then $\pi(\tilde{H})$ is isomorphic and homeomorphic to $\mathbb{R}$ and closed in G since $\Gamma \subseteq \tilde{N}$, $\pi(\tilde{N})$ is closed and normal in G since $G/\pi(\tilde{N})$ is homeomorphic to $\mathbb{R}$ [6; Theorems 5.21 and 5.34], and G is isomorphic to the semi-direct product of $\pi(\tilde{N})$ and $\pi(\tilde{H})$. The first case in (c) now follows from the theorem while an inductive argument on $\dim(G)$ together with the theorem will complete the proof of (b).

Turning to the proof of (d), let

$$F = \{\exp(t_1X_1 + \cdots + t_nX_n) : 0 \le t_j < 1 \text{ for } 1 \le j \le k \text{ and } t_j \in \mathbb{R} \text{ for } k < j \le n\}$$

and put

$$\tilde{A}_a = \{\exp(t_1X_1 + \cdots + t_nX_n) : |t_j| \le a \text{ for each } j\}$$

and $A_a = \pi(\tilde{A}_a)$ for each $a > 0$. The measure $\tilde{\lambda}$ on $\tilde{G}$ which corresponds to Lebesgue measure on $\mathfrak{g}$ under exp is a Haar measure on $\tilde{G}$ and the measure λ on G defined by

$$\lambda(A) = \tilde{\lambda}\big(F \cap \pi^{-1}(A)\big)$$

for Borel subsets A of G is a Haar measure on G. Let $\tilde{\alpha}_u$ and α_u be conjugation by $\exp(uT)$ and $\pi\big(\exp(uT)\big)$ on $\tilde{G}$ and G, resptively. Recall that as $\mathfrak{g}$ is nilpotent the multiplication on $\tilde{G}$ corresponds under exp to a polynomial map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ which, relative to the Jordan-Hölder basis, is additive in the first k coordinates [11; Theorems 3.6.1 and 3.6.2].

This implies that there is a polynomial f such that $\tilde{\alpha}_u(\tilde{A}_1)\tilde{A}_1 \subseteq \tilde{A}_{\max\{1,f(u)\}}$ for all $u \in \mathbb{R}$, and therefore $\tilde{\lambda}\big(\tilde{\alpha}_u(A_1)\tilde{A}_1\big) \leq be^{|cu|}$ for some $b > 0$, some $c \in \mathbb{R}$, and all $u \in \mathbb{R}$. But then

$$\begin{aligned}\lambda\big(\alpha_u(A_1)A_1\big) &= \tilde{\lambda}\Big(F \cap \pi^{-1}\big(\alpha_u(A_1)A_1\big)\Big)\\ &\leq \tilde{\lambda}\big(F \cap \tilde{\alpha}_u(\tilde{A}_1)\tilde{A}_1\big)\\ &\leq be^{|cu|},\end{aligned}$$

and this completes the proof of (d).

PROOF OF THEOREM 3. The first two steps will be to introduce and calculate the norms of certain functions on G and to reduce the proof of parts (a) and (b) to the case where H is either $\mathbb{R}$ or $\mathbb{T}$. It will be convenient to take $\mathbb{T} = [-1/2, 1/2)$ with addition modulo 1 and to let du denote both Lebesgue measure on $\mathbb{R}$ and normalized Lebesgue measure on $\mathbb{T}$.

Suppose, for the moment, that G is a locally compact group which is the semi-direct product of a closed normal subgroup N and a closed abelian subgroup H. Let δ and λ [resp., Δ and μ] be the modular function and left Haar measure on N [resp., G]. Put $\alpha_u(x) = uxu^{-1}$ for $x \in N$ and $u \in H$ and let ϑ be the continuous homomorphism from H into $(0, \infty)$ such that

$$\int_N f\big(\alpha_u(x)\big)\, d\lambda(x) = \vartheta(u) \int_N f(x)\, d\lambda(x)$$

for all $u \in H$ and all $f \in C_c(G)$. Then one may take G to be the set $N \times H$ with the group multiplication

$$(x, u)(y, v) = \big(x\alpha_u(y), u + v\big)$$

and it is easy to verify that $\Delta = \delta \otimes \vartheta$ and that one may take $d\mu(x, u) = \vartheta(u)\, d\lambda(x)\, du$, where du is Haar measure on H.

