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A COMBINATORIAL INTERPRETATION OF THE WREATH PRODUCT OF SCHUR FUNCTIONS

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1. Introduction. A combinatorial interpretation of Schur functions in terms of Young tableaux is well-known. (For example, see Littlewood [**1**] or Thomas [**4**]). The purpose of this paper is to present a combinatorial interpretation of the *wreath product* (or *plethysm*) of two Schur functions.

Read [**3**] has described a wreath product as analogous to a process of substitution. The main result of this paper shows clearly that, combinatorially speaking, a wreath product is very much a substitution process. The notation used is taken from Read [**3**].

2. Definitions and notation. Let $x_1, x_2, \ldots$ be an infinite set of indeterminates. We define the *symmetric power sums* of these indeterminates by

$$s_r = \sum_{i=1}^{\infty} x_i^{\,r} \quad \text{for } r = 1, 2, \ldots .$$

Let $(\rho) = (1^{\rho_1}, 2^{\rho_2}, \ldots, n^{\rho_n})$ be a partition of the integer n. We now define

$$s_\rho = s_1^{\,\rho_1} s_2^{\,\rho_2} \ldots s_n^{\,\rho_n}.$$

In addition, define

$$g_\rho = \frac{n!}{1^{\rho_1}\rho_1!\, 2^{\rho_2}\rho_2! \ldots n^{\rho_n}\rho_n!},$$

that is, g_ρ is the number of elements in the conjugacy class (ρ) of the symmetric group $\mathscr{S}_n$ of degree n.

Finally, for each partition (λ) of n, we define the *Schur function*

$$\{\lambda\} = \frac{1}{n!} \sum_{\rho} \chi_\rho^{\lambda} g_\rho s_\rho$$

where the summation is over all partitions (ρ) of n.

The coefficient χ_ρ^{λ} is the characteristic of the conjugacy class (ρ) in the irreducible representation (λ) of $\mathscr{S}_n$.

3. Young tableaux. Another interpretation of Schur functions is in terms of Young tableaux. Given a partition (λ) of n, we define the *frame of* (λ) as

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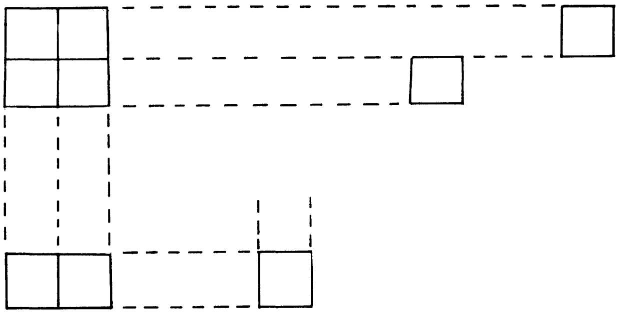
a pattern of unit squares or "boxes" as shown below

(1)



where the first row contains λ_1 squares, the second row, λ_2 squares, etc., and the rows are aligned on the left hand side. We denote the frame of (λ) by $F(\lambda)$.

We now *number* the squares of $F(\lambda)$ by placing an indeterminate x_i in each square such that in each row, the suffixes are in *non-decreasing* order from left to right, and in each column, the suffixes are in *strictly increasing* order from top to bottom. A frame $F(\lambda)$ with such a numbering will be called a *Young tableau of* (λ).

Example. $(\lambda) = (1, 2, 4^2)$. An example of a Young tableau of (λ) would be

x_1	x_1	x_1	x_5
x_2	x_3	x_3	x_7
x_6	x_7		
x_7			

Given a Young tableau, Y^λ of (λ), we associate a monomial

$$M(Y^\lambda) = x_1^{t(1)}x_2^{t(2)} \ldots$$

where, for $i = 1, 2, \ldots,$ the indeterminate x_i appears $t(i)$ times in Y^λ. For example, in the Young tableau in (1) above,

$$M(Y^\lambda) = x_1^3x_2x_3^2x_5x_6x_7^3.$$

We can now say

$$(2) \quad \{\lambda\} = \sum_{Y^\lambda} M(Y^\lambda)$$

where the summation is over all Young tableaux Y^λ of (λ).

