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FUNDAMENTAL BIORTHOGONAL SEQUENCES AND K-NORMS ON ϕ

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1. Introduction. A biorthogonal sequence is a double sequence (x_i, f_i) where each x_i is from some locally convex space X, each f_i is from X^* and $f_i(x_j) = \delta_{ij}$. A biorthogonal sequence is called total if the functionals (f_i) are total over X and is called fundamental if $\text{sp}(x_i)$ is dense in X. If a biorthogonal sequence is both total and fundamental we refer to it as a Markushivich basis or, more simply, an M-basis.

If (x_i, f_i) is a total biorthogonal sequence for X, then X can be identified with the space of all scalar sequences $(f_i(x))$ under the correspondence $x \leftrightarrow (f_i(x))$. We refer to this space as the associated sequence space with respect to (x_i, f_i). With this correspondence, x_i corresponds to $e_i = (\delta_{ij})_{j=1}^{\infty}$ and f_i corresponds to E_i, the ith coordinate functional. If X is Frechêt, then the associated sequence space, with the identification topology, is an FK-space with (e_i, E_i) as a total biorthogonal sequence. For a discussion of the basic properties of FK-spaces, see [**6**, p. 202].

The multiplier algebra of a total biorthogonal sequence is the algebra of all scalar sequences t with the property that $tx = (t(i)x(i))$ is in the associated sequence space whenever x is in the associated sequence space. If the space is a Banach space, then the multiplier algebra can be given a BK-topology [**3**, Corollary 3.3]. Multiplier algebras of Schauder bases in Banach spaces have been investigated by Yamazaki [**7**; **8**] and by McGivney and Ruckle [**3**]. In [**3**], McGivney and Ruckle have characterized those BK-algebras which arise as multiplier algebras of a Schauder basis in a Banach space. Multiplier algebras of various types of M-bases (in particular, series summable M-bases; *cf.* [**5**, Theorems 6.4 and 7.2]), have been investigated by Ruckle in [**4**].

The central result of this paper is the following characterization of those algebras which are multiplier algebras of various kinds of biorthogonal systems.

A BK algebra X containing ϕ and e is the multipler algebra of a K-norm on ϕ if and only if X is the dual sequence space of a K-norm on ϕ. Here, K-norm on ϕ can be replaced by any of the following: series summable K-norm on ϕ, strongly series summable K-norm on ϕ, Schauder basis, unconditional Schauder basis, series summable M-basis, or strongly series summable M-basis.

The problem of characterizing multiplier algebras of M-bases is still open.

We have found it natural and convenient to include fundamental biorthogonal sequences (or, equivalently, K-norms on ϕ), and have therefore

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generalized the main results of [**5**] to this setting. Such straight forward generalizations are labeled propositions and proofs are included only if they are substantially different from those of [**5**].

In § 3 we give some preliminary results on the construction of sequence spaces which contain a given set A as a bounded subset. As an immediate corollary of a theorem in § 4, a necessary and sufficient condition (Theorem 4.7) is given for a space to be the dual of a separable Banach space.

Finally, using the Main Theorem we have been able to construct a series summable M-basis (Example 4.24) which is not strongly series summable. This solves a problem left open by Ruckle in [**5**]. It is of interest to note that this M-basis is not norming (see [**2**]).

2. Notation and terminology. Let ω denote the space of all scalar sequences. With the topology of coordinatewise convergence, ω is an FK-space. For $s \in \omega$, we denote the ith element of s by $s(i)$. For $A \subseteq \omega$ and $B \subseteq \phi$, A^ϕ and B^ω are defined as follows:

$$A^\phi = \left\{x \in \phi: \left|\sum_i x(i)y(i)\right| \leqq 1, \text{ for each } y \in A\right\},$$

and

$$B^\omega = \left\{x \in \omega: \left|\sum_i x(i)y(i)\right| \leqq 1, \text{ for each } y \in B\right\}.$$

Thus, A^ϕ is the absolute polar of A in ϕ and B^ω is the absolute polar of B in ω, where ω and ϕ are placed in duality by means of the pairing

$$(x, y) = \sum_i x(i)y(i).$$

If A and B are subsets of ω, then we say that A absorbs B if there exists $k > 0$ such that $B \subseteq kA$. If A absorbs B and B absorbs A, then we say that A and B are equivalent and write $A \sim B$.