For any function φ on N and any function f on H define a function $\varphi \,\square f$ on G by the formula

$$(\varphi \,\square f)(x, u) = \varphi\big(\alpha_{-u}(x)\big)f(u).$$

Now consider two functions φ and ψ on N and two functions f and g on H, and assume that each of these four functions is bounded and integrable. Then it is easy to verify that

$$\|\varphi \otimes \vartheta^{-1/p} f\|_p = \|\varphi\|_p \|f\|_p \quad \text{and} \quad \|\psi \,\square g\|_q = \|\psi\|_q \|g\|_q,$$

that

$$\begin{aligned}&[(\varphi \otimes \vartheta^{-1/p} f) * \Delta^{1/p'}(\psi \,\square g)](x, u)\\ &= \int_H \int_N (\varphi \otimes \vartheta^{-1/p} f)\big(x\alpha_u(y), u + v\big)\Delta(y, v)^{-1/p}(\psi \,\square g)\big(\alpha_{-v}(y^{-1}), -v\big)\vartheta(v)\, d\lambda(y)\, dv\\ &= \vartheta^{-1/p}(u) \int_H \int_N \varphi\big(x\alpha_u(y)\big)\delta^{1/p'}(y^{-1})\psi(y^{-1})f(u + v)g(-v)\, d\lambda(y)\, dv\\ &= \vartheta^{-1/p}(y)[(\varphi \circ \alpha_u) * \delta^{1/p'}\psi]\big(\alpha_{-u}(x)\big)(f * g)(u),\end{aligned}$$

and hence that

$$\|(\varphi \otimes \vartheta^{-1/p} f) * \Delta^{1/p'}(\psi \Box g)\|_s^s$$
$$= \int_H \int_N \left|[(\varphi \circ \alpha_u) * \delta^{1/p'}\psi]\big(\alpha_{-u}(x)\big)\right|^s |(f * g)(u)|^s \vartheta^{1-s/p}(u)\, d\lambda(x)\, du$$
$$= \int_H \|(\varphi \circ \alpha_u) * \delta^{1/p'}\psi\|_s^s\, |(f * g)(u)|^s \vartheta^{-s/p}(u)\, du$$

for $1 \leq s < \infty$.

Now suppose that H itself is the product of two closed normal subgroups H_1 and H_2. Then NH_1 is closed and normal in G (by [6; Theorem 5.21] and the fact that G/NH_1 is homeomorphic to H_2) and G is the semi-direct product of NH_1 and H_2. This observation and Beckner's theorem (see above) implies that to prove (a) it is enough to consider the case in which H is $\mathbb{R}$ and to prove (b), the case in which H is either $\mathbb{R}$ or $\mathbb{T}$.

Suppose that $H = \mathbb{R}$ and that φ and ψ are two bounded and integrable functions on N with $\|\varphi\|_p = \|\psi\|_q = 1$. If t is the number determined by the equation $p^{-1} + t^{-1} = 1 + s^{-1}$ then $1 \leq t < \infty$ and hence Young's inequality and a standard argument will show that $\|(\varphi \circ \alpha_u) * \delta^{1/p'}\psi\|_s \vartheta^{-1/p}(u)$ is a continuous function of u bounded by $\|\psi\|_t$. Put $f_a(u) = \exp(-au^2)$ for $a > 0$ and $u \in \mathbb{R}$. Then the above calculation together with the lemma below and Beckner's theorem implies that the expression

