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CONVEX DIRECTED SUBGROUPS OF A GROUP OF DIVISIBILITY

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Introduction. If D is an integral domain with quotient field K, the group of divisibility $G(D)$ of D is the partially ordered group of non-zero principal fractional ideals with $aD \leqq bD$ if and only if aD contains bD. If K^* denotes the multiplicative group of K and $U(D)$ the group of units of D, then $G(D)$ is order isomorphic to $K^*/U(D)$, where $aU(D) \leqq bU(D)$ if and only if $b/a \in D$.

The study of divisibility of elements of D amounts essentially to the study of $G(D)$. In fact, D is a UFD if and only if $G(D)$ is a cardinal sum of copies of $\mathbf{Z}$.

But, $G(D)$ reflects more than the properties of factorization of elements of D. Krull [**14**] observed that D is a valuation ring if and only if $G(D)$ is totally ordered. Then Jaffard [**13**] proved that D is a GCD domain if and only if $G(D)$ is lattice-ordered.

Furthermore, there are two theorems which establish one-to-one correspondences between certain subsets of an integral domain D and certain subsets of $G(D)$. The first theorem ([**14**, p. 167; **8**, p. 184], or [**21**, p. 40]) is of a rather special character, yielding a correspondence between prime ideals of a valuation ring D and convex subgroups of the totally ordered group $G(D)$. The second is more general [**18**], and establishes a correspondence between prime ideals of a Bezout domain D and prime filters of the positive cone of $G(D)$. (In particular, the Krull dimension of a Bezout domain D is revealed by $G(D)$.)

We ask: does there exist a similar correspondence for arbitrary integral domains? If some correspondence exists, does it reduce to the familiar correspondences for Bezout domains and for valuation rings?

Theorem 2.1 and its corollaries answer these questions. Theorem 2.1 establishes a one-to-one correspondence between saturated multiplicative systems in an integral domain D and convex directed subgroups of the group of divisibility of D. We define the dimension of a partially ordered group and prove the Krull dimension of a Prüfer domain D is equal to the dimension of $G(D)$. Since the dimension of an l-group G is equal to the prime filter dimension of G_+, we reach Sheldon's conclusion [**18**]: there is a one-to-one correspondence between prime ideals of a Bezout domain D and prime filters in the positive cone of $G(D)$.

In § 3 we obtain information about certain rings, not so much from knowledge of their internal structure as from analysis of their groups of divisibility. We compute the dimensions of two rings constructed by Heinzer [**9**] and

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Sheldon [**18**]. By the factorization theorem of Weirstrass, we can embed the cardinal product of countably many copies of $\mathbf{Z}$ in the group of divisibility of the ring E of entire functions. By this means, we achieve another proof that E has infinite Krull dimension. Moreover, that each prime ideal of E is contained in a unique maximal ideal follows since $G(E)$ is a complete l-group.

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1. Definitions and notations. The notation and terminology will essentially be the same as that of Ohm's paper [**16**]. In this paper, all groups are abelian; an 0-group is partially ordered, and an l-group is lattice-ordered. A cartesian product of 0-groups G_λ is called the *cardinal product* (sometimes called the ordered direct product or vector group) if $x = (x_\lambda) \geqq y = (y_\lambda)$ if and only if $x_\lambda \geqq y_\lambda$ for each λ. An 0-group G is a *subcardinal product* (usually called a subdirect sum) of the groups G_λ if there is an 0-embedding ϕ of G into $\prod_\lambda G_\lambda$ such that $p_\lambda \phi(G) = G_\lambda$ for each λ, where p_λ is the canonical projection map of $\prod_\lambda G_\lambda$ onto G_λ. The *cardinal sum* of the groups G_λ, denoted by $\sum_\lambda G_\lambda$, is the subset of $\prod_\lambda G_\lambda$ of all elements with finite support. Let $G \oplus_L H$ denote the lexicographic sum of the 0-groups G and H.