If $\lambda \in \omega$ and $(x_n) \subseteq \omega$, then, unless otherwise stated, the statement $y = \sum_i \lambda(i)x_i$ means that $\sum_{i=1}^n \lambda(i)x_i$ converges coordinatewise to y. For X a normed space, $D_1(X)$ will denote the closed unit ball of X, and for A a subset of a linear space, $K(A)$ denotes the absolutely convex hull of A.

If (x_i, f_i) is an M-basis for a normed space X, then the dual sequence space of this M-bais, denoted by X^δ, is defined to be the space of all sequences $(f(x_i))$ as f ranges over X^*. For an arbitrary normed K-space Y containing ϕ, Y^δ is defined to be the dual sequence space of the M-basis (e_i, E_i) in the K-space $Y^0 = \bar{\phi}$.

A linear subspace of ω, with a locally convex topology which yields continuous coordinates, will be called a K-space. The spaces

$$l_1 = \left\{x \in \omega: \sum_i |x_i| \text{ converges}\right\}$$

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and

$$bv = \left\{ x \in \omega : \sum_i |x_i - x_{i+1}| \text{ converges} \right\},$$

are BK-spaces with respective norms

$$||x||_{l_1} = \sum_i |x_i|$$

and

$$||x||_{bv} = |x_1| + \sum_i |x_i - x_{i+1}|.$$

For a detailed discussion of multiplier algebras of M-basis, see [**3**] and for definitions and a duscussion of series summable M-basis and the series space, see [**5**].

3. Preliminary results. For $A \subseteq \omega$, let $E(A) = \bigcup_{n=1}^{\infty} nK(A)$, and give $E(A)$ the topology of p_A, the gauge of $K(A)$. Then $E(A)$ is a semi-normed space which contains A as a bounded subset. If A is coordinatewise bounded, then $E(A)$ is a normed K-space and if in addition A contains a multiple of e_i for each i, then $E(A)$ is a normed K-space which contains ϕ. If in turn we require that $\sum_i \lambda(i) x_i \in K(A)$, for each $\lambda \in D_1(l_1)$ and $x_i \in A$, then $E(A)$ is a BK-space containing ϕ. However, to guarantee completeness we can do with the following

PROPOSITION 3.1. *Let A be an absolutely convex, coordinatewise bounded, closed subset of ω which contains a multiple of e_i, for each i. Then $E(A)$ is a BK-space containing ϕ.*

If A is a coordinatewise bounded subset of ω, then there is a smallest BK-space containing A as a bounded subset. We will denote this space by $S(A)$ and it can be characterized as follows:

$$S(A) = \left\{ \sum_i \lambda(i) x_i : \lambda \in l_1, x_i \in A, \text{ for each } i \right\},$$

with norm

$$||x||_A = \inf \left\{ ||\lambda||_{l_1} : x = \sum_i \lambda(i) x_i, x_i \in A \right\}.$$

This is equivalent to the formulation of $S(A)$ given by Ruckle in [**5**] and so we omit the argument that $S(A)$ is a BK-space. Note that if $A \sim B$, then $S(A) = S(B)$.

If A is an absolutely convex, coordinatewise bounded, closed subset of ω which contains a multiple of e_i for each i, then ϕ will be dense in $E(A)$ if and only if $E(A) = S(A \cap \phi)$. If A is an absolutely convex radial subset of ϕ, then, by [**5**, Theorem 5.4], the coordinates will be norming on $E(A) = S(A)$ if and only if $A^{\omega} \sim A^{\phi\phi\omega}$, which in turn is true if and only if $A \sim A^{\phi\phi}$.

