ON INTERSECTIONS AND UNIONS OF RADICAL CLASSES

YU-LEE LEE and R. E. PROPES [1]

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1. Introduction

Let $\mathscr{A}$ be a class of rings, and let $L(\mathscr{A})$ denote the lower radical class determined by $\mathscr{A}$. In [3] Yu-Lee Lee showed that $L(\mathscr{A})$ may be constructed in the following manner: Let $H(\mathscr{A})$ be the class of all homomorphic images of rings in $\mathscr{A}$. For each ring R, let $D_1(R)$ be the set of all ideals of R, and by induction define $D_{n+1}(R)$ to be the family of all rings which are ideals of some ring in $D_n(R)$ and set $D(R) = \cup \{D_n(R): n = 1, 2, 3, \cdots\}$ which is commonly known as the hereditary closure of R. A ring R is called an $L(\mathscr{A})$ – ring if $D(R/I)$ contains a non-zero ring which is isomorphic to a ring in $H(\mathscr{A})$ for each ideal I of R and $I \neq R$ i.e., if each non-zero homomorphic image of R contains an accessible subring isomorphic to a ring in $H(\mathscr{A})$. In [4] Yu-Lee Lee proved that any class $\mathscr{A}$ of rings determines an upper radical property $\mathfrak{S}(\mathscr{A})$.

The following theorem was conjectured by Yu-Lee Lee: Let $\mathscr{A}_i$ be a homomorphically closed and hereditary class of rings ($i = 1, 2$). Then $L(\mathscr{A}_1 \cap \mathscr{A}_2) = L(\mathscr{A}_i) \cap L(\mathscr{A}_2)$. The purpose of this paper is to prove this theorem and, in addition, to prove an "intersection theorem" for upper radicals. In this paper we shall use the following notation: $I \leqq R$ signifies that I is an ideal of the ring R.

We shall use the following theorem which is due to A. E. Hoffman and W. G. Leavitt [2]. Theorem. If $\mathscr{A}$ is a hereditary class, then $L(\mathscr{A})$ is hereditary.

Since we shall be concerned with the intersections of radical classes, we shall often employ (without specifically noting it) the following useful proposition. We mention that T. L. Jenkins in [1] proved an analogous proposition for hereditary radicals.

PROPOSITION. *Let P_1 and P_2 be radical classes in some universal class $\mathscr{W}$ of rings, and define $T(R) = P_1(R) \cap P_2(R)$, and set $T = \{R \in \mathscr{W} : T(R) = R\}$. Then $T = P_1 \cap P_2$.*

PROOF.
$$R \in T \text{ iff } R = T(R) = P_1(R) \cap P_2(R)$$
$$\text{iff } R = P_1(R) = P_2(R)$$
$$\text{iff } R \in P_1 \cap P_2.$$

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THEOREM 1. *Let $\mathscr{A}_i$ be a homomorphically closed and hereditary class of rings* ($i = 1, 2$). *Then* $L(\mathscr{A}_1 \cap \mathscr{A}_2) = L(\mathscr{A}_1) \cap L(\mathscr{A}_2)$.

PROOF. Since $L(\mathscr{A}_1 \cap \mathscr{A}_2) \subseteq L(\mathscr{A}_i)$ for $i = 1, 2$, we have $L(\mathscr{A}_1 \cap \mathscr{A}_2) \subseteq L(\mathscr{A}_1) \cap L(\mathscr{A}_2)$. Thus let $R \in L(\mathscr{A}_1) \cap L(\mathscr{A}_2)$ and let I be a proper ideal of R. Now $R \in L(\mathscr{A}_1)$ implies $D(R/I) \cap \mathscr{A}_1 \neq 0$, hence let $A \in D(R/I) \cap \mathscr{A}_1$. Since $L(\mathscr{A}_1) \cap L(\mathscr{A}_2)$ is hereditary [2] and since $R/I \in L(\mathscr{A}_1) \cap L(\mathscr{A}_2)$, we have $D(R/I) \subseteq L(\mathscr{A}_1) \cap L(\mathscr{A}_2)$; and so $A \in L(\mathscr{A}_1) \cap L(\mathscr{A}_2)$, viz., $A \in L(\mathscr{A}_2)$. But $A \in L(\mathscr{A}_2)$ implies $D(A) \cap \mathscr{A}_2 \neq 0$. Thus let $0 \neq B \in D(A) \cap \mathscr{A}_2$. Now $\mathscr{A}_1$ is hereditary, and $A \in \mathscr{A}_1$, so that $D(A) \subseteq \mathscr{A}_1$. Hence $B \in \mathscr{A}_1 \cap \mathscr{A}_2$. But $D(A) \subseteq D(R/I)$ so that $B \in D(R/I) \cap (\mathscr{A}_1 \cap \mathscr{A}_2)$. Therefore $D(R/I) \cap (\mathscr{A}_1 \cap \mathscr{A}_2) \neq 0$. Thus $R \in L(\mathscr{A}_1) \cap L(\mathscr{A}_2)$ implies $R \in L(\mathscr{A}_1 \cap \mathscr{A}_2)$. This completes the proof of Theorem 1.