$$\lim_{a\to\infty} \frac{\|(\varphi \otimes \vartheta^{-1/p} f_{ap'}) * \Delta^{1/p'}(\psi \Box f_{aq'})\|_s^s}{\|\varphi \otimes \vartheta^{-1/p} f_{ap'}\|_p^s\, \|\psi \Box f_{aq'}\|_q^s}$$
$$= \lim_{a\to\infty} \frac{1}{\|f_{ap'}\|_p^s\, \|g_{aq'}\|_q^s} \int_H \|(\varphi \circ \alpha_u) * \delta^{1/p'}\psi\|_s^s\, |(f_{ap'} * f_{aq'})(u)|^s \vartheta^{-s/p}(u)\, du$$
$$= \lim_{a\to\infty} \left(\frac{p^{1/p} q^{1/q} r'}{p'^{1/p'} q^{1/q'}}\right)^{s/2} \left(\frac{a}{\pi}\right)^{s/2r} \int_H \|(\varphi \circ \alpha_u) * \delta^{1/p'}\psi\|_s^s \vartheta^{-s/p}(u) \exp(-ar'su^2)\, du$$

is equal to

$$\|\varphi * \delta^{1/p'}\psi\|_r^r (A_p A_q A_{r'})^r = \|\varphi * \delta^{1/p'}\psi\|_r^r c_{p,q}(\mathbb{R})^r$$

if $s = r$ and to ∞ if $s > r$ and $\|\varphi * \delta^{1/p'}\psi\|_s > 0$. This completes the proof of (a) in the case $H = \mathbb{R}$, and the proof of (b) in this same case will follow from two known arguments, one due to Zelazko and the other to Fournier.

First of all, if $L^p(G) * \Delta^{1/p'} L^q(G) \subseteq L^s(G)$ for some $s > r$ then $\|f * \Delta^{1/p'} g\|_s \leq c\|f\|_p \|g\|_q$ for all $f \in L^p(G)$, all $g \in L^q(G)$, and some $c > 0$ (*cf.* [12; Lemma 4]). But the preceding calculations show that this is not the case for any $s > r$, and thus $L^p(G) * \Delta^{1/p'} L^q(G) \not\subseteq L^s(G)$ for each $s > r$. The Baire category theorem then implies that the set

$$V = \{(f, g) \in L^p(G) \times L^q(G) : f * \Delta^{1/p'} g \notin L^{r+1/n}(G) \text{ for } n = 1, 2, \ldots\}$$

is a dense G_δ subset of $L^p(G) \times L^q(G)$ (*cf.* the proof of the Banach-Steinhaus Theorem). Now suppose that (f, g) is an element of V with the property that $f * \Delta^{1/p'} g \in L^s(G)$ for some $s > r$. Then $f * \Delta^{1/p'} g \in L^r(G)$ by Young's inequality and hence $f * \Delta^{1/p'} g \in$

$L^{r+1/n}(G)$ for some n by a well-known convexity property of the norm. This shows that the set

$$\{f * \Delta^{1/p'} g : (f, g) \in V\}$$

is disjoint from the union $\bigcup_{r<s<\infty} L^s(G)$ and completes the proof of (b) in the case $H = \mathbb{R}$ (*cf.* [4; p. 268]).

Now suppose that $s > r$ and that $H = \mathbb{T}$, and let φ and ψ continue to denote two bounded integrable functions on N with $\|\varphi\|_p = \|\psi\|_q = 1$. Notice that $\vartheta(u) = 1$ for all $u \in \mathbb{T}$. For $0 < a < 1/4$ put $g_a(u) = 1_{[-a,a]}(u)$ for $u \in \mathbb{T}$. Then

$$\frac{\|(\varphi \otimes g_a) * \Delta^{1/p'}(\psi \,\square\, g_a)\|_s^s}{\|\varphi \otimes g_a\|_p^s \, \|\psi \,\square\, g_a\|_q^s} = (2a)^{-s-s/r} \int_{\mathbb{T}} \|(\varphi \circ \alpha_u) * \delta^{1/p'} \psi\|_s^s \, |(g_a * g_a)(u)|^s \, du.$$