Definition 3.2. Let A be a coordinatewise bounded subset of ϕ which contains a multiple of e_i, for each i. If $|| \ ||_A$ agrees with p_A on $E(A)$ then we say that A is consistent.

For properties of consistent sets, see [**5**]; particularly, Theorem 4.2. A discussion of related concepts can be found in [**4**].

Definition 3.3. We say that a norm on ϕ is a K-norm if the E_i's are continuous on $(\phi, || \ ||)$ and is consistent if $D_1(\phi, || \ ||)$ is consistent.

In light of [**5**, Theorem 4.2], we have:

PROPOSITION 3.4. *A K-norm $|| \ ||$ on ϕ is consistent if and only if, in any completion Y of $(\phi, || \ ||)$, the extensions of the E_i's to all of Y form a total family. In particular, if $|| \ ||$ is consistent, then the completion of $(\phi, || \ ||)$ can be realized as a BK-space.*

The following technical lemma will be useful in § 4.

LEMMA 3.5. *If A is an absolutely convex radial subset of ϕ, then A absorbs $A^{\omega\phi}$.*

Proof. The weak topology on ϕ by ω, with respect to the pairing $(x, y) = \sum_i x(i)y(i)$, is, in fact, the strongest locally convex topology on ϕ. Thus, p_A is a continuous seminorm on ϕ with this topology. Since $A^{\omega\phi} = \bar{A}$ and $p_A(A) \subseteq [0, 1]$, it follows that $p_A(\bar{A}) \subseteq [0, 1]$.

4. Main results. Let (x_i, f_i) be a fundamental biorthogonal sequence for the Banach space X. Then $\mathrm{sp}(x_i)$ can be identified with ϕ under the correspondece $x \leftrightarrow s_x$, where $s_x = \sum_i f_i(x)e_i$. The induced norm, defined by $||s_x|| = ||x||$, is a K-norm on ϕ since each f_i is continuous on X. Thus, each fundamental biorthogonal sequence gives rise to a K-norm on ϕ. Conversely, each K-norm on ϕ corresponds to at least one fundamental biorthogonal sequence for a Banach space X (e.g., consider the completion of $(\phi, || \ ||)$).

PROPOSITION 4.1. *Let (x_i, f_i) be a fundamental biorthogonal sequence for the Banach space X, and let $|| \ ||$ denote the induced K-norm on ϕ. Then (x_i, f_i) is an M-basis for X if and only if $A = D_1(\phi, || \ ||)$ is consistent.*

Definition 4.2. Let $|| \ ||$ be a K-norm on ϕ. By the δ-dual of this K-norm we mean the δ-dual of the normed space $(\phi, || \ ||)$.

PROPOSITION 4.3. *If $|| \ ||$ is a K-norm on ϕ and $A = D_1(\phi, || \ ||)$, then*

$$(\phi, || \ ||)^\delta = \bigcup_{n=1}^{\infty} nA^\omega.$$

COROLLARY 4.4. *If (x_i, f_i) is an M-basis for the Banach space X, then $(\phi, || \ ||)^\delta$ is the dual sequence space of (x_i, f_i).*

Notice that the δ-dual determines whether or not the fundamental biorthogonal sequence (x_i, f_i) is an M-basis, since $A^{\omega\phi}$ is consistent if and only if A is consistent.

The following theorem is used later to characterize multiplier algebras and has, as an immediate corollary, a characterization of duals of separable Banach spaces.

THEOREM 4.5. *A BK-space Y containing ϕ is the δ-dual of a K-norm on ϕ if and only if* $D_1(Y) \sim D_1(Y)^{\phi\omega}$.

Proof. If Y is the δ-dual of a K-norm on ϕ, then it is of the form $\bigcup_{n=1}^{\infty} nA^{\omega}$, and $A^{\omega} = A^{\omega\phi\omega}$. Assume that $D_1(Y) \sim D_1(Y)^{\phi\omega}$, and let X be a completion of $\bigcup_{n=1}^{\infty} nD_1(Y)^{\phi}$. Then

$$\begin{aligned} Y &= \bigcup_{n=1}^{\infty} nD_1(Y) \\ &= \bigcup_{n=1}^{\infty} nD_1(Y)^{\phi\omega} \\ &= (\phi, \|\ \|)^{\delta}, \end{aligned}$$

where $\|\ \|$ is the gauge of $D_1(Y)^{\phi}$.

