STRUCTURE THEORY FOR MONTGOMERY-SAMELSON FIBERINGS BETWEEN MANIFOLDS. II

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1. Introduction. Let $f\colon M^n \to N^p$ be the projection map of an MS-fibering of manifolds β with finite non-empty singular set A and simply connected total space (see **1**). Results of Timourian (**10**) imply that $(n, p) = (4, 3)$, $(8, 5)$ or $(16, 9)$, while a theorem of Conner (**2**) yields that $\#(A)$, the cardinality of the singular set, is equal to the Euler characteristic of M^n. We give an elementary proof of this fact and, in addition, prove that $\#(A)$ is actually determined by $b_{n/2}(M^n)$, the middle betti number of M^n, or what is the same, by $b_{n/2}(N^p - f(A))$. It is then shown that β is topologically the suspension of a (Hopf) sphere bundle when N^p is a sphere and $b_{n/2}(M^n) = 0$. It follows as a corollary that β must also be a suspension when M^n is $n/4$-connected with vanishing $b_{n/2}$. Examples where $b_{n/2}$ is not zero are constructed and we state a couple of conjectures concerning the classification of such objects.

2. The cardinality of the singular set.

PROPOSITION 2.1. *If $f\colon M^n \to N^p$ is the projection map of an MS-fibering of manifolds with finite non-empty singular set $A \subseteq M^n$ and M^n is 1-connected, then*

$$\#(A) = X(M^n).$$

Proof. Via Timourian's result (**10**), $(n, p) = (4, 3)$, $(8, 5)$ or $(16, 9)$, and the fiber F is a homotopy 1-, 3- or 7-sphere. Then $M^n - A$ is 1-connected and of course, $X(F) = 0$.

Utilizing the fiber-homotopy sequence, $N^p - f(A)$, and hence, N^p, is 1-connected. Thus, in particular,

$$X(M^n - A) = X(N^p - f(A)) \cdot X(F) = 0. \tag{1}$$

The connectivity of $M^n - A$ and the orientability of M^n imply that

$$b_0(M^n - A) = 1, \qquad b_n(M^n - A) = 0. \tag{2}$$

By the cohomology exact sequence for the pair (M^n, A) we obtain

$$H^q(M^n) = H^q(M^n, A), \qquad q \geqq 2.$$

Received August 4, 1967. The material in this paper constitutes a portion of the author's doctoral dissertation directed by Professor E. Hemmingsen and submitted to Syracuse University. This research was partially supported by a NASA Graduate Fellowship and an NSF Summer Graduate Fellowship and NSF contract #. GP-5420.

But, Lefschetz duality yields

$$H_{n-q}(M^n - A) = H^q(M^n, A),$$

and therefore

$$\dim_Q H_{n-q}(M^n - A) = \dim_Q H^q(M^n)$$

so that we arrive at

$$b_q(M^n) = b_q(M^n - A) \tag{3}$$

for $n - 2 \geqq q \geqq 2$ via Poincaré duality.

When $q = 1$, we have the cohomology sequence

$$\leftarrow H^1(A) \leftarrow H^1(M^n) \leftarrow H^1(M^n, A) \leftarrow H^0(A) \leftarrow H^0(M) \leftarrow,$$

where the groups are reduced in dimension zero. However,

$$H^1(A) = 0 = H^0(M)$$

so that we obtain a short exact sequence of finite-dimensional vector spaces. This sequence must split, therefore

$$\begin{aligned} \dim_Q H^1(M^n, A) &= \dim_Q H^1(M^n) + \dim_Q H^0(A) \\ &= \dim_Q H_{n-1}(M^n) + \#(A) - 1 \\ &= b_{n-1}(M^n) + \#(A) - 1 \end{aligned}$$

while Lefschetz duality yields

$$\dim_Q H^1(M^n, A) = \dim_Q H_{n-1}(M^n - A) = b_{n-1}(M^n - A).$$

Therefore,

$$b_{n-1}(M^n - A) = b_{n-1}(M^n) + \#(A) - 1, \tag{4}$$

where $n = 2, 4$ or 8. By (1) we know that

$$\sum_{q=0}^{n} (-1)^q b_q(M^n - A) = C$$

so that (2) and (3) together apply to yield

$$\sum_{q=0}^{n-2} (-1)^q b_q(M^n) - b_{n-1}(M^n - A) = 0.$$

This relation, coupled with (4), then yields

$$\sum_{q=0}^{n-2} (-1)^q b_q(M^n) = b_{n-1}(M^n) + \#(A) - 1,$$

which proves the proposition.

