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OSCILLATIONS OF SECOND ORDER NEUTRAL EQUATIONS

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1. Introduction. Consider the second order neutral differential equation

$$(1) \quad \frac{d^2}{dt^2}[y(t) + py(t - \tau)] + qy(t - \sigma) = 0$$

where the coefficients p and q and the deviating arguments τ and σ are real numbers. The characteristic equation of Eq. (1) is

$$(2) \quad F(\lambda) \equiv \lambda^2 + p\lambda^2 e^{-\lambda\tau} + qe^{-\lambda\sigma} = 0.$$

The main result in this paper is the following necessary and sufficient condition for all solutions of Eq. (1) to oscillate.

THEOREM. *The following statements are equivalent*:
(a) *Every solution of Eq.* (1) *oscillates.*
(b) *Equation* (2) *has no real roots.*

On the basis of the analysis which was presented in **[3]**, it suffices to give the proof of this theorem in the special case where

$$(3) \quad q > 0, \tau < 0, \sigma < 0 \quad \text{and} \quad p < 0.$$

2. Proof of the theorem. The proof that (a) $\Rightarrow$ (b) is obvious. However, the proof that (b) $\Rightarrow$ (a) is quite complicated and will be accomplished by examining various cases and by establishing a series of lemmas.

In the sequel we will assume, without further mention, that (3) holds. We will also assume that Eq. (2) has no real root and, for the sake of contradiction, we assume that Eq. (1) has an eventually positive solution $y(t)$.

LEMMA 1. (a)

$$(4) \quad \sigma < \tau.$$

(b) *There exists a positive constant m such that*

$$(5) \quad F(\lambda) \equiv \lambda^2 + \lambda^2 pe^{-\lambda\tau} + qe^{-\lambda\sigma} \geqq m, \quad \text{for all } \lambda \in \mathbf{R}.$$

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Proof. (a) Otherwise, $\sigma \geqq \tau$ and so

$$F(\infty) = -\infty.$$

But $F(0) = q > 0$. This is impossible because the characteristic equation $F(\lambda) = 0$ has no real roots.

(b) We have $F(-\infty) = F(\infty) = \infty$ and so $F(\lambda) > 0$ for all $\lambda \in \mathbf{R}$. Hence

$$m \equiv \min_{\lambda \in \mathbf{R}} F(\lambda)$$

exists and is a positive number which satisfies (5).

Set

$$(6) \quad v(t) = -[y(t) + py(t - \tau)].$$

LEMMA 2. (a) $v(t)$ *is a twice continuously differentiable solution of* Eq. (1). *That is,*

$$(7) \quad \ddot{v}(t) + p\ddot{v}(t - \tau) + qv(t - \sigma) = 0.$$

(b) *Either*

$$(8) \quad v(t) > 0, \dot{v}(t) < 0, \ddot{v}(t) > 0 \quad \textit{and} \quad \lim_{t \to \infty} v(t) = \lim_{t \to \infty} v(t) = 0$$

or

$$(9) \quad v(t) > 0, \dot{v}(t) > 0, \ddot{v}(t) > 0 \quad \textit{and} \quad \lim_{t \to \infty} v(t) = \lim_{t \to \infty} \dot{v}(t) = \infty.$$

(c) *When* (8) *holds then* $p < -1$.

Proof. (a) It follows immediately from the linearity and the autonomous character of Eq. (1).

(b) We have

$$(10) \quad \ddot{v}(t) = qy(t - \sigma) > 0$$

which implies that $\dot{v}(t)$ is strictly increasing and so either

$$(11) \quad \lim_{t \to \infty} \dot{v}(t) = \infty$$

or

$$(12) \quad \lim_{t \to \infty} \dot{v}(t) \equiv l \in \mathbf{R}.$$

Clearly (11) implies (9). Now let (12) hold. First we will show that $l = 0$. Indeed, by integrating (10) from t_0 to t and letting $t \to \infty$ we see that

$$l - \dot{v}(t_0) = q \int_{t_0}^{\infty} y(s - \sigma) ds$$

which shows that

$$y \in L^1[t_0, \infty).$$

Hence,

(13) $\quad v \in L^1[t_0, \infty)$

and so $l = 0$. Thus $\dot{v}(t)$ increases to zero which implies that eventually