Let $\mathbf{Z}$ denote the group of integers under the natural order and let $\mathbf{R}$ denote the additive group of real numbers.

If $a_0, a_1, \ldots, a_n$ are elements of an 0-group G, $a_0 \in \sup(\inf_G\{a_1, \ldots, a_n\})$ means a_0 is an upper bound of the set of all lower bounds of $a_1, \ldots, a_n$ (in Ohm's notation $a_0 \geqq \inf_G\{a_1, \ldots, a_n\}$). If G is an l-group, let cup $(\vee)$ and cap $(\wedge)$ denote sups and infs. If $a, b \in G$ then $a \| b$ means $a \nleqq b$ and $b \nleqq a$.

The reader should consult Ohm's paper [**16**] for the definition of semi-valuation and Gilmer's book [**8**] for the definition of GCD-domain, and Prüfer domain.

2. Main result. A multiplicative system S in an integral domain D is *saturated* if S contains along with an element x all divisors of x. If S is saturated, S is equal to $U(D_S) \cap D$, where $U(D_S)$ is the set of units of the quotient ring D_S. Moreover, each unit of D_S is of the form s_1/s_2 where $s_i \in S$.

An 0-group G is *directed* if for each pair of elements $g_1, g_2 \in G$ there is an element g exceeding both, or equivalently, if each $g \in G$ is the difference of two elements of the positive cone G_+ of G.

THEOREM 2.1. *Suppose D is an integral and $G(D)$ is its group of divisibility. There is a one-to-one correspondence σ between saturated multiplicative systems in D and convex directed subgroups of $G(D)$. Furthermore, if S and G_S correspond under σ, then the group of divisibility of D_S is G/G_S.*

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Proof. Suppose S is a saturated multiplicative system of D. Let v be the canonical semi-valuation of K^* onto $G(D)$. Clearly, $v(S)$ is subsemigroup of G_+. Further, if $0 \leqq g \leqq v(s)$, then $g = v(x)$ for some $x \in D$, where $xD \supseteq sD$. Thus, x divides s, and $x \in S$ since S is saturated. If

$$G_S = \{g_1 - g_2 | g_i \in v(S)\},$$

then G_S is a convex directed subgroup of G with positive cone $v(S)$.

Next, suppose H is a convex directed subgroup of G. Let $S = v^{-1}(H_+)$. Clearly, S is a multiplicative system in D. If $aD \supseteq sD$ for $s \in S$, then $0 \leqq v(a) \leqq v(s)$, where $v(s) \in H$. Since H is convex, $v(a) \in H$ so that $a \in S$. Therefore, S is saturated.

Note that the correspondence as described is between saturated multiplicative systems and the positive cones of convex directed subgroups (there is an obvious correspondence between positive cones and convex directed subgroups). If S is a saturated multiplicative system, then

$$S \xrightarrow{\sigma} v(S) = (G_S)_+.$$

If H is a convex directed subgroup of G, then

$$H_+ \xrightarrow{\tau} v^{-1}(H_+).$$

We can show that the correspondence is one-to-one if we observe that $\sigma\tau$ and $\tau\sigma$ are identity maps on the set of positive cones of convex directed subgroups and on the set of saturated multiplicative systems, respectively. Obviously, $\sigma\tau$ is the identity map. If S is a saturated multiplicative system, $\tau\sigma(S) = S'$ is a saturated multiplicative system of D containing S. If $x \in S'$, then $v(x) \in v(S)$ and $v(x) = v(s)$ for some $s \in S$. Consequently, x and s are associates, and $x \in S$ since S is saturated.

Finally, if S is a saturated multiplicative system of D, and if G_S is the convex directed subgroup of G generated by $v(S)$, observe that the group of divisibility of D_S is G/G_S. Clearly $G(D_S)$ is an order homorphic image of $G(D)$ under the map α defined by $\alpha(xD) = xD_S$. Next, observe that the kernel of α is G_S. Since each element $x \in S$ becomes a unit in D_S, $xD_S = D_S$, and $\alpha(v(S))$ is the identity element of $G(D_S)$. Hence, G_S is contained in the kernel of α.

