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A CLASS OF FUNCTIONAL EQUATIONS WHICH HAVE ENTIRE SOLUTIONS

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We consider the Abelian functional equation

$$g(\phi(z)) = g(z) + 1$$

where ϕ is a given entire function and g is to be found. The inverse function $f = g^{-1}$ (if one exists) must satisfy

$$f(w + 1) = \phi(f(w)).$$

We show that for a wide class of entire functions, which includes $\phi(z) = e^z - 1$, the latter equation has a non-constant entire solution.

1. INTRODUCTION

A functional equation of the form

$$g(\phi(z)) = g(z) + 1 \tag{1}$$

where ϕ is given, and g is to be found, is said to be of Abelian type, following the paper of Abel [1].

The inverse function $f = g^{-1}$ satisfies

$$f(w + 1) = \phi(f(w)) \tag{2}$$

where we have put $w = g(z)$.

Solutions of these equations are of importance in studying the flow in a set X determined by a map ϕ of X to itself, since the family of functions

$$\phi_t(z) = f(g(z) + t), \qquad t \in \mathbb{R},$$

satisfies the formal identities

$$\phi_0(z) = z,\ \phi_1(z) = \phi(z), \text{ and } \phi_t(\phi_u(z)) = \phi_{t+u}(z).$$

When $X = \mathbb{C}$ and ϕ is entire, there are obvious difficulties in the analytic continuation of solutions of (1) because of the complicated nature of the singularities of g

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which occur near any fixed point of ϕ. By contrast we show in this paper that for a reasonably wide class of entire functions ϕ, the equation (2) has an entire solution: the result is stated as Theorem 2 below.

An important special case is given by $\phi_0(z) = e^z - 1$. Solutions $g_0(z)$ of (1) are constructed in [3] for real positive argument, and in [4] for certain regions in $\mathbf{C}$. Some other special cases where one can give explicit solutions of (2) are given for constants a, $c > 1$, by $\phi(z) = cz$, $f(z) = c^z$, and by $\phi(z) = z^c$, $f(z) = a^{c^z}$. These illustrate the general situation in which solutions of (2) tend to increase much more rapidly than ϕ itself.

2. Construction of Solutions

We begin by stating the following important theorem of Fatou.

Theorem A. (Fatou [2]) *Let $\phi(z) = z + \sum_{n=1}^{\infty} c_n z^{n+1}$ be an entire function with $c_1 > 0$, and let N be a neighbourhood of 0 on which ϕ is invertible.*

Then there is an open subset S of N with the following properties:

(i) *the origin is a boundary point of S, and $(0,t) \subseteq S$ for some $t > 0$;*

(ii) *$\phi^{-1}(S) \subseteq S$;*

(iii) *if for any $z \in S$, we put $z_0 = z$, $z_{n+1} = \phi^{-1}(z_n)$, $n \geqslant 0$, then we have the asymptotic expansion*

$$\frac{1}{z_n} = an + b \log n - ag(z) + 0\left(\frac{\log n}{n}\right). \tag{3}$$

In (3) we have $a = c_1$, $b = \frac{1}{c_1}\left(c_2 - c_1^2\right)$, and g is an analytic function on S which satisfies $g\left(\phi^{-1}(z)\right) = g(z) - 1$ for all $z \in S$. The order term is uniform on compact subsets of S.

Note:. Fatou proves the result in much greater generality: the above is sufficient for our needs. One can be more explicit about the set S, whose boundary is the image of a parabolic arc $x + y^2 =$ constant(> 0) under the inversion mapping $z \to \frac{1}{z}$; thus the boundary of S is tangent to the negative real axis at the origin.

We can now state our method for constructing solutions of (2). Equation (3) defines $g(z)$ as a limit

$$\begin{aligned} w = g(z) &= \lim_{n\to\infty} [n + \frac{b}{a} \log n - (az_n)^{-1}] \\ &= \lim_{n\to\infty} [n + \frac{b}{a} \log n - \{a(\phi^{-1})^{[n]}(z)\}^{-1}] \\ &= \lim_{n\to\infty} g_n(z) \text{ say.} \end{aligned}$$

(For any function, we use $f^{[n]}$ to denote the n-fold iterate of f.)

We invert this relation to get

$$z = f(w) = g^{-1}(w) = \lim_{n\to\infty} g_n^{-1}(w) = \lim_{n\to\infty} \phi^{[n]}\left(\{a(n-w)+b\log n\}^{-1}\right). \tag{4}$$

Thus our aim is to show the existence of the limit in (4) for all $w \in \mathbb{C}$, which then defines a non-constant entire solution of (2).

We begin with the following result.