$$\dot{v}(t) < 0.$$

But then $v(t)$ decreases and in view of (13),

$$\lim_{t\to\infty} v(t) = 0.$$

Therefore, $v(t)$ decreases to zero which implies that

$$v(t) > 0.$$

(c) For the sake of contradiction assume that (8) holds and that $p \geqq -1$. From parts (a) and (b) we see that $v(t)$ is a positive solution of Eq. (1) and, therefore,

$$-[v(t) + pv(t - \tau)] > 0,$$

which implies that

$$v(t) < - pv(t - \tau) \leqq v(t - \tau).$$

But this contradicts the fact that $v(t)$ is a decreasing function.

Next, we will define two sets corresponding to whether (12) or (13) is satisfied. Let W^- and W^+ be the set of all functions of the form

$$w(t) = -[v(t) + pv(t - \tau)]$$

where $v(t)$ is a twice continuously differentiable solution of Eq. (1) which satisfies (8) and (9) respectively. In view of Lemma 2, either W^- or W^+ is nonempty. Also, an argument similar to that of Lemma 2 shows that each function $w \in W^- \cup W^+$ is a four times continuously differentiable solution of Eq. (1), that is, $w \in C^4$ and

(14) $\quad \ddot{w}(t) + p\ddot{w}(t - \tau) + qw(t - \sigma) = 0.$

Also, there is a solution $v \in C^2$ of Eq. (1) which satisfies (8) if $w \in W^-$ or (9) if $w \in W^+$ such that

(15) $\quad \ddot{w}(t) = qv(t - \sigma).$

Clearly, every function $w \in W^-$ satisfies

(16) $\quad w(t) > 0, \dot{w}(t) < 0, \ddot{w}(t) > 0 \quad \text{and} \quad \lim_{t\to\infty} w(t) = \lim_{t\to\infty} \dot{w}(t) = 0$

while every function $w \in W^+$ satisfies

(17) $w(t) > 0, \dot{w}(t) > 0, \ddot{w}(t) > 0$ and $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \dot{w}(t) = \infty.$

Furthermore,

$$w(t) \in W^- \Rightarrow -[w(t) + pw(t - \tau)] \in W^-$$

and

$$w(t) \in W^+ \Rightarrow -[w(t) + pw(t - \tau)] \in W^+.$$

Finally, w_1 and $w_2 \in W^-$ (respectively in W^+) and $a, b > 0 \Rightarrow aw_1 + bw_2 \in W^-$ (respectively in W^+).

With each function $w \in W^- \cup W^+$ define the set

$$\Lambda(w) = \{\lambda \geqq 0: \ddot{w}(t) - \lambda^2 w(t) \geqq 0\}.$$

Clearly, $0 \in \Lambda(w)$ and if $\lambda \in \Lambda(w)$ then $[0, \lambda] \subseteq \Lambda(w)$. That is, $\Lambda(w)$ is a nonempty subinterval of $\mathbf{R}^+$.

First, we will assume that $W^- = \emptyset$ (i.e., $W^+ \neq \emptyset$) and we will show that this leads to a contradiction.

LEMMA 3. (a) *Let* $w \in W^+$. *Then* $\lambda_0 \equiv (q/-p)^{1/2} \in \Lambda(w)$.

(b) $\Lambda(w)$ *is bounded above by a positive constant* μ, *for any* $w \in W^+$.

(c) *Let* $w \in W^+$ *and* $\lambda \in \Lambda(w)$. *Then*

$$\dot{w}(t) - \lambda w(t) \geqq 0.$$

Proof. (a) From (14) and the fact that $\ddot{w}(t) > 0$ we have that

(18) $p\ddot{w}(t - \tau) + qw(t - \sigma) < 0$

or

$$\ddot{w}(t) + \frac{q}{p}w(t + (\tau - \sigma)) > 0.$$

The increasing nature of $w(t)$ and the fact that $\tau > \sigma$ imply that

$$\ddot{w}(t) + \frac{q}{p}w(t) > 0$$

which shows that

$$\lambda_0 \equiv \left(\frac{q}{-p}\right)^{1/2} \in \Lambda(w).$$

(b) By integrating (18) from $t - \alpha$ to t, with $\alpha > 0$ we find

$$\dot{w}(t) - \dot{w}(t - \alpha) + \frac{q}{p}\int_{t-\alpha}^{t} w(s + (\tau - \sigma))ds > 0$$