Now since $\|(\varphi \circ \alpha_u) * \delta^{1/p'} \psi\|_s$ is continuous and bounded, since $g_a * g_a = 0$ off the interval $[-2a, 2a]$, and since $\|g_a * g_a\|_s^s = \frac{2(2a)^{s+1}}{s+1}$ it follows easily that

$$\lim_{a \to 0} \frac{\|(\varphi \otimes g_a) * \Delta^{1/p'}(\psi \,\square\, g_a)\|_s^s}{\|\varphi \otimes g_a\|_p^s \, \|\psi \,\square\, g_a\|_q^s} = \lim_{a \to 0} 2^{2-s/r} a^{1-s/r} \|\varphi * \delta^{1/p'} \psi\|_s^s = \infty.$$

Now, just as in the previous case, one can use the arguments of Zelazko and Fournier to complete the proof of (b) in the case $H = \mathbb{T}$.

Turning now to the proof of (c), let $H = \mathbb{R}^m \times \mathbb{T}^n$ for some $m \geq 1$ and $n \geq 0$. One may as well assume that $\lambda(A) = 1$. Fix an $s \in [1, r)$ and put $\varphi = 1_A$ and $\psi = \delta^{-1/p'} 1_A$ and for each $u \in \mathbb{R}^m$ and $v \in \mathbb{T}^n$ put

$$\varepsilon_{u,v} = \frac{s\vartheta(u, v)}{(s+1)\lambda\big(\alpha_{-(u,v)}(A)A\big)}$$

and

$$S_{u,v} = \{x \in N : [(1_A \circ \alpha_{(u,v)}) * 1_A](x) > \varepsilon_{u,v}\}.$$

Then $(1_A \circ \alpha_{(u,v)}) * 1_A = 0$ off $\alpha_{-(u,v)}(A)A$ and

$$\begin{aligned} \vartheta(u, v) &= \int_{S_{u,v}} [(1_A \circ \alpha_{(u,v)}) * 1_A](x) \, d\lambda(x) + \int_{G-S_{u,v}} [(1_A \circ \alpha_{(u,v)}) * 1_A](x) \, d\lambda(x) \\ &\leq \vartheta(u, v)\lambda(S_{u,v}) + \varepsilon_{u,v}\lambda\big(\alpha_{-(u,v)}(A)A\big), \end{aligned}$$

and thus

$$\begin{aligned} \|(1_A \circ \alpha_{(u,v)}) * 1_A\|_s^s &\geq \int_{S_{u,v}} |[(1_A \circ \alpha_{(u,v)}) * 1_A](x)|^s \, d\lambda(x) \\ &\geq (\varepsilon_{u,v})^s \Big(1 - \varepsilon_{u,v}\vartheta(u, v)^{-1}\lambda\big(\alpha_{-(u,v)}(A)A\big)\Big) \\ &= \frac{s^s \vartheta(u, v)}{(s+1)^{s+1}\lambda\big(\alpha_{-(u,v)}(A)A\big)^s} \\ &\geq b_1 \vartheta(u, v)^s e^{-s|\langle c,u\rangle|} \end{aligned}$$

for some constant $b_1 > 0$ and all $u \in \mathbb{R}^m$ and $v \in \mathbb{T}^n$. Now put $f_a(u,v) = \exp(-a\|u\|^2)$ for $a > 0$, $u \in \mathbb{R}^m$, and $v \in \mathbb{T}^n$ (here $\|\cdot\|$ is the usual 2-norm on $\mathbb{R}^m$). There will be a point $\omega \in \mathbb{R}^m$ such that $\vartheta(u,v) = e^{\langle\omega,u\rangle}$ for all $u \in \mathbb{R}^m$ and $v \in \mathbb{T}^n$. The preceding calculations and the fact that $\inf_u \exp(ar's\|u\|^2 - s|\langle c,u\rangle|) > 0$, where the infimum is over $u \in \mathbb{R}^m$, then implies that