COROLLARY 4.6. *A BK-space containing ϕ is the dual sequence space of an M-basis in a Banach space if and only if $D_1(X) \sim D_1(X)^{\phi\omega}$ and $D_1(X)^{\phi}$ is consistent.*

THEOREM 4.7. *A Banach space X is the dual of a separable Banach space if and only if X admits a total biorthogonal sequence (x_i, f_i) such that $B \sim B^{\phi\omega}$ and such that B^{ϕ} is consistent, where $B = \{(f_i(x)) : x \in D_1(X)\}$.*

Definition 4.8. The multiplier algebra of a K-norm $\|\ \|$ on ϕ is defined to be the set of all sequences $s \in \omega$ for which

$$\sup \{\|st\| : t \in \phi, \|t\| \leqq 1\} < \infty.$$

We denote the multiplier algebra of a K-norm $\|\ \|$ on ϕ by $M(\phi, \|\ \|)$. By the multiplier algebra of a fundamental biorthogonal sequence we mean the multiplier algebra of the induced K-norm on ϕ. In the terminology of [**3**],

$$M(\phi, \|\ \|) = M_c[(\phi, \|\ \|), e_i, E_i],$$

and is a BK-space algebra with norm

$$\|t\|_M = \sup_{\|s\| \leqq 1} \|ts\|.$$

PROPOSITION 4.9 (cf. [**3**, Theorem 4.2; **5** Theorem 7.1]). *If $\|\ \|$ is a K-norm on ϕ, then*

$$M(\phi, \|\ \|) = \bigcup_{n=1}^{\infty} n(AA^{\omega})^{\omega},$$

where $A = D_1(\phi, \|\ \|)$.

COROLLARY 4.10. *If (x_i, f_i) is an M-basis for X and $\| \; \|$ is the induced K-norm on ϕ, then $M(\phi, \| \; \|)$ is the multiplier algebra of the M-basis (x_i, f_i) in X.*

Proof. This follows from 4.9 and [**5**, Theorem 7.1].

If a K-norm $\| \; \|$ on ϕ is consistent, then, by the above, $M(\phi, \| \; \|)$ is the multiplier algebra of the M-basis (e_i, E_i) in X, where X is the BK realization of the completion of $(\phi, \| \; \|)$.

Question. If a K-norm on ϕ is not consistent, must there exist a consistent K-norm on ϕ with the same multiplier algebra?

THEOREM 4.11. *A BK-algebra X containing ϕ and e is the multiplier algebra of a K-norm on ϕ if and only if $D_1(X) \sim D_1(X)^{\phi\omega}$.*

Proof. If X is the multiplier algebra of a K-norm on ϕ, then by 4.9 it is of the form $\bigcup_{n=1}^{\infty} n(AA^{\omega})^{\omega}$. The result follows since $(AA^{\omega})^{\omega} \sim (AA^{\omega})^{\omega\phi\omega}$.

Suppose that $D_1(X) \sim D_1(X)^{\phi\omega}$. Then

$$\begin{aligned} X &= \bigcup_{n=1}^{\infty} nD_1(X)^{\phi\omega} \\ &= \bigcup_{n=1}^{\infty} n[D_1(X)^{\phi}D_1(X)]^{\omega} \\ &= \bigcup_{n=1}^{\infty} n[D_1(X)^{\phi}D_1(X)^{\phi\omega}]^{\omega} \\ &= M(\phi, \| \; \|), \end{aligned}$$

where $\| \; \|$ is the gauge of $D_1(X)^{\phi}$. We have used the fact that $D_1(X)^{\phi}D_1(X) \sim D_1(X)^{\phi}$. (It is clear that $D_1(X)^{\phi}D_1(X)$ absorbs $D_1(X)^{\phi}$, since $D_1(X)$ contains a multiple of e. Let $z, x \in D_1(X)$ and $y \in D_1(X)^{\phi}$. Then $|(xy, z)| = |(y, zx)| \leqq K$, where K is such that $D_1(X)D_1(X) \subseteq KD_1(X)$. Thus, $xy \in KD_1(X)^{\phi}$.)