which yields

$$\dot{w}(t) + \frac{q}{p}\alpha w(t - \alpha + (\tau - \sigma)) > 0.$$

By integrating again from $t - \beta$ to t, with $\beta > 0$ we find

$$w(t) - w(t - \beta) + \frac{q}{p}\alpha \int_{t-\beta}^{t} w(s - \alpha + (\tau - \sigma)ds) > 0$$

which implies

$$w(t) + \frac{q}{p}\alpha\beta w(t - (\alpha + \beta) + (\tau - \sigma)) > 0.$$

Choose

$$\alpha = \beta = (\tau - \sigma)/2 > 0.$$

Then

$$w(t) + \frac{q}{p}\frac{(\tau - \sigma)^2}{16} w\left(t + \frac{\tau - \sigma}{2}\right) > 0$$

or

$$w\left(t + \frac{\tau - \sigma}{2}\right) < Aw(t) \tag{19}$$

where

$$A = \frac{-16p}{q(\tau - \sigma)^2} > 0.$$

Now let $k \in \mathbf{N}$ be such that $-\sigma \leqq (\tau - \sigma/2)k$. Then (19) and the increasing nature of $w(t)$ imply that

$$\begin{aligned} w(t - \sigma) &\leqq w\left(t + \frac{\tau - \sigma}{2}k\right) < Aw\left(t + \frac{\tau - \sigma}{2}(k - 1)\right) \\ &< \ldots < A^k w(t). \end{aligned} \tag{20}$$

By integrating (15) from $t - \alpha$ to t, with $\alpha > 0$ we find

$$\dot{w}(t) - \dot{w}(t - \alpha) = \int_{t-\alpha}^{t} qv(s - \sigma)ds > q\alpha v(t - \alpha - \sigma)$$

or

$$\dot{w}(t) > q\alpha v(t - \alpha - \sigma).$$

By integrating again from $t - \beta$ to t, with $\beta > 0$ we find

$$\begin{aligned} w(t) - w(t - \beta) &> \int_{t-\beta}^{t} q\alpha v(s - \alpha - \sigma)ds \\ &> q\alpha\beta v(t - (\alpha + \beta) - \sigma) \end{aligned}$$

or

$$w(t) > q\alpha\beta v(t - (\alpha + \beta) - \sigma).$$

Choose $\alpha = \beta = -\sigma/2 > 0$. Then

(21) $$w(t) > \frac{q\sigma^2}{4} v(t).$$

By combining (15), (20) and (21) we have that

$$\ddot{w}(t) = qv(t - \sigma) < \frac{4}{\sigma^2} w(t - \sigma) < \frac{4A^k}{\sigma^2} w(t)$$

or

$$\ddot{w}(t) - \left(\frac{2A^{k/2}}{-\sigma}\right)^2 w(t) < 0$$

which shows that

$$\mu \equiv \frac{2A^{k/2}}{-\sigma} \notin \Lambda(w) \quad \text{for any } w \in W^+$$

proving (b).

(c) Set

$$\theta(t) = e^{-\lambda t} w(t).$$

Then

$$\dot{\theta}(t) = e^{-\lambda t}[\dot{w}(t) - \lambda w(t)],$$
$$\ddot{\theta}(t) = e^{-\lambda t}[\ddot{w}(t) - 2\lambda \dot{w}(t) + \lambda^2 w(t)]$$

and

(22) $$\ddot{\theta}(t) + 2\lambda\dot{\theta}(t) = e^{-\lambda t}[\ddot{w}(t) - \lambda^2 w(t)] \geqq 0.$$

From (22) we see that $\dot{\theta}(t)e^{2\lambda t}$ is a nondecreasing function and so if the conclusion in part (c) were false, then

(23) $$\dot{\theta}(t) < 0.$$

From (22) and (23) we see that

$$\ddot{\theta}(t) > 0$$

and so

$$\ddot{w}(t) - 2\lambda\dot{w}(t) + \lambda^2 w(t) > 0$$

which together with the hypothesis that

$$\ddot{w}(t) - \lambda^2 w(t) \geqq 0$$

implies that

(24) $\ddot{w}(t) - \lambda\dot{w}(t) > 0.$

Set

$$u(t) = -[\dot{w}(t) - \lambda w(t)].$$

Then $u(t)$ is a solution of Eq. (1) and because of (23) and (24)