If $g \in \ker \alpha$ and if $x \in K^*$ is such that $v(x) = g$, then $xD_S = D_S$ and x is a unit of D_S. Therefore, $x = s_1/s_2$ where $s_i \in S$, $v(x)$ is an element of G_S, and $G_S = \ker \alpha$.

The correspondence, established in Theorem 2.1, generalizes Krull's correspondence for valuation rings.

COROLLARY 2.2. *If D is a valuation ring, there is a one-to-one correspondence between prime ideals of D and convex subgroups of $G(D)$.*

Proof. In a valuation ring the only saturated multiplicative systems are complements of prime ideals.

Suppose D is an integral domain, G is its group of divisibility, and G/H is totally ordered, where H is a convex directed subgroup of G. Theorem 2.1 implies that G/H is the value group of some valuation ring of the form D_S, where $S = v^{-1}(H_+)$. By [**8**, p. 319], S is the complement of a prime ideal P of D. On the other hand, if P is a prime ideal of D, the multiplicative system $S = D\backslash P$ corresponds to a convex subgroup H generated by $v(S)$. If, in addition, D_P is a valuation ring, then $G/H = G(D_P)$ is totally ordered. Motivated by this observation, we define a convex directed subgroup H of an 0-group G to be a *prime subgroup* of G if G/H is totally ordered under the order inherited from G. Thus, Theorem 2.1 yields a one-to-one correspondence between prime ideals P of D such that D_P is a valuation ring and prime subgroups of $G(D)$.

For a totally ordered group G, the dimension of G is n if G contains exactly n distinct convex subgroups ($\neq G$); otherwise, the dimension of G is infinite. For an arbitrary 0-group G, define the dimension of G by

$$\dim G = \sup\{\dim G/H | H \text{ is a prime subgroup of } G\}.$$

If P is a prime ideal of an arbitrary domain D such that D_P is a valuation ring, then $\dim G(D_P) =$ height of P. Corollary 2.2 implies that the Krull dimension of a valuation ring D is equal to the dimension of $G(D)$. In particular, for a Prüfer domain D the above can be summarized: there is a one-to-one correspondence between prime ideals of D and prime subgroups of $G(D)$. Then, the following corollary to Theorem 2.1 is immediate.

COROLLARY 2.3. *If D is a Prüfer domain, the Krull dimension of D is equal to the dimension of the group of divisibility of D.*

In [**8**, p. 348], Gilmer defines the valuative dimension of an integral domain D as the supremum of the dimension of all valuative overrings of D. Clearly for a Prüfer domain D, the valuative dimension of D is equal to the dimension of $G(D)$. Thus, a result implicitly contained in Corollary 2.3 is that the Krull dimension of a Prüfer domain D is equal to the valuative dimension of D.

The notion of a prime subgroup of an l-group is not new [**2**; **5**]. The definition can also be given in terms of prime filters [**2**, p. 114]. If G is an l-group and G_+ is the positive cone of G, a *filter* in G_+ is a non-void subset F of G_+ such that

(1) $x \wedge y \in F$ for all $x, y \in F$, and

(2) $x \in F$ if $x \geqq y$ and $y \in F$.

A *proper filter* in G_+ is contained in $G_+\backslash\{0\}$. A proper filter in G_+ is a *prime filter* if

(3) $x + y \in F$, x and y in G_+, implies $x \in F$ or $y \in F$.

Here, we use the term "prime" with respect to the group operation rather than the lattice operation of forming join. However, a prime filter in this sense is also prime with respect to join since $x + y \geqq x \vee y$.

Let us summarize what is known. In an l-group G, there is a one-to-one correspondence between prime subgroups of G and prime filters in G_+. If H

is a prime subgroup of G, $G_+\backslash H$ is a prime filter in G_+. If F is a prime filter, $H = \{a - b | a, b \in G_+\backslash F\}$ is a prime subgroup of G.