THEOREM 4.12. *Let X be a BK-algebra containing ϕ and e; then X is the multiplier algebra of a K-norm if and only if X is the δ-dual of a K-norm.*

Proof. This is an immediate corollary of 4.5 and 4.11.

The concept of a multiplier algebra defines an equivalence relation on the set of all K-norms on ϕ. Two norms are in the same equivalence class if they have the same multiplier algebra. Theorem 4.11 constructs a distinguished element in each equivalence class, namely, the gauge of $D_1(X)^{\delta}$. We denote this K-norm by $\| \; \|_{(X)}$.

Definition 4.13. Let $\| \; \|$ be a K-norm on ϕ. The series space, $\mathcal{S}(\phi, \| \; \|)$, is defined by $\mathcal{S}(\phi, \| \; \|) = S(AA^{\omega})$ where $A = D_1(\phi, \| \; \|)$.

Note that, for A consistent, this coincides with the definition of series space given by Ruckle in [**5**].

Definition 4.14. A K-norm $\| \ \|$ on ϕ is called series summable if $e \in \mathcal{S}(\phi, \| \ \|)^\delta$.

PROPOSITION 4.15. *Let $\| \ \|$ be a K-norm on ϕ. Then the following are equivalent:*

(i) $e \in \mathcal{S}(\phi, \| \ \|)^\delta$;

(ii) $M(\phi, \| \ \|) = \mathcal{S}(\phi, \| \ \|)^\delta$;

(iii) AA^ω *is consistent, where* $A = D_1(\phi, \| \ \|)$;

(iv) $D_1(M(\phi, \| \ \|))^\phi$ *is consistent.*

Proof. The equivalence of the first 3 conditions follows much as in [**5**].

(iii) $\Leftrightarrow$ (iv). This follows since:

$$D_1(M(\phi, \| \ \|))^\phi = (AA^\omega)^{\omega\phi}$$

is consistent if and only if AA^ω is consistent.

By (iv), the multiplier algebra determines whether or not a K-norm on ϕ is series summable. This theorem also shows that the multiplier algebra does not determine whether or not a K-norm is consistent, since there exist BK-spaces with M-bases which are not series summable. The distinguished K-norm $\| \ \|$ in the equivalence class determined by such an M-basis is not consistent, by Proposition 4.15. An example is given on page 524 of [**5**].

THEOREM 4.16. *Let X be a BK-algebra containing ϕ and e. Then the following are equivalent*:

(i) *X is the multiplier algebra of a series summable K-norm on ϕ*;

(ii) $D_1(X) \sim D_1(X)^{\phi\omega}$ *and* $D_1(X)^\phi$ *is consistent*;

(iii) *X is the dual sequence space of an M-basis*;

(iv) *X is the δ-dual of a series summable K-norm on ϕ.*

Proof. (i) $\Leftrightarrow$ (ii). The necessity follows by 4.15. By 4.11, X is the multiplier algebra of a K-norm on ϕ and so by 4.15 this K-norm is series summable.

(ii) $\Leftrightarrow$ (iii). This is Corollary 4.6.

(i) $\Rightarrow$ (iv). Suppose that $X = M(\phi, \| \ \|)$, where $\| \ \|$ is a series summable K-norm on ϕ. Then by 4.15, $X = \mathcal{S}(\phi, \| \ \|)^\delta$, and as in the proof of 4.15, $M(\mathcal{S}(\phi, \| \ \|) = M(\phi, \| \ \|)$. Therefore, (e_i, E_i) is a series summable M-basis for $\mathcal{S}(\phi, \| \ \|)$. Thus, $X = (\phi, \| \ \|')^\delta$, where $\| \ \|'$ is the norm on $\mathcal{S}(\phi, \| \ \|)$ restricted to ϕ.