(25) $u(t) > 0$ and $\dot{u}(t) < 0.$

Now using u instead of y in (6) and the hypothesis that $W^- = \emptyset$ we see, as in the proof of (9), that

$$\lim_{t\to\infty} [-[u(t) + pu(t - \tau)]] = \infty.$$

But (25) implies that

$$\lim_{t\to\infty} u(t) \in \mathbf{R},$$

and this contradiction completes the proof of Lemma 3.

By integrating both sides of (14) from $t_0 + \sigma$ to t we find

$$[\dot{w}(t) + p\dot{w}(t - \tau)] - [\dot{w}(t_0 + \sigma) + p\dot{w}(t_0 + \sigma - \tau)]$$
$$+ q\int_{t_0+\sigma}^{t} w(s - \sigma)ds = 0$$

or

(26) $$-[\dot{w}(t) + p\dot{w}(t - \tau)] = c + q\int_{t_0+\sigma}^{t-\sigma} w(s)ds,$$

where

(27) $c = -[\dot{w}(t_0 + \sigma) + p\dot{w}(t_0 + \sigma - \tau)].$

As $\dot{w}(t)$ is a solution of Eq. (1), with $w(t)$ satisfying (17), it follows from (26) that if $w \in W^+$ then

$$c + q\int_{t_0}^{t-\sigma} w(s)ds \in W^+$$

where c is the constant given by (27).

LEMMA 4. *Let* $w \in W^+$ *and* $\lambda \in \Lambda(w)$. *Set*

$$N = \frac{m}{2(-pe^{-\mu\tau} + e^{-\mu\sigma})} > 0$$

where m *is the constant defined in Lemma* 1(b), *and* μ *is the constant defined in Lemma* 3(b). *Then*

$$(\lambda^2 + N)^{1/2} \in \Lambda(z).$$

where

$$(28)\quad z(t) = -[w(t) + pw(t-\tau)] + \lambda \int_{t_0}^{t-\sigma} w(s)ds + \frac{c\lambda}{q}$$

and c is the constant given by (27).

Proof. Clearly $z(t)$ is an element of W^+. From Lemma 3(c), we have

$$(29)\quad \dot{w}(t) - \lambda w(t) \geqq 0.$$

This, together with (28), yields

$$(30)\quad \ddot{z}(t) = qw(t-\sigma) + \lambda \dot{w}(t-\sigma) \geqq (q + \lambda^2)w(t-\sigma).$$

By integrating (29) from t_0 to t, we find

$$0 \leqq w(t) - \lambda \int_{t_0}^{t-\sigma} w(s)ds$$

$$= w(t) - \lambda \int_{t_0}^{t-\sigma} w(s)ds + \lambda \int_{t}^{t-\sigma} w(s)ds$$

and so

$$(31)\quad -w(t) + \lambda \int_{t_0}^{t-\sigma} w(s)ds \leqq \lambda \int_{t}^{t-\sigma} w(s)ds.$$

Using (28), (30), (31), the increasing nature of $w(t)$ and the fact that

$$t_0 < t < t - \tau < t - \sigma,$$

we see that

$$\ddot{z}(t) - (\lambda^2 + N)z(t) \geqq (q + \lambda^2)w(t-\sigma)$$
$$- (\lambda^2 + N)\lambda \int_{t}^{t-\sigma} w(s)ds + (\lambda^2 + N)pw(t-\tau) + c_1$$

where

$$c_1 = -(\lambda^2 + N)\frac{c\lambda}{q}.$$

Set

$$\varphi(t) = w(t)e^{-\lambda t}.$$

Then

$$\dot{\varphi}(t) = e^{-\lambda t}[\dot{w}(t) - \lambda w(t)] \geqq 0$$

which shows that $\varphi(t)$ is nondecreasing and so

$$\lambda \int_{t}^{t-\sigma} w(s)ds = \lambda \int_{t}^{t-\sigma} e^{\lambda s}\varphi(s)ds \leqq \varphi(t-\sigma) \int_{t}^{t-\sigma} \lambda e^{\lambda s}ds$$
$$= \varphi(t-\sigma)e^{\lambda t}[e^{-\lambda\sigma} - 1].$$