Let D_λ denote the set of all Young tableaux of (λ). D_λ is countable since it is a subset of a finite product of countable sets. Hence, the summation in (2)

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is sensible. Also, because D_λ is countable, we may totally order the elements of D_λ. Therefore, we may write all the Young tableaux of (λ) in a sequence $Y_1^\lambda, Y_2^\lambda, Y_3^\lambda, \ldots$ and hence we may write

$$\{\lambda\} = \sum_{r=1}^{\infty} M(Y_r^\lambda)$$

4. Wreath products. Let (λ) and (μ) be partitions of n and m respectively, and consider the Schur functions

$$\{\lambda\} = \frac{1}{n!} \sum_\rho \chi_\rho^\lambda g_\rho s_\rho$$

and

$$\{\mu\} = \frac{1}{m!} \sum_\nu \chi_\nu^\mu g_\nu s_\nu = \frac{1}{m!} \sum_\nu \chi_\nu^\mu g_\nu s_1^{\nu_1} s_2^{\nu_2} \ldots s_m^{\nu_m}.$$

We form the *wreath product* $\{\lambda\}[\{\mu\}]$ as follows. Firstly, define the functions

(3) $$S_r = \frac{1}{m!} \sum_\nu \chi_\nu^\mu g_\nu s_r^{\nu_r} s_{2r}^{\nu_2} \ldots s_{rm}^{\nu_m} \quad \text{for } r = 1, 2, \ldots.$$

(i.e. to form S_r, multiply the suffixes of the s_i's in $\{\mu\}$ by r).

Now define

$$\{\lambda\}[\{\mu\}] = \frac{1}{n!} \sum_\rho \chi_\rho^\lambda g_\rho S_\rho$$

where $S_\rho = S_1^{\rho_1} S_2^{\rho_2} \ldots S_n^{\rho_n}$ as before.

This process Read [**3**] refers to as substituting $\{\mu\}$ into $\{\lambda\}$.

Example. $\{\lambda\} = \frac{1}{2}(s_1^2 + s_2)$, $\{\mu\} = \frac{1}{3}(s_1^3 - s_3)$. The substitution is effected by replacing s_1 by $\frac{1}{3}(s_1^3 - s_3)$ and s_2 by $\frac{1}{3}(s_2^3 - s_6)$ in $\{\lambda\}$. Thus

$$\{\lambda\}[\{\mu\}] = \frac{1}{2}\left(\frac{1}{9}(s_1^3 - s_3)^2 + \frac{1}{3}(s_2^3 - s_6)\right)$$

$$= \frac{1}{18}(s_1^6 - 2s_1^3 s_3 + s_3^2 + 3s_2^3 - 3s_6).$$

The wreath product $\{\lambda\}[\{\mu\}]$ is sometimes written $\{\mu\} \otimes \{\lambda\}$ and is termed a *plethysm*. Read [**3**] points out that although these two operations have different origins, they are in fact the same.

5. Theorem. *Suppose*

$$\{\mu\} = \sum_{Y^\mu} M(Y^\mu) = \sum_{r=1}^{\infty} M(Y_r^\mu)$$

and suppose

$$\{\lambda\} = \sum_{Y^\lambda} M(Y^\lambda) = \sum_{Y^\lambda} x_1^{t(1)} x_2^{t(2)} \ldots.$$

Then

$$\{\lambda\}[\{\mu\}] = \sum_{Y^\lambda} M(Y_1^\mu)^{t(1)} M(Y_2^\mu)^{t(2)} \ldots .$$

(In other words, the wreath product $\{\lambda\}[\{\mu\}]$ is simply the Schur function $\{\lambda\}$ in which the indeterminates in which it is expressed are Young tableaux of (μ)).

Proof.

$$\{\mu\} = \frac{1}{m!} \sum_{\nu} \chi_\nu^\mu g_\nu s_\nu = \sum_{r=1}^{\infty} M(Y_r^\mu).$$

Using the notation in (3), $S_1 = \{\mu\}$. Therefore $S_1 = \sum_{r=1}^{\infty} M(Y_r^\mu)$.
Now consider S_k.

$$S_k = \frac{1}{m!} \sum_{\nu} \chi_\nu^\mu g_\nu s_k^{\nu_1} s_{2k}^{\nu_2} s_{3k}^{\nu_3} \ldots s_{mk}^{\nu_m}.$$

But $s_{pk} = \sum_{i=1}^{\infty} x_i^{pk} = \sum_{i=1}^{\infty} (x_i^k)^p$ for $p = 1, 2, \ldots, m$, that is, S_k is simply $\{\mu\}$ expressed in terms of the indeterminates $x_1^k, x_2^k, x_3^k, \ldots$. Therefore, $S_k = \sum_{r=1}^{\infty} M(Y_r^\mu)^k$. Hence

$$\{\lambda\}[\{\mu\}] = \frac{1}{n!} \sum_{\rho} \chi_\rho^\lambda g_\rho S_\rho = \sum_{\bar{Y}^\lambda} M(\bar{Y}^\lambda)$$

where the $\bar{Y}^\lambda$ are Young tableaux of (λ) formed in the indeterminates $M(Y_1^\mu)$, $M(Y_2^\mu)$, $M(Y_3^\mu)$, ... and hence the result follows.