In the proof of Corollary 2.3, we observed a one-to-one correspondence between prime ideals of a Prüfer domain D and prime subgroups of $G(D)$. There is a second one-to-one correspondence between prime subgroups of an l-group G and prime filters in G_+. The composition of both correspondences establishes Sheldon's correspondence. Combining this result (Corollary 2.4) with Corollary 2.2, we conclude: the correspondence of Theorem 2.1 extends each of the familiar correspondences.

COROLLARY 2.4 (Sheldon). *If D is a Bezout domain, there is a one-to-one correspondence between prime ideals of D and prime filters in the positive cone of $G(D)$.*

Added in proof. An earlier proof of this fact is due to I. Yakabe, *On semi-valuations II*, Mem. Fac. Sci. Kyushu Univ. Ser. A *17* (1963), 10–28.

Following Sheldon [**18**], we define the *prime filter dimension* of an l-group G to be the number of terms in the longest finite chain of prime filters in G_+, or infinity if there is no such longest chain. The one-to-one correspondence between prime subgroups of G and prime filters shows that the dimension of G is equal to the prime filter dimension. A conclusion follows at once: for a Bezout domain D, the Krull dimension of D = dimension of $G(D)$ = prime filter dimension of the positive cone of $G(D)$.

Each proper filter of an l-group G is contained in an ultrafilter of G_+, a maximal proper filter. Moreover, an ultrafilter is a prime filter. In the correspondence of Corollary 2.4, maximal ideals of a Bezout domain correspond to ultrafilters and hence to prime subgroups H such that $G_+\backslash H$ is an ultrafilter. Such a prime subgroup is designated a *minimal prime subgroup* by Conrad and McAlister in [**5**, p. 198]. (In the case where D is a valuation ring, the ultrafilter corresponding to the maximal ideal of D is $G_+\backslash\{0\}$.) If G is a lattice-ordered group and $g \in G$, $g \neq 0$, then a *value* of g is a convex l-subgroup M_g such that for any convex l-subgroup $H \supset M_g$, $g \in H$. A Zorn's lemma argument shows the existence of a value for any non-zero $g \in G$. It is well-known that M_g is a prime subgroup of G [**5**, p. 188]. In the correspondence between prime ideals of a Bezout domain D and prime subgroups of $G(D)$, observe that minimal prime ideals of a principal ideal xD correspond to values of $v(x)$ in $G(D)$.

Suppose that D is a GCD-domain and that v is the canonical semi-valuation from K^* onto $G(D)$. It is easy to see that D is a Bezout domain if and only if $v(Q^*)$ is a filter for each ideal Q of D. We communicated this result to Sheldon and he responded that he had observed that Theorem 2.1 could be used to prove the following extension of Corollary 2.4. If D is a GCD-domain, then these are equivalent:

(1) D is Bezout.

(2) $v(P^*)$ is a prime filter for each prime ideal P of D.

(3) $v(M^*)$ is an ultrafilter for each maximal ideal M of D.

Indeed, this is obvious from Theorem 2.1 for (3) implies that D_M is a valuation ring for each maximal ideal M of D. Sheldon has obtained several results on the prime ideal structure of GCD-domains including the following corollary to Theorem 2.1.

COROLLARY 2.5. *Suppose that D is a GCD-domain such that for each pair of prime ideals P and Q of D either $P \subseteq Q$, $Q \subseteq P$, or $P + Q = D$. Then D is a Bezout domain.*

Proof. The conclusion follows if we show that D is a Prüfer domain. If M is a maximal ideal of D, the prime ideals of D_M are linearly ordered by containment. Thus, the set of saturated multiplicative systems in D_M is linearly ordered by containment. Theorem 2.1 implies the convex directed subgroups of the lattice ordered group $G(D_M)$ are linearly ordered by containment. Therefore, $G(D_M)$ is totally ordered, D_M is a valuation ring, and D is Prüfer.