We shall now show that the number of singular points is completely determined by the middle betti number of M^n, for M^n simply connected.

https://doi.org/10.4153/CJM-1969-017-1 Published online by Cambridge University Press

THEOREM 2.2. *If f: $M^n \to N^p$ is the projection map of an MS-fibering of manifolds with finite non-empty singular set $A \subseteq M^n$ and M^n is* 1*-connected, then*

$$\#(A) = b_{n/2}(M^n) + 2.$$

Proof. Our argument splits naturally into three cases. In each of these cases the fiber homotopy sequence implies that both $N^p - f(A)$ and N^p are 1-connected. Furthermore, via Proposition 2.1, it suffices to prove that

$$X(M^n) = 2 + b_{n/2}(M^n).$$

Case (1). $(n, p) = (4, 3)$.

This case follows trivially from Poincaré duality. Thus,

$$X(M^4) = 2 + b_2.$$

Case (2). $(n, p) = (8, 5)$.

As in the first case, $X(M^8) = 2 + b_4 - 2(b_1 - b_2 + b_3)$ via Poincaré duality, and we have that $b_1(M^8) = 0$. It will suffice to show that

$$b_2(M^8) = b_3(M^8). \tag{5}$$

The mapping

$$M^8 - A \xrightarrow{f} N^5 - f(A)$$

is a fibering with a 1-connected base and a homotopy 3-sphere for fiber. The simple connectivity of $N^5 - f(A)$ implies that $\pi_1(N^5 - f(A))$ acts simply on $H*(F; Q)$ so that we may apply the Gysin sequence (rational coefficients)

$$\to H_{m+1}(N^5 - f(A)) \to H_{m-3}(N^5 - f(A)) \to H_m(M^8 - A) \to H_m(N^5 - f(A)) \to;$$

letting $m = 5$ we obtain

$$H_2(N^5 - f(A)) = H_5(M^8 - A) \tag{6}$$

since

$$H_6(N^5 - f(A)) = 0 = H_5(N^5 - f(A)).$$

The group on the left is zero since $N^5 - f(A)$ has the homotopy type of a complex without any 6-simplices, while the group on the right vanishes via the orientability of N^5.

Lefschetz duality and (6) yield

$$H^3(M^8, A) = H_5(M^8 - A) = H_2(N^5 - f(A)) = H^3(N^5, f(A)), \tag{7}$$

while the cohomology sequences for the pairs (M^8, A) and $(N^5, f(A))$ yield the equations

$$H^3(M^8) = H^3(M^8, A), \qquad H^3(N^5) = H^3(N^5, f(A)). \tag{8}$$

Therefore, (6), (7), and Poincaré duality yield

$$H_5(M^8) = H^3(M^8) = H^3(N^5) = H_2(N^5)$$

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so that

$$\dim_Q H_5(M^8) = \dim_Q H_2(N^5),$$

and hence $b_5(M^8) = b_2(N^5)$. But, duality for betti numbers yields $b_3(M^8) = b_5(M^8)$ so that we finally arrive at

$$(9) \qquad b_3(M^8) = b_2(N^5).$$

Letting $m = 6$ in the Gysin sequence we have that
$\to H_7(N^5 - f(A)) \to H_3(N^5 - f(A)) \to H_6(M^8 - A) \to H_6(N^5 - f(A)) \to$
from which we obtain the isomorphism

$$(9') \qquad H_3(N^5 - f(A)) = H_6(M^8 - A),$$

since

$$H_7(N^5 - f(A)) = 0 = H_6(N^5 - f(A)).$$

From Lefschetz duality and (9′) we obtain

$$H^2(N^5, f(A)) = H_3(N^5 - f(A)) = H_6(M^8 - A) = H^2(M^8, A)$$

and as in (8) we have that

$$H^2(N^5) = H^2(N^5, f(A)), \qquad H^2(M^8) = H^2(M^8, A)$$

so that $H^2(N^5) = H^2(M^8)$. Then Poincaré duality applies to yield

$$(9'') \qquad b_3(N^5) = b_2(M^8),$$

and finally (9), (9″), and Poincaré duality yield

$$b_2(M^8) = b_3(M^8),$$

as desired.

Case (3). $(n, p) = (16, 9)$.