THEOREM 1. *Let ϕ be an entire function of the form $\phi(z) = z + \sum_1^\infty c_n z^{n+1}$, with $c_1 > 0$ and $c_n \geqslant 0$ for $n \geqslant 2$. Put $a = c_1$, $b = \frac{1}{c_1}(c_2 - c_1^2)$, $r_n = n + \frac{b}{a}\log n$, and define*

$$f_n(w) = \phi^{[n]}\left(\{a(n-w)+b\log n\}^{-1}\right).$$

Then f_n is analytic on $\mathbb{C}\setminus\{r_n\}$, and for any $M > 0$, the sequence $(f_n)_{r_n > M}$ is uniformly bounded on $\overline{S}(0, M) = \{z\colon |z| \leqslant M\}$, provided that either (i) $c_2 \neq c_1^2$, or (ii) $c_3 < c_1^3$.

PROOF: Since $c_n \geqslant 0$ for all n, the Maclaurin coefficients of f_n are also non-negative. In particular for $|w| < r_n$, we have $|f_n(w)| \leqslant f_n(|w|)$. Thus it will be sufficient to show that the sequence $(f_n(w))$ is convergent for $w > 0$. In fact we shall show, subject to either of the conditions (i) or (ii), that for $w > 0$ the sequence $(f_n(w))$ is eventually decreasing. Since ϕ is monotone increasing on $[0, \infty)$ it is sufficient to prove, for $w > 0$ and sufficiently large n, that

$$\phi(\alpha_n) < \alpha_{n-1} \tag{5}$$

where we have put $\alpha_n = \{a(n-w)+b\log n\}^{-1}$.

To prove (5), we expand both sides asymptotically and compare terms. For ease of calculation, we put $k = \frac{b}{a}$, and $w_n = w - k\log n$, so that $\alpha_n = \frac{1}{a(n-w_n)}$, and the result to be proved is that

$$\alpha_{n-1}/\alpha_n > \phi(\alpha_n)/\alpha_n = 1 + \sum_1^\infty c_r(\alpha_n)^r \qquad \text{or}$$

$$\frac{n-w_n}{n-1-w_{n-1}} > 1 + c_1\alpha_n + c_2\alpha_n^2 + \dots. \tag{6}$$

Now $w_{n-1} = w - k\log(n-1) = w - k\log n - k\log\frac{n-1}{n} = w_n + ks_n$, say, where $s_n = -\log\left(\frac{n-1}{n}\right) = \frac{1}{n} + \frac{1}{2n^2} + \dots = 0\left(\frac{1}{n}\right)$.

Hence on the left hand side of (6) we have

$$\frac{n-w_n}{n-w_n-1-ks_n}=\left(1-\frac{w_n}{n}\right)\left(1-\frac{w_n+1+ks_n}{n}\right)^{-1},$$

which we expand as far as terms in n^{-3}, to get

$$(*)\qquad \left(1-\frac{w_n}{n}\right)\Bigg[1+\frac{1}{n}(w_n+1+ks_n)+\frac{1}{n^2}\left((w_n+1)^2+2ks_n(w_n+1)\right)$$
$$+\frac{1}{n^3}(w_n+1)^3+0\left(\left(\frac{\log n}{n}\right)^4\right)\Bigg]$$
$$=1+\frac{1}{n}+\frac{1}{n^2}\{(w_n+1)^2-w_n(w_n+1)\}+\frac{k}{n}s_n$$
$$+\frac{1}{n^3}\{(w_n+1)^3-w_n(w_n+1)^2\}-\frac{k}{n^2}s_nw_n+\frac{2k}{n^2}s_n(w_n+1)$$
$$+0\left(\left(\frac{\log n}{n}\right)^4\right)$$
$$=1+\frac{1}{n}+\frac{1}{n^2}(w_n+1)+\frac{k}{n}\left(\frac{1}{n}+\frac{1}{2n^2}\right)+\frac{1}{n^3}(w_n+1)^2-\frac{kw_n}{n^3}$$
$$+\frac{2k}{n^3}(w_n+1)+0\left(\left(\frac{\log n}{n}\right)^4\right)$$
$$=1+\frac{1}{n}+\frac{1}{n^2}(w_n+1+k)+\frac{1}{n^3}\left((w_n+1)^2+k\left(w_n+\frac{5}{2}\right)\right)$$
$$+0\left(\left(\frac{\log n}{n}\right)^4\right).$$

Similarly on the right hand of (6), we substitute $\alpha_n=\frac{1}{a(n-w_n)}$ and $c_1=a$, $c_2=a^2(1+k)$ to get

(**)

$$1+c_1\alpha_n+c_2\alpha_n^2+c_3\alpha_n^3+0(\alpha_n^4)$$
$$=1+(n-w_n)^{-1}+(1+k)(n-w_n)^{-2}+\frac{c_3}{a^3}(n-w_n)^{-3}+0(n^{-4})$$
$$=1+\frac{1}{n}\left(1+\frac{w_n}{n}+\frac{w_n^2}{n^2}\right)+\left(\frac{1+k}{n^2}\right)\left(1+\frac{2w_n}{n}\right)+\frac{c_3}{a^3n^3}+0\left(\frac{(\log n)^3}{n^4}\right)$$
$$=1+\frac{1}{n}+\frac{1}{n^2}(w_n+1+k)+\frac{1}{n^3}\left(w_n^2+2(1+k)w_n+\frac{c_3}{a^3}\right)+0\left(\frac{(\log n)^3}{n^4}\right).$$

If we compare (*) and (**) we see that (6) is equivalent (for sufficiently large n) to the inequality

$$(w_n+1)^2+k\left(w_n+\frac{5}{2}\right)>w_n^2+2(1+k)w_n+\frac{c_3}{a^3},$$

or to $1+5k/2>k(w-k\log n)+\frac{c_3}{a^3}$.