3. Examples. In this section we will compute the dimensions of some lattice-ordered groups. All examples have appeared in the literature, and, in some cases (Examples 2 and 3), their dimensions are known.

1. The cardinal product of countably many copies of $\mathbf{Z}$: If $P = \prod_{i<\omega} \mathbf{Z}_i$ is the cardinal product of countably many copies of $\mathbf{Z}$, then P is infinite dimensional. Actually, we prove that

$$G = \prod_{i<\omega} \mathbf{Z}_i / \sum_{i<\omega} \mathbf{Z}_i$$

is infinite dimensional. For any integer $k \geqq 0$, define $g_k \in P$ by $g_k(i) = i^k$ for each integer $1 \leqq i < \omega$.

Let σ denote the canonical homomorphism of P onto G. For each positive integer k and all positive integers n, $i^{k+1} \geqq ni^k$ except for a finite number of values of i. Thus $n\sigma(g_k) \leqq \sigma(g_{k+1})$ for all positive integers n.

If $M_{\sigma(g_0)}$ is a value of $\sigma(g_0)$ in G, then $G_1 = G/M_{\sigma(g_0)}$ is totally ordered and infinite dimensional since G_1 contains infinitely many positive elements $\{g_i'\}_{i=1}^{\infty}$ such that $ng_k' \leqq g_{k+1}'$ for each positive integer n, and $g_l' \notin M_{gk'}$ for $l \geqq k$, where $M_{gi'}$ is the value of g_i' in G_1.

2. Complete integral closure and the examples of Heinzer and of Sheldon: If G is an 0-group and g is a non-zero element of G_+, then g is said to be *bounded* if there is an element $h \in G$ such that $ng \leqq h$ for each positive integer n. The set of all bounded elements forms a convex semigroup of G, and generates a convex directed subgroup,

$$B(G) = \{a - b \mid a, b \text{ are bounded elements of } G\},$$

called the *bounded closure* of G. (Note that if G is an archimedean l-group, then $B(G) = 0$.) Thus, if D is an integral domain, $G(D)$ its group of divisibility, and v the canonical semi-valuation, define $x \in D$ to be bounded if $v(x)$ is a bounded element of $G(D)$. Theorem 2.1 implies the set $B(D)$ of all bounded elements of D is a saturated multiplicative system and $G/B(G)$ is the group of divisibility of $D_{B(D)}$. A slight modification of the proof of a theorem of Sheldon [**18**] shows that if the complete integral closure D^* of D is a quotient ring of D, then $D^* = D_{B(D)}$. Since every overring of a Bezout domain D is a quotient ring of D, we conclude the following corollary of Theorem 2.1.

COROLLARY 2.6. *If D is a Bezout domain, G is its group of divisibility, and $B(G)$ is the bounded closure of G, then $G/B(G)$ is the group of divisibility of the complete integral closure of D.*

Obviously, the complete integral closure of a Bezout domain D is completely integrally closed if and only if $G/B(G)$ has trivial bounded closure.

Define the groups $G^{(n)}$ recursively by

$$G^{(n+1)} = G^{(n)}/B(G^{(n)})$$

for each non-negative integer n ($G^0 = G$).

Corollary 2.6 has particular relevance to two integral domains–one constructed by Heinzer [**9**] and the other by Sheldon [**18**].

These domains are similar in several respects. Each domain D_i ($i = 1, 2$) is constructed using the Krull-Kaplansky-Jaffard-Ohm Theorem [**13**, p. 64] and an l-group G_i such that $G_i^{(1)}$ has nontrivial bounded closure; thus, each example shows that the complete integral closure of a Bezout domain need not be completely integrally closed. Moreover, each group G_i is such that $G_i^{(2)}$ has trivial bounded closure; thus, D_i^{**} is completely integrally closed for each i.

Nevertheless, the two domains have some different characteristics. In particular, Heinzer's example is infinite dimensional, while Sheldon's is two-dimensional. We proceed to compute their dimensions.