Then

$$\begin{aligned}
&\ddot{z}(t) - (\lambda^2 + N)z(t) \\
&\geqq (q + \lambda^2)\varphi(t - \sigma)e^{\lambda(t-\sigma)} - (\lambda^2 + N)\varphi(t - \sigma)e^{\lambda t}[e^{-\lambda\sigma} - 1] \\
&+ (\lambda^2 + N)p\varphi(t - \sigma)e^{\lambda(t-\tau)} + c_1 \\
&= \varphi(t - \sigma)e^{\lambda t}\Big[qe^{-\lambda\sigma} - Ne^{-\lambda\sigma} + \lambda^2 + N + \lambda^2 pe^{-\lambda\tau} \\
&\qquad\qquad + Npe^{-\lambda\tau} + \frac{c_1 e^{-\lambda\sigma}}{w(t - \sigma)}\Big] \\
&\geqq w(t - \sigma)e^{\lambda\sigma}\Big[(\lambda^2 + pe^{-\lambda\tau} + qe^{-\lambda\sigma}) + \frac{c_1 e^{-\lambda\sigma}}{w(t - \sigma)} \\
&\qquad\qquad - N(-pe^{-\lambda\tau} + e^{-\lambda\sigma})\Big] \\
&\geqq w(t - \sigma)e^{\lambda\sigma}\Big[m + \frac{c_1 e^{-\lambda\sigma}}{w(t - \sigma)} - N(-pe^{-\mu\tau} + e^{-\mu\sigma})\Big].
\end{aligned}$$

As

$$\lim_{t\to\infty} w(t) = \infty$$

we see that for sufficiently large t,

$$m + \frac{c_1 e^{-\lambda\sigma}}{w(t - \sigma)} \geqq \frac{m}{2}.$$

Then

$$\ddot{z}(t) + (\lambda^2 + N)z(t) \geqq w(t - \sigma)e^{\lambda\sigma}\left[\frac{m}{2} - \frac{m}{2}\right] = 0$$

which completes the proof of Lemma 4.

Now consider the sequence of functions

$$z_n(t) = -[z_{n-1}(t) + pz_{n-1}(t - \tau)] + \lambda_n \int_{t_0}^{t-\sigma} z_{n-1}(s)ds + \frac{c_n\lambda_n}{q},$$
$$n = 1, 2, \dots$$

where $z_0(t)$ is the function defined by (28), λ_0 is the number defined in Lemma 3(a),

$$N = \frac{m}{2(-pe^{-\mu\tau} + e^{-\mu\sigma})},$$

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$$\lambda_n = (\lambda_{n-1}^2 + N)^{1/2}$$

and

$$c_n = -[\dot{z}_n(t_0 + \sigma) + p\dot{z}(t_0 + \sigma - \tau)].$$

A repeated application of Lemma 4 shows that

$$\lambda_n \in \Lambda(z_{n-1}), \quad \text{for } n = 1, 2, \dots .$$

Clearly

$$\lim_{n\to\infty} \lambda_n = \infty$$

which contradicts the fact proved in Lemma 3(b) that

$$\lambda_n \leqq \mu \quad \text{for all } n = 1, 2, \dots .$$

This completes the proof of the Theorem when $W^- = \emptyset$.

Next we assume that $W^- \neq \emptyset$. Then in view of Lemma 2 (c), $p < -1$.

LEMMA 5. (a) *Let* $w \in W^-$ *and* $k \in \mathbf{N}$ *be such that* $-k\tau > \tau - \sigma$. *Then*

$$\lambda_1 \equiv \left[\frac{q}{(-p)^{k+1}}\right]^{1/2} \in \Lambda(w).$$

(b) *Let* $w \in W^-$ *and* $\lambda \in \Lambda(w)$. *Then*

$$\dot{w}(t) + \lambda w(t) \leqq 0.$$

(c) $\Lambda(w)$ *is bounded above by a positive constant* λ_2, *for any* $w \in W^-$.