NOTE. By an inductive argument $L(\bigcap_{i=1}^n \mathscr{A}_i) = \bigcap_{i=1}^n L(\mathscr{A}_i)$, where $\mathscr{A}_i$ is a class of rings which is both homomorphically closed and hereditary for $i = 1, 2, \cdots, n$. Next we provide an example for which $L(\mathscr{A} \cap \mathscr{B})$ is a proper subset of $L(\mathscr{A}) \cap L(\mathscr{B})$.

EXAMPLE. Let Z denote the ring of integers, and let (4) denote the principal ideal of Z generated by the integer 4. Let $Z/(4)$ denote the ordinary quotient ring, and let $R = \{0+(4), 2+(4)\}$. Let $\mathscr{A} = H(\{(Z/(4))/R, R\})$ and $\mathscr{B} = H(\{Z/(4)\})$ denote the homomorphic closures of the classes $\{(Z/(4))/R, R\}$ and $\{Z/(4)\}$ respectively. It is easy to see that the only proper ideal of $Z/(4)$ is R and the ring $Z/(4)$ cannot be mapped homomorphically onto its ideal R. We also note that the only subrings of $Z/(4)$ are $Z/(4)$, R, $0 \cdots$ none of which is a field (hence none is isomorphic with $(Z/(4))/R$).

Now $\mathscr{A} \cap \mathscr{B} = H(\{(Z/(4))/R\})$, and $(Z/(4))/R$ is simple; therefore each ring in $\mathscr{A} \cap \mathscr{B}$ is either 0 or else isomorphic with $(Z/(4))/R$. Since $Z/(4)$ has no subring isomorphic to the field $(Z/(4))/R$, then $Z/(4) \notin L(\mathscr{A} \cap \mathscr{B})$, in fact, $L(\mathscr{A} \cap \mathscr{B})(Z/(4)) = 0$. Certainly $Z/(4) \in L(\mathscr{B})$, and also $Z/(4) \in L(\mathscr{A})$, because each non-zero homomorphic image of $Z/(4)[(Z/(4))/R, Z/(4)]$ contains a non-zero subring in $\mathscr{A}$. Thus $Z/(4) \in L(\mathscr{A}) \cap L(\mathscr{B})$ and hence $L(\mathscr{A} \cap \mathscr{B})$ is a proper subset of $L(\mathscr{A}_1) \cap L(\mathscr{A}_2)$.

We note that $\mathscr{A}$ is hereditary, because both $(Z/(4))/R$ and R are simple rings. However, $\mathscr{B}$ is not hereditary, because $R \leqq Z/(4) \in \mathscr{B}$, but $R \notin \mathscr{B}$.

For a class $\mathscr{M}$ of rings, let $\mathfrak{S}(\mathscr{M})$ denote the upper radical class determined by the class $\mathscr{M}$. The following theorem is similar to 2.3.3 of [1, p. 28].

THEOREM 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be classes of rings. Then* $\mathfrak{S}(\mathscr{A} \cup \mathscr{B}) = \mathfrak{S}(\mathscr{A}) \cap \mathfrak{S}(\mathscr{B})$.

PROOF. First, since each ring in $\mathscr{A}$ is in $\mathscr{A} \cup \mathscr{B}$, then each ring in $\mathscr{A}$ is

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$\mathfrak{S}(\mathscr{A} \cup \mathscr{B})$ – semi-simple. Then since $\mathfrak{S}(\mathscr{A})$ is the largest radical for which every ring in $\mathscr{A}$ is semi-simple, we must have $\mathfrak{S}(\mathscr{A} \cup \mathscr{B}) \subseteq \mathfrak{S}(\mathscr{A}.)$ Similarly $\mathfrak{S}(\mathscr{A} \cup \mathscr{B}) \subseteq \mathfrak{S}(\mathscr{B})$, and so $\mathfrak{S}(\mathscr{A} \cup \mathscr{B}) \subseteq \mathfrak{S}(\mathscr{A}) \cap \mathfrak{S}(\mathscr{B})$. Now let $R \in \mathscr{A} \cup \mathscr{B}$. If $R \in \mathscr{A}$, then $(\mathfrak{S}(\mathscr{A}))(R) = 0$ and so $(\mathfrak{S}(\mathscr{A}) \cap \mathfrak{S}(\mathscr{B}))(R) = 0$. Similarly, for $R \in \mathscr{B}$, $(\mathfrak{S}(\mathscr{A}) \cap \mathfrak{S}(\mathscr{B}))(R) = 0$. Thus every ring in $\mathscr{A} \cup \mathscr{B}$ is $\mathfrak{S}(\mathscr{A}) \cap \mathfrak{S}(\mathscr{B})$-semi-simple. Since $\mathfrak{S}(\mathscr{A} \cup \mathscr{B})$ is the largest radical for which every ring in $\mathscr{A} \cup \mathscr{B}$ is semi-simple, we must have $\mathfrak{S}(\mathscr{A}) \cap \mathfrak{S}(\mathscr{B}) \subseteq \mathfrak{S}(\mathscr{A} \cup \mathscr{B})$. Hence $\mathfrak{S}(\mathscr{A} \cup \mathscr{B}) = \mathfrak{S}(\mathscr{A}) \cap \mathfrak{S}(\mathscr{B})$.

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Kansas State University
and
New York State University College
at Potsdam