Heinzer [**9**] considers the group H of all functions $f : \mathbf{Z}_+ \rightarrow \mathbf{Z} \oplus_L \mathbf{Z}$ such that if $f(n) = (a_n, b_n)$, then $a_n = 0$ for all but a finite number of values of n. Alternately, H is a subgroup of the cardinal product of countably many copies of $\mathbf{Z} \oplus_L \mathbf{Z}$ containing the cardinal product P of countably many copies of $\mathbf{Z}$ such that H/P is a cardinal sum of copies of $\mathbf{Z}$. More precisely, if $(\mathbf{Z} \oplus_L \mathbf{Z})_i$ is generated by positive elements c_i and d_i, where d_i generates the only nontrivial convex subgroup of $(\mathbf{Z} \oplus_L \mathbf{Z})_i$, then P is the cardinal product of the groups (d_i), and H/P is the cardinal sum of the groups (c_i).

That H is infinite dimensional follows since P is infinite dimensional. In fact, if G is any lattice-ordered group containing an infinite dimensional subgroup K as a sublattice, then G is infinite dimensional. We translate the problem to the more familiar totally ordered situation. Thus, if n is any positive integer, there is an element k_n of K and a value K_n of k_n in K such that K/k_n is totally ordered of dimension $\geqq n$. Thus, if G_n is a value of K_n in G such that $K_n = G_n \cap K$, then G/G_n is totally ordered and contains K/K_n. That G/G_n has dimension $\geqq n$ is clear since each convex subgroup of K/K_n is the intersection of a convex subgroup of G/G_n with K/K_n.

In [**18**] Sheldon considered the group S of all functions $f : \mathbf{Z}_+ \to \mathbf{Z} \oplus_L \mathbf{Z}$ for which there exists $a, b \in \mathbf{Z}$ such that $f(n) = (0, an + b)$ for all positive integers n outside a finite set. Sheldon showed that S is two-dimensional by exhibiting all possible prime filters of the group. We offer an easier proof that S is two-dimensional.

We observe that S is a subcardinal product of $(\mathbf{Z} \oplus_L \mathbf{Z})_i$ for $i \in \mathbf{Z}^+$ and contains $\sum_i (\mathbf{Z} \oplus_L \mathbf{Z})_i$ (for let $a = 0$ and $b = 0$ in the above description of an $f \in S$). Furthermore, $S/\sum_i (\mathbf{Z} \oplus_L \mathbf{Z})_i$ is isomorphic to $\mathbf{Z} \oplus_L \mathbf{Z}$ since the map $\sigma : S \to \mathbf{Z} \oplus_L \mathbf{Z}$, where $\sigma(f) = (a, b)$ if $f(n) = (0, an + b)$ for n outside a finite set, is an l-map with kernel $\sum_i (\mathbf{Z} \oplus_L \mathbf{Z})_i$.

That S is two-dimensional follows from the following general proposition.

PROPOSITION 3.1. *Suppose G is a lattice-ordered subcardinal product of G_λ such that $\sum_\lambda H_\lambda \subseteq G$, where H_λ is a convex subgroup of the totally ordered group G_λ for each λ. If each totally ordered l-homorphic image of $G/\sum_\lambda H_\lambda$ is an l-homomorphic image of some G_λ, then the same is true for G.*