$$\begin{aligned}
&\frac{\|(\varphi \otimes \vartheta^{-1/p} f_{ap'}) * \Delta^{1/p'}(\psi \,\Box\, f_{aq'})\|_s^s}{\|\varphi \otimes \vartheta^{-1/p} f_{ap'}\|_p^s \,\|\psi \,\Box\, f_{aq'}\|_q^s} \\
&\quad= \frac{1}{\|f_{ap'}\|_p^s \,\|f_{aq'}\|_q^s \,\|\psi\|_q^s} \int_H \|(\varphi \circ \alpha_w) * \delta^{1/p'}\psi\|_s^s \,(f_{ap'} * f_{aq'})(w)|^s \vartheta^{-s/p}(w)\,dw \\
&\quad\geq b_2 a^{ms/2r} \int_{\mathbb{R}^m} \exp\big(-2ar's\|u\|^2 + (s/p')\langle\omega,u\rangle\big)\,du \\
&\quad= b_3 a^{(m/2)(s/r-1)} \exp\left(\frac{s\|\omega\|^2}{8ar'p'^2}\right)
\end{aligned}$$

for some positive constants b_2 and b_3 and all $a > 0$, and this last expression clearly tends to ∞ as $a \longrightarrow 0+$.

One can now, just as in the proof of part (b), deduce that there is a dense G_δ subset W of $L^p(G) \times L^q(G)$ such that the set

$$\{f * \Delta^{1/p'} g : (f,g) \in W\}$$

is disjoint from the union $\bigcup_{1 \leq s < r} L^s(G)$, and this clearly completes the proof of (c).

The formulations of Theorem 3 and its Corollary are not completely satisfactory. One would obviously like to prove part (a) of the theorem without the hypothesis that $n = 0$ and part (c) without the hypothesis involving the set A (Notice that in (c) one cannot avoid the assumption that $m \geq 1$). If the first of these desiderata could be achieved then part (b) of the corollary would be true with nilpotent replaced by solvable (thus eliminating the need for (a)). Now it is easy to see that part (a) of the theorem is true if, for each $\varepsilon > 0$, there exist bounded integrable functions φ and ψ on N with $\|\varphi\|_p = \|\psi\|_q = 1$, with $\|\varphi * \delta^{1/p'}\psi\|_r > c_{p,q}(N) - \varepsilon$, and with $\varphi \circ \alpha_u = \varphi$ for all $u \in \mathbb{T}$. This condition is satisfied in a number of examples, and the author does not know if it is always the case.

LEMMA. *If f is a continuous bounded function on $\mathbb{R}$ then*

$$\lim_{a \longrightarrow \infty} a^{1/2} \int_{-\infty}^{\infty} f(u) e^{-abu^2}\,du = f(0)(\pi/b)^{1/2}$$

for any $b > 0$.

PROOF. Suppose that an $\varepsilon > 0$ is given. Then there will be a $\delta > 0$ such that $|f(u) - f(0)| < \varepsilon$ whenever $|u| < \delta$, and hence

$$\begin{aligned}
\left|a^{1/2} \int_{-\infty}^{\infty} f(u) e^{-abu^2}\,du - f(0)(\pi/b)^{1/2}\right| &\leq a^{1/2} \int_{-\infty}^{\infty} |f(u) - f(0)| e^{-abu^2}\,du \\
&\leq a^{1/2} \int_{|u| \geq \delta} |f(u) - f(0)| e^{-abu^2}\,du + \varepsilon(\pi/b)^{1/2}.
\end{aligned}$$

But since the last integral tends to zero as a tends to infinity by the Lebesgue dominated convergence theorem the Lemma is clear.

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