(iv) $\Rightarrow$ (i). This follows from 4.12.

We note that the M-basis in (iii) above will necessarily be series summable and that the K-norm in (iv) will be consistent.

Definition 4.17 (*cf.* [**5**, Theorem 7.2]). A K-norm $\| \ \|$ on ϕ is said to be strongly series summable if $M(\phi, \| \ \|) = M(\phi, \| \ \|)^{\delta\delta}$.

PROPOSITION 4.18. *Let* $\|\ \|$ *be a K-norm on* ϕ. *Then the following are equivalent*:

(i) $M(\phi, \|\ \|) = M(\phi, \|\ \|)^{\delta\delta}$;

(ii) $e \in M(\phi, \|\ \|)^{\delta\delta}$;

(iii) *there is a sequence* $(u_n) \subseteq \phi$ *such that* $\lim_n u_n(k) = 1$, *for all* k, *and* $\sup_n \|u_n\|_M < \infty$.

Proof. (i) $\Rightarrow$ (ii). $e \in M(\phi, \|\ \|)$.

(ii) $\Rightarrow$ (i). Since $\bigcup_{n=1}^{\infty} n(AA^{\omega})^{\phi\omega}$ is a BK-space containing AA^{ω} as a bounded subset (Proposition 3.1), we have

$$\mathscr{S}(\phi, \|\ \|) \subseteq \bigcup_{n=1}^{\infty} n(AA^{\omega})^{\phi\omega} = M(\phi, \|\ \|)^{\delta},$$

and, therefore, $e \in M(\phi, \|\ \|)^{\delta\delta} \subseteq \mathscr{S}(\phi, \|\ \|)^{\delta}$. By 4.15, $(\phi, \|\ \|)$ is series summable and the distinguished element in the equivalence class determined by $M(\phi, \|\ \|)$ is consistent. Now, since $M(\phi, \|\ \|)$ is known to be the multiplier algebra associated with an M-basis, the result follows by [**5**, Theorem 7.2].

(ii) $\Leftrightarrow$ (iii). This follows exactly as in [**5**, Theorem 7.2].

COROLLARY 4.19. *Every strongly series summable K-norm on* ϕ *is series summable.*

We know that there are inconsistent K-norms in every equivalence class of K-norms which are not series summable. The next result shows that there are no inconsistent K-norms in any strongly series summable equivalence class.

THEOREM 4.20. *Every strongly series summable K-norm on* ϕ *is consistent.*

Proof. Assume that $(\phi, \|\ \|)$ is strongly series summable. Then there exists $(u_n) \subseteq \phi$ such that $u_n(k) \to 1$ on n, for each k, and $\sup_n \|u_n\|_M < \infty$. It suffices to show that there exists a $K > 0$ such that $\| \sum_i \lambda(i) s_i \| \leqq K$, for all $\lambda \in D_1(l_1)$ and all $(s_i) \subseteq A = D_1(\phi, \|\ \|)$. Let $\lambda \in D_1(l_1)$ and $(s_i) \subseteq A$. Then for fixed n,

$$\lim_m \left\| u_n \sum_{i=1}^{\infty} \lambda(i) s_i - u_n \sum_{i=1}^{m} \lambda(i) s_i \right\| = \lim_m \left\| \sum_{i=m+1}^{\infty} \lambda(i) s_i u_n \right\| = 0,$$

since u_n has finitely many non-zero coordinates and

$$\lim_m \sum_{i=m+1}^{\infty} \lambda(i) s_i(k) u_n(k) = 0,$$

for each k. Since

$$\left\| u_n \sum_{i=1}^{m} \lambda(i) s_i \right\| \leqq K \left\| \sum_{i=1}^{m} \lambda(i) s_i \right\| \leqq K,$$

this gives that

$$\left\| u_n \sum_{i=1}^{\infty} \lambda(i) s_i \right\| \leqq K.$$

But $\lim_n ||u_n s - s|| = 0$, for each $s \in \phi$, so

$$\left\| \sum_{i=1}^{\infty} \lambda(i) s_i \right\| \leqq K.$$

LEMMA 4.21. *A BK-space X containing ϕ is the dual sequence space of a norming M-basis in a Banach space if and only if $D_1(X)^{\phi\phi\omega} \sim D_1(X)$.*