But this inequality is evidently satisfied for all n sufficiently large, if either (i) $k\neq 0$, (equivalently $c_2\neq c_1^2$), or (ii) if $k=0$, then $c_3<a^3=c_1^3$. Hence either condition (i) or (ii) is sufficient to establish (6), which completes the proof of Theorem 1. ∎

The uniform boundedness which we have just proved enables us to deduce our main theorem on existence of solutions of (2).

THEOREM 2. *Let* ϕ *be an entire function of the form* $\phi(z)=z+\sum_1^\infty c_n z^{n+1}$, *where* $c_1>0$, $c_n\geqslant 0$ *for all* n, *and either (i)* $c_2\neq c_1^2$ *or (ii)* $c_3<c_1^3$.

Then the sequence (f_n) *defined in Theorem 1 converges uniformly on every* $\overline{S}(0,M)$ *to a function* f *which is an entire non-constant solution of (2).*

PROOF: Theorem 1 shows that the sequence (f_n) forms a normal family on each $\overline{S}(0,M)$. In the course of the proof we also showed that for any $M>0$ and sufficiently large n, the restrictions of f_n to $[-M,M]$ form a sequence of positive functions which decreases with increasing n, and so converges on $[-M,M]$ to a limit ψ say. Hence any subsequence of (f_n) which converges on $\overline{S}(0,M)$ must have a limit which agrees with ψ on the real axis, from which we deduce the convergence of the whole sequence to an entire function f, whose restriction to $[-M,M]$ is ψ. Moreover, since f_n is defined as the inverse of the function g_n for which $g_n(z)\to g(z)$ for $z\in S$ (Fatou's Theorem A), f must equal g^{-1}, on some open subset U, say, of $g(S)$, (for instance a neighbourhood of $g(S\cap(0,\infty))$), so f cannot be constant. Again since $f=g^{-1}$, we must have (2) at least on $g(S\cap(0,\infty))$. But both sides of (2) are entire, and so the equation must hold generally and the proof of Theorem 2 is complete. ∎

To conclude, we mention some general properties of the function f which we have constructed. Since $f_n(w)=\phi^{[n]}\left(\frac{1}{a(n-w)+b\log n}\right)$, and $\phi(t)=t+\sum_1^\infty c_n t^n$, $c_n\geqslant 0$, it follows that f is a positive increasing function on $\mathbf{R}$, whose Maclaurin coefficients are again non-negative. We can deduce the asymptotic rate at which $f(x)\to 0$ as $x\to-\infty$, from the corresponding expansion for $g(t)$ as $t\to 0_+$, in the following way. First simplify the asymptotic expansion (3) of Fatou's Theorem to read $\frac{1}{t_n}=an+b\log n-ag(t)+o(1)$, for $t>0$, $t\in S$. The functional equation satisfied by g shows that $g(t_n)=g(t)-n$, and hence if we put $x=t_n$, $y=g(t_n)$ so that $x\to 0_+$,

$y \to -\infty$ as $n \to \infty$, then we obtain

$$\begin{aligned} g(x) = g(t) - n &= -\frac{1}{ax} + k\log n + o(1) \\ &= -\frac{1}{ax} + k\log\left(\frac{1}{ax} - k\log n + ag(t) + o(1)\right) \\ &= -\frac{1}{ax} + k\log\left(\frac{1}{ax}\right) + o(1) \text{ as } x \to 0_+. \end{aligned}$$

Similarly, we can show that $\lim_{x \to 0_+} x^2 g'(x) = \frac{1}{a}$, which is sufficient for the unique determination of a solution of (1) (up to an additive constant), as is pointed out by Szekeres in [**3**, Lemma 1].

Then the get the asymptotic expansion of $f(x)$ as $x \to -\infty$, we invert the above expansion for g to obtain

$$af(x) = -\frac{1}{x} + \frac{k}{x^2}\log|x| + o(x^{-2}) :$$

in particular $xf(x) \to -\frac{1}{a}$ as $x \to -\infty$.

In the special case when $\phi(t) = e^t - 1$, the hypotheses of Theorem 2 are satisfied and we can deduce the existence of an entire non-constant solution of the equation $f(w+1) = e^{f(w)} - 1$. This function is inverse to the function g constructed in [**4**], Theorem 2, which is analytic on $S = \mathbb{C} \setminus (-\infty, 0]$ and satisfies $g(\log(1+z)) = g(z) - 1$ for all z in S. Hence the family of mappings $\phi_t(z) = f(g(z) - t)$, $t \geqslant 0$, determines the flow of the map $z \to \log(1+z)$ in S.

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