Now, $X(M^{16}) = 2 + b_8 - 2(b_1 - b_2 + \ldots - b_6 + b_7)$. The procedure of proof is as in Case (2). We prove that

(a) $b_2(M^{16}) = b_7(M^{16})$,
(b) $b_3(M^{16}) = b_6(M^{16})$,
(c) $b_4(M^{16}) = b_5(M^{16})$,

and then by the 1-connectivity of M^{16}, $b_1 = 0$ so that we finally obtain $X(M^{16}) = 2 + b_8(M^{16})$ as desired.

As before, $N^9 - f(A)$ is 1-connected, so we may apply the Gysin sequence with rational coefficients. Since the fiber is a 7-sphere, the sequence has the form

$$\to H_{m+1}(N^9 - f(A)) \to H_{m-7}(N^9 - f(A)) \to H_m(M^{16} - A) \to H_m(N^9 - f(A)) \to$$

First letting $m = 9$ and proceeding as in Case (2) we obtain (a) above. Similarly, letting $m = 10, 11$ we obtain (b) and (c) above. We shall omit these details.

https://doi.org/10.4153/CJM-1969-017-1 Published online by Cambridge University Press

In Theorem 2.2 the cardinality of the singular set A was related to the betti numbers of the total space M^n. In the following proposition $\#(A)$ is related to the betti numbers of $N^p - f(A)$. The proof follows by similar techniques and is omitted.

PROPOSITION 2.3. *If f: $M^n \to N^p$ is the projection map of an MS-fibering of manifolds with finite non-empty singular set A and M^n is 1-connected, then*

$$\#(A) = b_{n/2}(N^p - (A))f + 1.$$

3. The structure theorem.

THEOREM 3.1. *If f: $M^n \to S^p$, M^n 1-connected, $b_{n/2}(M^n) = 0$ is the projection map of an MS-fibering of manifolds β with non-empty finite singular set A, then β is topologically $s(\alpha)$ for some Hopf spine α.*

Proof. By Theorem 2.2 we must have that $\#(A) = 2$. We may as well suppose the set $f(A)$ to consist of antipodal points on S^p. In this case we have $S^p - f(A)$ topologically equivalent to $S^{p-1} \times (0, 1)$. We may then conclude that $M^n - A$ is topologically just $f^{-1}(S^{p-1}) \times (0, 1)$. This then implies that M^n is the suspension of $f^{-1}(S^{p-1}) = X(\alpha)$. Moreover, $X(\alpha)$ must be a closed connected topological manifold since it is the total space of a fibering in which both the base $Y(\alpha) = S^{p-1}$ and the fiber $F(\alpha) = F(\beta)$ are closed connected manifolds.

It is clear from definitions that α is a spine of β and that $S(\alpha) = \beta$. It therefore remains to show that α is a Hopf spine. Now, by our definition of $Y(\alpha)$, it is a $(p - 1)$-sphere; thus, it will be sufficient to demonstrate that $X(\alpha)$ is a homotopy sphere (modulo the usual restrictions on the Poincaré conjecture in dim 3, 4).

We have the following isomorphisms:

$$\pi_q(M^n) = \pi_q(M^n - A) = \pi_q(X(\alpha) \times (0, 1)) = \pi_q(X(\alpha))$$

for $q \leqq n - 2$. Therefore, if we show that $\pi_q(M^n) = 0$ for this range of integers, the fact that $\operatorname{codim}(\beta; \alpha) = 1$ will imply, via (**4**, p. 357), that $X(\alpha)$ is a homotopy sphere since it is a compact finite-dimensional absolute neighbourhood retract (ANR).

Let f: $S^q \to M^n$ be a representative of a class $[f]$ in $\pi_q(M^n)$ for $q \leqq n - 2$. We wish to show that $[f] = 0$. We may suppose that f is simplicial via the simplicial approximation theorem so that f does not raise dimension. This means that f cannot be onto, so there exists a point $X_0 \in M^n - f(S^q)$. Now $M^n = S(X(\alpha))$ is a connected manifold so that we may suppose that X_0 is one of the two suspension points of $S(X(\alpha))$.

Since $f(S^q) \subseteq M^n - X_0$ and $M^n - X_0$ is collapsible onto the cone over $X(\alpha)$ and finally onto the cone point itself, the mapping f must be inessential. Hence, $[f] = 0$ and the proof is complete.

It should be noted that Theorem 3.1 takes care of the $p = 0$ case of the structure Theorem 5.2 of (**1**), without the use of tameness or flatness in the hypothesis.

COROLLARY 3.2. *If $f\colon M^n \to N^p$ is the projection map of an MS-fibering of manifolds β with finite non-empty singular set A and $b_{n/2}(M^n) = 0$ with M^n $n/4$-connected, then β admits a Hopf spine α and $\beta = S(\alpha)$.*

This result is modulo the possibility that the Poincaré conjecture may be false in dim 3, 4.

