Let M be the mid point of PQ.

$AM^2 + PM^2 = AO^2 + r^2$,

$\therefore\ AM^2 + r^2 - OM^2 = AO^2 + r^2$,

$\therefore\ \angle AOM = 90°$

But $\angle OMP = 90°$

$\therefore$ PQ is parallel to OA.

I venture to say that 99 per cent. of casual readers will see nothing wrong about this. But what if M is at the centre?

(The student of elementary geometrical conics will easily prove that if, more generally, $AP^2 + AQ^2 = c^2$, then PQ envelopes a parabola with focus at O. In the special case the parabolic envelope breaks down into a couple of points, one at infinity, the other the centre).

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The solution of "homogeneous" quadratics.

$$\left.\begin{aligned} 2x^2 - 5xy + 4y^2 &= 4 \\ 3y^2 - x^2 &= 3 \end{aligned}\right\} \quad\ldots\ldots\ldots(1)$$

A common method is to put $y = mx$.

$$\left.\begin{aligned} x^2(2 - 5m + 4m^2) &= 4 \\ x^2(3m^2 - 1) &= 3 \end{aligned}\right\} \quad\ldots\ldots\ldots(2)$$

By division $$\frac{x^2(2-5m+4m^2)}{x^2(3m^2-1)} = \frac{4}{3} \quad\ldots\ldots\ldots(3)$$

or $$\frac{2-5m+4m^2}{3m^2-1} = \frac{4}{3} \quad\ldots\ldots\ldots(4)$$

$$m = \frac{2}{3},$$

and from either of (1), $x = \pm 3$, $y = \pm 2$.

The point is that we have missed the obvious solutions $x = 0$, $y = \pm 1$. We dropped them at the passage from (3) to (4). In fact from (3) we can only infer (4), *if x is not zero*, so that we should say, *either* $x = 0$, *or*

$$\frac{2-5m+4m^2}{3m^2-1} = \frac{4}{3},$$

and then try $x = 0$ in (1).

Instead of putting $y = mx$, it is therefore perhaps preferable to form a homogeneous equation from the two given equations, by multiplying them by 3 and 4 and subtracting.

Thus $$6x^2 - 15xy + 12y^2 = 12y^2 - 4x^2,$$
$$\text{or} \quad x(2x - 3y) = 0,$$
$$x = 0, \text{ or } 2x - 3y = 0.$$

The example illustrates the danger of cancelling a common factor from numerator and denominator of a fraction without considering the possibility of that factor being zero.

It may also be found useful (as the Editor remarks to me) as the basis of a lesson on infinite roots of an equation. The equation for m, which we expect to be a quadratic, turns out to be of the first degree. The second value of $\frac{y}{x}$ is here $\frac{1}{0}$ or infinity.

It may even happen that both values of m are infinite. A boy with $y = mx$ as his only resource would be rather nonplussed with the example
$$x^2 + xy + y^2 = 1,$$
$$2x^2 + 3xy + 3y^2 = 3.$$

Infinite roots appear in another way in this class of equations, namely, in the case when the quadratic functions in the two given equations have a common factor.

Take $$(x + y)(x - y) = 3, \qquad (1)$$
$$(x + y)(2x + y) = 15. \qquad (2)$$

Here $$(x + y)(2x + y) = 5(x + y)(x - y),$$
$$x + y = 0 \quad \text{or} \quad x = 2y.$$

If we put $x + y = 0$ in (1) we get $0 = 3$, and we say that the equations have no solution which makes $x + y = 0$.

But if we use the $y = mx$ method, we find $m = -1$ or $\frac{1}{2}$.

Then from (1), $x^2(1 - m^2) = 3$,
$$x^2 = \frac{3}{1 - m^2}.$$

If $m = \frac{1}{2}$, this gives $x = \pm 2$, $y = \pm 1$; but if $m = -1$, $x = \pm\infty$, $y = \mp\infty$.

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