Proof. (a) Let $k \in \mathbf{N}$ be such that $-k\tau > \tau - \sigma > 0$. For $w \in W^-$ we have

$$-[w(t) + pw(t - \tau)] > 0 \tag{32}$$

and so

$$w(t) < -pw(t - \tau) < (-p)^k w(t - k\tau) < (-p)^k w(t + (\tau - \sigma))$$

which together with (18) implies that

$$\ddot{w}(t) - \left[\frac{q}{(-p)^{k+1}}\right] w(t) \geqq 0.$$

(b) Set

$$\psi(t) = \dot{w}(t) + \lambda w(t).$$

Then

$$\dot{\psi}(t) - \lambda\psi(t) = \ddot{w}(t) - \lambda^2 w(t) \geqq 0$$

and so

$$\frac{d}{dt}[\psi(t)e^{-\lambda t}] \geqq 0$$

showing that $\psi(t)e^{-\lambda t}$ is a nondecreasing function.

Now observe that

$$\lim_{t\to\infty} [\psi(t)e^{-\lambda t}] = 0$$

which implies that

$$\psi(t)e^{-\lambda t} \leqq 0$$

and so

$$\psi(t) \leqq 0.$$

(c) Otherwise

$$\lambda_2 \equiv \frac{1}{-\tau} \ln(-p) \in \Lambda(w), \quad \text{for some } w \in W^-.$$

Then from part (b)

$$\dot{w}(t) + \lambda_2 w(t) \leqq 0$$

which shows that the function

$$w(t)e^{\lambda_2 t}$$

is nonincreasing and so

$$w(t)e^{\lambda_2 t} \geqq w(t - \tau)e^{\lambda_2(t-\tau)}$$

or

$$w(t) \geqq e^{-\lambda_2\tau}w(t - \tau) = -pw(t - \tau)$$

contradicting (32). The proof of Lemma 5 is complete.

For any function $w \in W^-$ by integrating (14) from t to t_1, twice, and by letting $t_1 \to \infty$ and using (16), we find that

$$-[w(t) + pw(t - \tau)] = q \int_t^\infty \int_s^\infty w(\xi - \sigma)d\xi ds$$

or

(33) $$-[w(t + \sigma - \tau) + pw(t + \sigma - 2\tau)] = q \int_{t-\tau}^\infty \int_s^\infty w(\xi)d\xi ds.$$

This shows that the right-hand-side of (33) is an element of W^-, and so for any $w \in W^-$ the function

(34) $$z(t) = -[w(t) + pw(t - \tau)] + q \int_{t-\tau}^\infty \int_s^\infty w(\xi)d\xi ds$$

is an element of W^-.

LEMMA 6. *Let* $w \in W^-$ *and let* $\lambda \in \Lambda(w)$ *with*

$$\lambda \geqq \lambda_0 \equiv [q/(-p)^{k+1}]^{1/2}.$$

Set

$$K = \frac{m}{-p + q/\lambda_0^2}$$

where m *is the constant defined in Lemma* 1(b). *Then*

$$(\lambda^2 + K)^{1/2} \in \Lambda(z)$$

where z *is the function defined by* (34).

Proof. Clearly $z(t) \in W^-$. Also as $\lambda \in \Lambda(w)$,

$$w(t - \sigma) - \lambda^2 \int_{t-\sigma}^{\infty} \int_{s}^{\infty} w(\xi)d\xi ds$$

$$\geqq w(t - \sigma) - \int_{t-\sigma}^{\infty} \int_{s}^{\infty} \ddot{w}(\xi)d\xi ds = 0$$

or

$$w(t - \sigma) \geqq \lambda^2 \int_{t-\sigma}^{\infty} \int_{s}^{\infty} w(\xi)d\xi ds. \tag{35}$$

By using (35) we have

$$\ddot{z}(t) - (\lambda^2 + K)z(t)$$

$$= qw(t - \sigma) + qw(t - \tau) + (\lambda^2 + K)w(t) + (\lambda^2 + K)pw(t - \tau)$$

$$- (\lambda^2 + K)q \int_{t-\tau}^{\infty} \int_{s}^{\infty} w(\xi)d\xi ds$$

$$\geqq q\lambda^2 \int_{t-\sigma}^{\infty} \int_{s}^{\infty} w(\xi)d\xi ds + qw(t - \tau) + \lambda^2 w(t) + \lambda^2 pw(t - \tau)$$