Example. Consider $(\lambda) = (2, 1)$, and suppose $\{\lambda\}$ is expressed in terms of the indeterminates $y_1, y_2, y_3, \ldots$. Therefore $\{\lambda\}$ is formed by summing over tableaux such as

$$\begin{array}{|c|c|}\hline y_1 & y_1 \\ \hline y_2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline y_1 & y_2 \\ \hline y_2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline y_1 & y_1 \\ \hline y_3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline y_1 & y_2 \\ \hline y_3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline y_1 & y_3 \\ \hline y_2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline y_2 & y_2 \\ \hline y_3 \\ \cline{1-1}\end{array}$$

(4)

$$\begin{array}{|c|c|}\hline y_1 & y_3 \\ \hline y_3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline y_2 & y_3 \\ \hline y_3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline y_1 & y_1 \\ \hline y_4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline y_1 & y_2 \\ \hline y_4 \\ \cline{1-1}\end{array} \qquad \text{etc.}$$

Now consider $(\mu) = (2^2)$. Therefore $\{\mu\}$ is formed by summing over tableaux

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such as

$$\begin{array}{|c|c|}\hline x_1 & x_1 \\ \hline x_2 & x_2 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline x_1 & x_1 \\ \hline x_2 & x_3 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline x_1 & x_2 \\ \hline x_2 & x_3 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline x_1 & x_1 \\ \hline x_3 & x_3 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline x_1 & x_2 \\ \hline x_3 & x_3 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline x_2 & x_2 \\ \hline x_3 & x_3 \\ \hline \end{array}\ \text{etc.}$$

Hence, $\{\lambda\}[\{\mu\}]$ is formed by summing over the tableaux in (4) and making the substitutions

$$y_1 = x_1^2x_2^2,\ y_2 = x_1^2x_2x_3,\ y_3 = x_1x_2^2x_3,\ y_4 = x_1^2x_3^2,$$
$$y_5 = x_1x_2x_3^2,\ y_6 = x_2^2x_3^2,\ \text{etc.}$$

6. Applications. It is well-known that

$$\prod_{i=1}^{\infty} \frac{1}{(1 - x_iz)} = 1 + \sum_{r=1}^{\infty} h_rz^r$$

where $h_r = \{r\}$ are the homogenous product sums of the indeterminates $x_1, x_2, \ldots$.

It follows from the theorem that

$$\prod_{i_1\leqslant\ldots\leqslant i_n} \frac{1}{(1 - x_{i_1}x_{i_2}\ldots x_{i_n}z)} = 1 + \sum_{r=1}^{\infty} h_r[h_n]z^r.$$

In particular,

$$\prod_{i\leqslant j} \frac{1}{(1 - x_ix_jz)} = 1 + \sum_{r=1}^{\infty} h_r[h_2]z^r$$

$$\prod_{i<j} \frac{1}{(1 - x_ix_jz)} = 1 + \sum_{r=1}^{\infty} h_r[a_2]z^r$$

$$\prod_{i\leqslant j} (1 - x_ix_jz) = 1 + \sum_{r=1}^{\infty} (-1)^ra_r[h_2]z^r$$

$$\prod_{i<j} (1 - x_ix_jz) = 1 + \sum_{r=1}^{\infty} (-1)^ra_r[a_2]z^r.$$

We may now use results originally stated by Littlewood [**1**] and later proved completely by McConnell and Newell [**2**] to obtain the following identities.

(5) $h_n[h_2] = \sum \{2\lambda\}$

where the summation is over all partitions of $2n$ which are composed of even numbers only.

$$h_n[a_2] = \sum \{\widetilde{2\lambda}\}$$

where the summation is over all partitions of $2n$ in which each number occurs an even number of times. (i.e. partitions conjugate to those in (5)).

$$a_n[h_2] = \sum \{\xi\}$$

where the summation is over all partitions of $2n$ which have one of the following forms when expressed in Frobenius notation (see Read [**3**]).

$$(6) \quad \begin{pmatrix} a+1 \\ a \end{pmatrix}, \quad \begin{pmatrix} a+1 & b+1 \\ a & b \end{pmatrix}, \quad \begin{pmatrix} a+1 & b+1 & c+1 \\ a & b & c \end{pmatrix}, \text{ etc.}$$

$$a_n[a_2] = \sum \{\xi\}$$

where the summation is over the partitions of $2n$ which are conjugate to those in (6).

References

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