Proof. Assume that X is the dual sequence space of a norming M-basis. Then by [**5**, Theorem 4.4] there is a consistent balanced subset A of ϕ such that $X = \bigcup_{n=1}^{\infty} nA^{\omega}$, where $A^{\omega} \sim A^{\phi\omega}$. But $D_1(X) \sim A^{\omega}$, and so

$$D_1(X) \sim A^{\omega} \sim A^{\phi\omega} \sim (A^{\omega\phi})^{\phi\omega} \sim D_1(X)^{\phi\phi\omega}.$$

Conversely, assume that $D_1(X) \sim D_1(X)^{\phi\phi\omega}$. Thus,

$$D_1(X) \sim D_1(X)^{\phi\phi\omega} \sim D_1(X)^{\phi\phi\omega\phi\omega} \sim D_1(X)^{\phi\omega}.$$

Now,

$$D_1(X)^{\phi} \sim D_1(X)^{\phi\phi\omega\phi} \sim D_1(X)^{\phi\phi\phi}$$

and, hence, $D_1(X)^{\phi}$ is consistent. For,

$$D_1(X)^{\phi} \sim D_1(X)^{\phi\phi\phi} = D_1(\bigcup_n nD_1(X)^{\phi\omega}) \cap \phi$$

and, hence, $D_1(X)^{\phi}$ is the intersection of ϕ with the unit ball of a BK-space. By 4.6, X is the dual sequence space of an M-basis and this M-basis is norming by [**5**, Theorem 5.4].

THEOREM 4.22. *Let X be a BK-algebra containing ϕ and e. Then the following are equivalent*:

(i) *X is the multiplier algebra of a strongly series summable K-norm on ϕ;*

(ii) $D_1(X) \sim D_1(X)^{\phi\phi\omega}$;

(iii) *X is the dual sequence space of a norming M-basis;*

(iv) *X is the δ-dual of a strongly series summable K-norm on ϕ.*

Proof. (i) $\Rightarrow$ (ii). Since X is a multiplier algebra, $D_1(X) \sim D_1(X)^{\phi\omega}$, so $D_1(X) \cap \phi$ is equivalent to $D_1(X)^{\phi\phi}$. This, and the fact that $X = X^{\delta\delta}$, gives that $D_1(X) \sim D_1(X)^{\phi\phi\omega}$.

(ii) $\Rightarrow$ (i).

$$D_1(X)^{\phi\omega} \sim D_1(X)^{\phi\phi\omega\phi\omega} = D_1(X)^{\phi\phi\omega} \sim D_1(X).$$

Thus, by 4.11, X is the multiplier algebra of a K-norm on ϕ. Now

$D_1(X) \sim D_1(X)^{\phi\omega}$ implies that $D_1(X)^{\phi\phi} \sim D_1(X) \cap \phi$, so

$$\begin{aligned} X^{\delta\delta} &= \bigcup_{n=1}^{\infty} n(D_1(X) \cap \phi)^{\phi\omega} \\ &= \bigcup_{n=1}^{\infty} nD_1(X)^{\phi\phi\phi\omega} \\ &= \bigcup_{n=1}^{\infty} nD_1(X) \\ &= X. \end{aligned}$$

Thus, the K-norm is strongly series summable.

(ii) ⇔ (iii). This follows from Lemma 4.21.

(i) ⇔ (iv). The proof here is the same as the argument for the (i) ⇔ (iv) part of 4.16.

Thus, we have that a BK-algebra X containing ϕ and e is the multiplier algebra of a $\triangle$ if and only if X is the dual sequence space of a $\triangle$, where $\triangle$ can be any of: K-norm on ϕ, series summable K-norm on ϕ, strongly series summable K-norm on ϕ, unconditional Schauder basis, Schauder basis, series summable M-basis, or strongly series summable M-basis.