$$- q\lambda^2 \int_{t-\tau}^{\infty} \int_{s}^{\infty} w(\xi)d\xi ds$$

$$+ Kw(t) + Kpw(t - \tau) - Kq \int_{t-\tau}^{\infty} \int_{s}^{\infty} w(\xi)d\xi ds$$

$$\geqq q\lambda^2 \left[- \int_{t-\tau}^{t-\sigma} \int_{s}^{\infty} w(\xi)d\xi ds \right] + qw(t - \tau) + \lambda^2 w(t)$$

$$+ \lambda^2 pw(t - \tau) + Kpw(t - \tau) - \frac{Kq}{\lambda_0^2}\lambda^2 \int_{t-\tau}^{\infty} \int_{s}^{\infty} w(\xi)d(\xi)ds.$$

Now set

$$\varphi(t) = w(t)e^{\lambda t}.$$

Then

$$\dot{\varphi}(t) = e^{\lambda t}[\dot{w}(t) + \lambda w(t)] \leqq 0$$

which shows that $\varphi(t)$ is nonincreasing and so

(36) $$\lambda^2\left[-\int_{t-\tau}^{t-\sigma}\int_s^\infty w(\xi)d\xi ds\right]$$
$$= -\lambda^2 \int_{t-\tau}^{t-\sigma}\int_s^\infty e^{-\lambda\xi}\varphi(\xi)d\xi ds \geqq \varphi(t-\tau)\lambda \int_{t-\tau}^{t-\sigma} - e^{-\lambda s}ds$$
$$= \varphi(t-\tau)\int_{t-\tau}^{t-\sigma} - \lambda e^{-\lambda s}ds = \varphi(t-\tau)e^{-\lambda(t-\tau)}[e^{-\lambda(\tau-\sigma)} - 1]$$
$$= w(t-\tau)[e^{-\lambda(\tau-\sigma)} - 1].$$

Also the nonincreasing nature of $\varphi(t)$ implies that

$$w(t)e^{\lambda t} \geqq w(t-\tau)e^{\lambda(t-\tau)}$$

or

(37) $$w(t) \geqq w(t-\tau)e^{-\lambda\tau}.$$

By using (35), (36) and (37) we find

$$\ddot{z}(t) - (\lambda^2 + K)z(t)$$
$$\geqq qw(t-\tau)[e^{-\lambda(\tau-\sigma)} - 1] + qw(t-\tau) + \lambda^2 w(t-\tau)e^{-\lambda\tau}$$
$$+ \lambda^2 pw(t-\tau) + Kpw(t-\tau) - \frac{Kq}{\lambda_0^2}w(t-\tau)$$
$$= w(t-\tau)e^{-\lambda\tau}\left[(qe^{\lambda\sigma} + \lambda^2 + \lambda^2 pe^{\lambda\tau}) - K\left(\frac{q}{\lambda_0^2} - p\right)e^{\lambda\tau}\right]$$
$$\geqq w(t-\tau)e^{-\lambda\tau}\left[m - K\left(-p + \frac{q}{\lambda_0^2}\right)\right]$$
$$= w(t-\tau)e^{-\lambda\tau}[m - m]$$
$$= 0$$

which completes the proof of Lemma 6.

Now consider the sequence of functions

$$z_n(t) = -[z_{n-1}(t) + pz_{n-1}(t-\tau)] + q\int_{t-\tau}^\infty\int_s^\infty z_{n-1}(\xi)d\xi ds,$$
$$n = 1, 2, \dots,$$

where $z_0(t)$ is the function $z(t)$ defined in (34),

$$\lambda_0 = \left[\frac{q}{(-p)^{k+1}}\right]^{1/2},$$

$$K = \frac{m}{-p + q/\lambda_0^2}$$

and

$$\lambda_n = (\lambda_{n-1}^2 + K)^{1/2}.$$

A repeated application of Lemma 6 shows that

$$\lambda_n \in \Lambda(z_{n-1}) \quad \text{for } n = 1, 2, \ldots .$$

Clearly

$$\lim_{n \to \infty} \lambda_n = \infty$$

contradicting the fact proved in Lemma 5 (c), that $\Lambda(w)$ is bounded above for any $w \in W^-$. This contradiction completes the proof of the theorem.

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