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A GALOIS THEORY FOR THE FIELD EXTENSION $K((X))/K$

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Abstract. Let K be a field of characteristic 0, which is algebraically closed to radicals. Let $F = K((X))$ be the valued field of Laurent power series and let $G = Aut(F/K)$. We prove that if L is a subfield of F, $K \neq L$, such that L/K is a sub-extension of F/K and F/L is a Galois algebraic extension (L/K is Galois coalgebraic in F/K), then L is closed in F, F/L is a finite extension and $Gal(F/L)$ is a finite cyclic group of G. We also prove that there is a one-to-one and onto correspondence between the set of all finite subgroups of G and the set of all Galois coalgebraic sub-extensions of F/K. Some other auxiliary results which are useful by their own are given.

2010 *Mathematics Subject Classification.* Primary 12F10, 13F25, 12J20; Secondary 12J10, 12E99, 12F99.

1. Introduction. In [**5**], O. F. G. Schilling considered the automorphisms group $G = Aut(F/K)$, where $F = K((X))$ is the Laurent formal power series valued field with coefficients in a field K of characteristic 0. He proved that any $\sigma \in G$ is continuous relative to the X-order topology and $\sigma(X) = u_1X + u_2X^2 + \ldots$, where $u_1, u_2, \ldots \in K$ and $u_1 \neq 0$ (see also Part 1 of the present paper).

Using these basic ideas of Schilling, in this paper we try to construct a Galois type theory for some sub-extensions of F/K. For this, in Part 2 we use a version of Krasner Lemma (Lemma 1) to prove that if (T, w) is a Krull valued field and P is a subfield of it, such that T/P is a Galois algebraic extension and w is the only valuation of T which extends the restriction w_P of w to P, then P is closed in T (Corollary 1). Using this last result, we prove in Theorem 1 that for any Galois coalgebraic sub-extension $L/K \subset F/K$ (F/L is a Galois algebraic extension), L is closed, F/L is finite and $Gal(F/L)$ is a finite cyclic subgroup of G.

In Theorem 2, we prove that a subgroup H of G is finite if and only if its fixed subfield L_H of F is $\neq K$. Moreover, every finite subgroup of G is cyclic (Corollary 2). In Theorem 3 we provide a Galois type theory for all the Galois coalgebraic sub-extension

L/K of F/K. These last ones are in one-to-one and onto correspondence with the set of all finite (cyclic) subgroups of G. With the additional hypothesis that K be algebraically closed to radicals, we prove in Theorem 4 that for any two power series f and g of the same order $n \in \mathbb{N}$ does exist exactly n automorphisms σ in G such that $\sigma(f) = g$. Using this result, in Corollary 3 and Corollary 4 we prove (if K is algebraically closed to radicals) that any finite subgroup of G is conjugate with a Galois group of the type $Gal(F/K((X^n)))$ for a $n \in \mathbb{N}$.

We hope that these elementary results on $K((X))$ and on its group of K-automorphisms provide a serious starting point for a deeper study of the structure of all sub-extensions of $K((X))/K$ and of $\overline{K((X))}/K$, where $\overline{K((X))}$ is an algebraic closure of $K((X))$.

(1) Let K be field of characteristic 0. Let $F = K((X))$ be the field of Laurent power series $f = \sum_{j>-\infty}^{\infty} a_j X^j$ with coefficients $a_j \in K$. Let v be the usual X- order valuation on $F : v(f) = \min\{j : a_j \neq 0\}$. By $G = Aut(F/K)$ we mean all the field automorphisms φ of F such that $\varphi(\alpha) = \alpha$ for any $\alpha \in K$. For $\varphi \in G$ the map $v^\varphi : F^* \to \mathbb{R}$, $v^\varphi(f) = v(\varphi^{-1}(f))$ is a new valuation on F with the valuation ring $\varphi(K[[X]])$, where $K[[X]] = \{g \in F : v(g) \geq 0\}$ is the valuation ring of (F, v). It is easy to prove (see also [**5**, Proof of Lemma 1]) that F is also complete relative to this last valuation v^φ. Since F is complete relative to v and v^φ and since F is not algebraically closed, v and v^φ must be equivalent (see [**4**] or [**6**]). So there exists $s \in \mathbb{N}^* = \{1, 2, \ldots\}$ such that $v^\varphi(f) = sv(f)$ for any $f \in F$. In particular, φ is continuous relative to the topology induced by v on F. Hence φ is completely determined by $\varphi(X) = c_1 X + c_2 X^2 + \ldots$, where $c_1 \neq 0$ and G is isomorphic with the group U of all the series of this last type, with respect to the usual composition between two power series. Moreover, the above s is 1, so $v(\varphi(f)) = v(f)$ and so any $\varphi \in G$ preserves the order of series.