Proof. The space $M^n - A$ is $n/4$-connected, since $n - n/4 \geqq 2$. Using the fiber homotopy sequence for the associated fibering $\bar{\beta}$, namely
$$\ldots \to \pi_q(M^n - A) \to \pi_q(N^p - f(A)) \to \pi_{q-1}(F) \to \pi_{q-1}(M^n - A) \to \ldots,$$
we obtain the isomorphism

$$0 \to \pi_q(N^p - f(A)) \xrightarrow{\approx} \pi_{q-1}(F) \to 0 \tag{12}$$

for $q \leqq n/4$. But, F is a homotopy sphere of dim 1, 3 or 7 and dim $F \geqq n/4$ so that
$$\pi_{q-1}(F) = 0, \qquad q \leqq n/4,$$
and hence

$$\pi_q(N^p - f(A)) = 0, \qquad q \leqq n/4. \tag{13}$$

Then N^p is $n/4$-connected and the Hurewicz isomorphism yields

$$H_q(N^p; Z) = 0 \tag{14}$$

for $q \leqq n/4 = (p - 1)/2$. By (14) and duality for betti numbers and torsion coefficients we have that N^p is an integral homology sphere, and finally via the Hurewicz isomorphism that N^p is a homotopy sphere. Now apply Theorem 3.1 to complete the proof.

4. Examples where $b_{n/2}$ does not vanish. Let α denote the Hopf fibering $[S^n, f(\alpha), S^p, F(\alpha)]$ and $S(\alpha)$ its suspension. $S(\alpha)$ is clearly an MS-fibering of manifolds with singular set just two points. Let D^p be an open p-disc in a "trivializing" neighbourhood of the fibering $\overline{S(\alpha)}$ associated with $S(\alpha)$. (See **1** for notation.) We then have the homeomorphism
$$(Sf(\alpha))^{-1}(D^p) \approx D^p \times F(\alpha),$$
where $Sf(\alpha)$ denotes the suspension of $f(\alpha)$. Now, since $F(\alpha)$ is contained in a tubular neighbourhood, it is locally flatly embedded in the total space of $\overline{S(\alpha)}$, and hence in that of $S(\alpha)$, i.e., in S^{n+1}. Therefore, via results of Gluck (**7**) and Stallings (**15**), we may suppose that $F(\alpha)$ is an equatorial $(n - p)$-sphere in S^{n+1}.

Now, taking boundaries we obtain
$$\partial Q = \partial(XS(\alpha) - D^p \times F(\alpha)) = \partial D^p \times F(\alpha),$$
$$\partial Q' = \partial(YS(\alpha) - D^p) = \partial D^p,$$
where $XS(\alpha)$, $YS(\alpha)$ are the base and total spaces of $S(\alpha)$. Forming the doubles of Q and Q' (i.e., gluing two copies of Q and Q' along their boundaries) we

obtain manifolds $XD(\alpha)$, $YD(\alpha)$, where $YD(\alpha)$ is homeomorphic to S^{p+1}. The restriction of $Sf(\alpha)$ to Q is a fibering over Q' and extends in the natural way to a map $Df(\alpha)\colon DX(\alpha) \to DY(\alpha)$. This map is the projection map of an MS-fibering of manifolds $D(\alpha)$ whose singular set consists of exactly four points. Now,

$$Q \underset{\text{retract}}{\overset{\text{deform}}{\rightleftarrows}} XS(\alpha) - F(\alpha) \overset{\text{homeo}}{\approx} S^{n+1} - S^{n-p} \underset{\text{retract}}{\overset{\text{deform}}{\rightleftarrows}} S^p,$$

where the homeomorphism is given by the local flatness of $F(\alpha)$ in S^{n+1}. This means that Q is simply connected and via the Van Kampen theorem due to Olum (**8**), we have that $DX(\alpha)$ simply connected as well. Hence,

$$b_{n/2}(DX(\alpha)) \neq 0$$

by Theorem 2.2.

In closing we conjecture the following:

(A) For $n = 2, 4, 8$, $S^n \times S^n$ fibers over S^{n+1} with finite singular set;

(B) Any MS-fibering of manifolds β with projection $M^n \to S^p$ and M^n simply connected, admits a spine if and only if $b_{n/2}$ vanishes.

Of course, Theorem 3.1 takes care of (2) in one direction.

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