Since we know that every strongly series summable K-norm is consistent the following are consequences of [**3**, Theorem 4.7].

PROPOSITION 4.23. *Let* $\| \ \|$ *be a* K*-norm on* ϕ. *Then*:

(i) $bv \subseteq M(\phi, \| \ \|)$ *if and only if* (e_i, E_i) *is a Schauder basis for the* BK*-completion of* $(\phi, \| \ \|)$;

(ii) $m = M(\phi, \| \ \|)$ *if and only if* (e_i, E_i) *is an unconditional Schauder basis for the* BK*-completion of* $(\phi, \| \ \|)$.

Using the above results, we are now able to give an example of a series summable M-basis which is not strongly series summable, and thus answer a question rasied in [**5**].

Example 4.24. We will construct a BK-space X with a non-norming M-basis such that X^δ is an algebra containing e. By 4.12, 4.16, and 4.22, this M-basis is series summable but not strongly series summable.

Let X_n denote the set l_1 with the norm

$$||x||_{(n)} = \frac{1}{n} \sum_{i=1}^{\infty} |x_i| + \left| \sum_{i=1}^{\infty} x_i \right| .$$

This norm is equivalent to the usual l_1 norm, since

$$\frac{1}{n} \sum_{i=1}^{\infty} |x_i| \leqq ||x||_n \leqq \left(\frac{1}{n} + 1\right) \sum_{i=1}^{\infty} |x_i|.$$

Let Y_n denote the dual of X_n. Then Y_n is the set l_∞ with the norm $||\cdot||^{(n)}$, determined in the usual way by X_n. Let $M_1, M_2, \ldots, M_i = \{m_{ij}\}_{j=1}^{\infty}$ be a

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partition of the positive integers into countably many infinite subsets. For a sequence x, define $Q_n(x) = \{x(m_{nb})\}_{b=1}^{\infty}$.

Finally, define X to be the space of all sequences x such that

$$||x|| \equiv \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=1}^{\infty} |x(m_{nj})| + \left|\sum_{j=1}^{\infty} x(m_{nj})\right|\right) = \sum_{n=1}^{\infty} ||Q_n(x)||_{(n)} < \infty.$$

It is not difficult to see that $(X, ||\cdot||)$ is a BK-space containing l_1 densely. Furthermore, the dual, Y, is given by the set of all sequences y such that $||y|| \equiv \sup ||Q_n(y)||^{(n)} < \infty$, and is a BK space (see [**1**, p. 31]).

It can be shown that Y is an algebra containing e. It remains to show that the coordinate functionals are not norming on X. Let $A = D_1(X) \cap \phi$. As we remarked in § 3, it is sufficient to show that A does not absorb $A^{\phi\phi}$.

Let $A_n = D_1(X_n) \cap \phi$. Then

$$\inf\{\kappa > 0 : A_n^{\phi\phi} \subseteq \kappa A_n\} \geqq n/2,$$

for each n. For,

$$||(n/2)e_1||_{(n)} = (n/2)(1 + 1/n) > n/2.$$

Let $y_i = (n/2)(e_1 - e_i)$. Then $y_i \in A_n$ and $\lim_i y_i(\kappa) = (n/2)e_i(\kappa)$, for all κ. Since $A^{\phi\phi}$ is the coordinatewise closure of A in ϕ, $(n/2)e_1 \in A_0^{\phi\phi}$.) It follows that there is no $\kappa > 0$ such that $A^{\phi\phi} \subseteq \kappa A$.

Finally, we note that (e_i, E_i) is not a basis for X since it is not strongly series summable, but that X does possess a basis $\{z_{ij}\}_{i,j=1}^{\infty}$ where

$$Q_n(z_{ij}) = \begin{cases} e_1 & \text{if } i = n, j = 1 \\ e_j - e_1 & \text{if } n, j > 1 \\ 0 & \text{otherwise.} \end{cases}$$

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