In the following, we freely use the fact that any K-automorphism of $G = Aut(F/K)$ is continuous, i.e. whenever $\sigma \in G$, one has $\sigma(\sum_{j>-\infty}^{\infty} a_j X^j) = \sum_{j>-\infty}^{\infty} a_j \sigma(X)^j$.

(2) Let L/K be a sub-extension of F/K such that $L \neq K$ and L is (topologically) closed in F. Let v_L be the induced valuation (by v) on L and let $\Gamma(F)$, $\Gamma(L)$ be the value groups of F and L, respectively. Since $\Gamma(F) = \mathbb{Z}$, $\Gamma(L) = n\mathbb{Z}$ for a natural number $n \neq 0$. Since L is a complete discrete rank 1 valued field with residue field K, the valued field extension F/L is totally and tamely ramified and $n = [F : L]$, i.e. the codimension of L in F is finite and equal to the index of $\Gamma(L)$ in $\Gamma(F) = \mathbb{Z}$. A basic result in valuation theory (see e.g. [**3**, Proposition 4.4]) says that there is a power series $f \in K[[X]]$ with $v(f) = n$ such that $L = K((f))$. It is easy to see that $\{1, X, X^2, \ldots, X^{n-1}\}$ is a basis of the vector space F over L. Since the nth roots of unity are in K, the extension $L \subset F$ is a Galois extension with the Galois group $G_L = Gal(F/L)$ a cyclic group of order n, $F = L(\sqrt[n]{fu})$, u a unit in F and $v(f) = n$ (see [**7**, Proposition 3-4-3]).

We recall now a version of the known 'Krasner Lemma' (see [**1**] for the original version).

Let (T, w) be an arbitrary (Krull) non-trivial valued field and let P be a proper subfield of it such that T/P is a Galois algebraic extension and w is the unique valuation on T which extends w_P, the restriction of w to P. For any $\alpha \in T, \alpha \notin P$, one defines $\omega(\alpha) = \max\{w(\alpha - \alpha')\}$, where α' runs on the set of all conjugates α' of α, distinct of α.

LEMMA 1. *Let (T, w) and P be as above. Let $\alpha \in T \setminus P$ and $\beta \in T$ such that $w(\alpha - \beta) > \omega(\alpha)$. Then $P(\alpha) \subset P(\beta)$.*

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Proof. Assume that $\alpha \notin P(\beta)$. Then there exists $\sigma \in Gal(T/P)$ such that $\sigma(\beta) = \beta$ and $\sigma(\alpha) \neq \alpha$. Hence $\omega(\alpha) \geq w(\alpha - \sigma(\alpha)) \geq \min\{w(\alpha - \beta), w(\sigma(\beta) - \sigma(\alpha))\}$. But $w \circ \sigma$ is a new valuation on T which extends w_P, thus $w(\sigma(\alpha - \beta)) = w(\alpha - \beta)$, i.e. $\omega(\alpha) \geq w(\alpha - \beta)$, which contradicts the hypothesis. Hence we must conclude that $P(\alpha) \subset P(\beta)$. □

COROLLARY 1. *Let (T, w) and P like in Lemma 1 and let $P' \supset P$, $P' \subset T$ such that P is dense in P' relative to the topology induced by w. Then $P = P'$.*

Proof. Suppose that α is in P' and α is not in P. Since P is dense in P' one can choose a $\beta \in P$ such that $\omega(\alpha) < w(\alpha - \beta)$. By Lemma 1, $P(\alpha) \subset P(\beta) = P$ and so $\alpha \in P$, a contradiction. Hence $P = P'$. □

(3) Let us apply these results to our situation.

THEOREM 1. *Let L/K be a sub-extension of F/K, where $F = K((X))$. Suppose that either*

(i) *F/L is a Galois algebraic extension, or*

(ii) *v is the only extension to F of the restriction v_L of v to L.*

Then L is (topologically) closed, $[F : L] < \infty$ and $G_L = Gal(F/L)$ is a finite cyclic subgroup of $G = Aut(F/K)$.

Proof. Let $\widetilde{L}$ be the (topological) closure of L in F. From the above remarks (see Part 2), $\widetilde{L} = K((f))$, where $f \in K[[X]]$, $v(f) = n$ and $[F : \widetilde{L}] = n$, for a natural number $n \neq 0$. Since F/L is a Galois extension (in case [i]) any extension w of v_L to F is of the form: $w = v \circ \varphi$, where $\varphi \in Aut(F/L)$ (see [**3**, p. 167, Proposition 9.1]). But φ is also a continuous K-automorphism of F, so $w = v \circ \varphi = v$ (see Part 1). This means that v is the unique valuation on F which extends v_L (i.e. the case [ii]). Now, since L is dense in $\widetilde{L}$, one can apply Corollary 1 and find that $L = \widetilde{L}$. So that the other statements follow easily from this last equality and the remarks from the beginning of Part 2. □

In the proof of Theorem 1 we saw that if F/L is a Galois algebraic extension, then v is the unique extension of v_L to F. Conversely, if v is the unique extension of v_L to F, then we also proved in Theorem 1 that $L = \widetilde{L} = K((f))$ for a $f \in F$ and that F/L is a cyclic extension, i.e. a Galois algebraic extension. Hence $L = \widetilde{L}$ if and only if F/L is a Galois algebraic extension. Hence, in the statement of Theorem 1, the hypotheses (i) and (ii) are equivalent.

REMARK 1. If F/L is algebraic but not Galois, the statement of Theorem 1 may be not true. Let, for instance, $K = \mathbb{Q}(\sqrt[3]{2})$, $F = K((X))$ and $L = K(T)$, where T is a transcendence base of F over K, which contains X. The equation $Z^3 - 2X^3 = 0$ has only one root in L ($i \notin L$). So, the extension F/L is not Galois. If the conclusions of Theorem 1 were true, then $L = K((f))$, for an $f \in F$ (see Part 2). But this last result is not true since $(1 + f)^{1/2}$ would be in $L = K(T)$, which is not the case.

THEOREM 2. *Let H be a subgroup of $G = Aut(F/K)$, and let L_H be the fixed subfield by H. Then,*

(a) *H is infinite if and only if $L_H = K$, and*

(b) *if H is finite, then $L_H = K((f))$ for a $f \in K[[X]]$; H is a cyclic subgroup of G with $|H| = v(f)$ and $H = Gal(F/L_H)$.*

Proof. (a) Assume H is infinite and $L_H \supsetneq K$. Then $L_H = \widetilde{L}_H$ because any automorphism in H is continuous. Hence $L_H = K((f))$ and $[F : L_H] = n = v(f)$. From

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Part 2 and Theorem 1, $G_{L_H} = Gal(F/L_H)$ is cyclic and contains H. So H itself is cyclic and finite, a contradiction. Therefore, if H is infinite, then $L_H = K$. Conversely, if $L_H = K$, then H cannot be finite. Otherwise $[F : L_H] = |H| < \infty$ (see [**2**, Section 14.2, Theorem 9]) and in this last case F/L_H would be algebraic, which is a contradiction because $L_H = K$.

(b) If H is finite, $[F : L_H] = |H| < \infty$, like in (a) and $L_H = \widetilde{L}_H = K((f))$, so H becomes a subgroup of order $n = v(f) = [F : L_H]$ of $G_{L_H} = Gal(F/L_H)$. Since $\left|G_{L_H}\right| = n$ one gets that $H = G_{L_H}$. □

COROLLARY 2. *Every finite subgroup of $G = Aut(K((X))/K)$ is cyclic.*

A sub-extension L/K of F/K is called *coalgebraic if* F/L is algebraic, and it is called *Galois coalgebraic* if in addition F/L is Galois.

THEOREM 3. *Let $\mathcal{F}(G)$ be the set of all finite (cyclic) subgroups H of G and let $Gcoalg(F)$ be the set of all Galois coalgebraic sub-extensions L/K of F/K. Let $\varphi : \mathcal{F}(G) \to Gcoalg(F)$ be the mapping which carries H in L_H, the fixed subfield of H and let $\psi : Gcoalg(F) \to \mathcal{F}(G)$, $\psi(L) = G_L = Gal(F/L)$. Then $\varphi \circ \psi = 1_{Gcoalg}$ and $\psi \circ \varphi = 1_{\mathcal{F}(G)}$.*

Proof. Let H be $\psi(L)$ with F/L a Galois algebraic extension. Theorem 1 says that $L = \widetilde{L}$ and $G_L = Gal(F/L)$ is exactly the Galois group of F/L. So $\varphi(\psi(L)) = L_H = L$, because $[F : L_H] = n = [F : L]$. Let now $T = L_H$ be $\varphi(H)$ for a $H \in \mathcal{F}(G)$. Then H is cyclic and $[F : L_H] = |H|$, i. e. $\psi(\varphi(H)) = Gal(F/L_H) = H$ and the proof is complete. □

THEOREM 4. *Let K be algebraically closed to radicals and of characteristic 0. Let f and g be in $F = K((X))$ such that $n = v(f) = v(g)$. Then there exists n distinct automorphisms $\sigma \in Aut(F/K)$ such that $\sigma(f) = g$. In particular, $K((f))$ is K- isomorphic and homeomorphic with $K((g))$ by a restriction of an automorphism of G.*

Proof. It is enough to consider $g = X^n$ and to construct $\sigma \in G$ with $\sigma(X^n) = f$. To construct σ we need to find $u_1 \neq 0, u_2, \ldots$ in K such that if we define $\sigma(X) = u_1X + u_2X^2 + \ldots$ then $\sigma(X^n) = (u_1X + u_2X^2 + \ldots)^n = f = a_nX^n + a_{n+1}X^{n+1} + \ldots$, where $a_n \neq 0, a_{n+1}, \ldots$ are known elements in K. So we must determine $u_1, u_2, \ldots$ as functions of $a_n, a_{n+1}, \ldots$. Since $u_1^n = a_n$ and since K is algebraically closed to radicals, one has n distinct possibilities for u_1, namely the nth roots of a_n in K (K has the characteristic 0!). Let us fix such a root $u_1 \in K$. Considering the equality $(u_1X + u_2X^2 + \ldots)^n = a_nX^n + a_{n+1}X^{n+1} + \ldots$, $\mathrm{mod}(X^{n+2})$, $\mathrm{mod}(X^{n+3}), \ldots$, one can use mathematical induction to prove that $u_2, u_3, \ldots$ can be uniquely expressed in function of u_1 and $a_n, a_{n+1}, \ldots$. The other statements follow easily from this last observation. □

In the following, we assume that K is algebraically closed to radicals.

REMARK 2. G acts transitively on the set C_n of elements f of the same valuation $n = v(f)$.

COROLLARY 3. *Any Galois coalgebraic sub-extension L/K of F/K is conjugate relative to G with a subfield of F of the type $K((X^n))$, $n = 1, 2, \ldots$. In particular, if $D_n = Gal(F/K((X^n)))$, then $Gal(F/L) = \sigma D_n \sigma^{-1}$, where σ is an automorphism in G.*

COROLLARY 4. *Let $\tau \in G$ be an automorphism of order m. Let ζ_m be a primitive m-th root of 1 in K and let $\mu_{\zeta_m} \in G$ such that $\mu_{\zeta_m}(X) = \zeta_m X$. Then μ_{ζ_m} is a generator of $D_m = Gal(F/K((X^m)))$, the normalizer $N_G(D_m)$ of D_m in G is the infinite subgroup*

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*$\{\omega \in G \mid \omega(X) = aX,\ a \neq 0\}$ and there exists a $\sigma \in G$ such that $\tau = \sigma\mu_{\zeta_m}^h\sigma^{-1}$, where $(h, m) = 1$ and h depends only on τ. Moreover, h is unique (*mod m*) with the property that $\tau = \sigma\mu_{\zeta_m}^h\sigma^{-1}$ for a σ in G.*

Proof. Let $H = \langle\tau\rangle$ be the cyclic subgroup of G generated by τ, and let $L_H = K((f))$ be the fixed subfield of F relative to H. Let $\sigma \in G$ such that $\sigma(X^m) = f$ (the number of such σ's is finite, see Theorem 4). Since $Gal(F/L) = \sigma D_m\sigma^{-1}$ and $\tau \in Gal(F/L)$, one gets that $\tau = \sigma\mu_{\zeta_m}^h\sigma^{-1}$ with $(h, m) = 1$. If we change σ with σ' such that $\sigma'(X^m) = f$, then $\sigma'(X) = \sigma(\mu_{\zeta_m}^t(X)) = \mu_{\zeta_m}^t\sigma(X)$, where $(t, m) = 1$ (see Theorem 4). So $\tau = \sigma'\mu_{\zeta_m}^h(\sigma')^{-1}$, i. e. h depends only on τ and not on σ. This h is unique, because $\tau = \sigma_1\mu_{\zeta_m}^{h_1}\sigma_1^{-1} = \sigma\mu_{\zeta_m}^{h_1}\sigma^{-1} = \sigma\mu_{\zeta_m}^h\sigma^{-1}$ implies $h_1 = h \bmod m$. A simple computation shows us the structure of the normalizer $N_G(D_m)$. □

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