Proof. Suppose σ is an l-homomorphism of G onto a totally ordered group T with kernel K. If $K \supseteq \sum H_\lambda$, then T is an 0-homomorphic image of $G/\sum_\lambda H_\lambda$. If $K \not\supseteq \sum_\lambda H_\lambda$, then $H_{\lambda_0} \not\subseteq K$ for some λ_0. Then if $h_{\lambda_0} \in H_{\lambda_0} \backslash K$ and if h_{λ_0} is positive, let H be the subgroup of $\prod_\lambda G_\lambda$ of all elements with λ_0 coordinate 0. Note that $h_{\lambda_0} \wedge h = 0$ for any $h \in H$ and, in particular, for any $h \in H \cap G$. Thus, $\sigma(h_{\lambda_0}) \wedge \sigma(h) = 0$ and $\sigma(h) = 0$ since T is totally ordered. Hence, $H \cap G \subseteq K$ and $T = G/K$ is an image of $G/H \cap G$. Furthermore, since G is a subcardinal product of G_λ, $G/H \cap G = p_{\lambda_0}(G) = G_{\lambda_0}$, where p_{λ_0} is the projection of $\prod_\lambda G_\lambda$ onto G_{λ_0}. Consequently, T is an image of some G_λ, and the proposition is proved.

3. Eventually constant real sequences: By Proposition 3.1, the l-group G of all eventually constant real sequences is one-dimensional since $G/\sum_i \mathbf{R}_i \simeq \mathbf{R}$ where $\mathbf{R}$ and $\mathbf{R}_i$ denote the group of reals.

The l-group H of all integral valued eventually constant sequences is also one-dimensional. Since $H/\sum_i \mathbf{Z}_i \simeq \mathbf{Z}$, Proposition 3.1 implies that all totally ordered homomorphic images of H are isomorphic to $\mathbf{Z}$. One can also compute all prime subgroups of H as in [**5**, p. 202].

4. Free l-groups: Weinberg [**20**] has shown that a free abelian l-group of rank α exists. The definition and construction are as follows. If F is free abelian group on α generators, then a free abelian l-group over F is an l-group F' together with an isomorphism $f : F \to F'$, such that for each l-group G and each homomorphism $\sigma : F \to G$, there is an l-homomorphism $\tau : F' \to G$ such that $\tau f = \sigma$. The traditional model is obtained by taking all possible total orders T_λ on F and letting F' be the l-subgroup of the cardinal product of the totally ordered groups (F, T_λ) generated by the diagonal [**4**, p. 49]. Then, in this context, F is identified with the diagonal of $\prod_\lambda (F, T_\lambda)$ under the map f.

Each element of F' is of the form

$$\bigwedge_i \bigvee_j a_{ij}, \quad \text{where } a_{ij} \in F.$$

PROPOSITION 3.2. *If n is a positive integer, then any free abelian l-group on n generators has dimension n.*

Proof. Clearly, if F is free on n generators the totally ordered group T of dimension n with the lexicographic ordering is a homomorphic image of F. This map can be lifted to a lattice homomorphism of F' onto T. Thus, dim $F' \geqq n$. On the other hand, if σ is an l-map of F' onto a totally ordered group T', then $\sigma|F'$ the restriction of σ to F, is a homomorphism of F into T'. Moreover, any $x \in F'$ is of the form

$$\bigwedge_i \bigvee_j a_{ij}, \quad \text{where } a_{ij} \in F,$$

and

$$\sigma(x) = \bigwedge_i \bigvee_j \sigma(a_{ij}).$$

For some i and j, $\sigma(x) = \sigma(a_{ij})$, since T' is totally ordered. Therefore, $\sigma|F$ maps F onto T' and the rational rank of T' is less than or equal to n. By [**21**, p. 50], dim $T' \leqq n$, and dim $F' = n$.

5. The ring E of entire functions: The question of the dimensionality of E has an interesting history. Helmer [**10**] showed that E is a Bezout domain in 1940. In 1946, Schilling [**17**] claimed to have shown that E is one-dimensional, but, in 1952, Kaplansky showed that its dimension was at least two (Kaplansky's proof appears in [**11**]). Henriksen studied the ideal structure in [**11**; **12**], and in [**12**], he showed that P_M, the set of prime ideals contained in a free maximal ideal M, is linearly ordered by inclusion and has cardinality at least $2^{\aleph_1}$. This should have settled the question; but in 1965, Fuster [**7**] claimed to show that E is one-dimensional. Enochs reviewed Fuster's paper for Zentralblatt and, in 1969 [**6**], published another proof that E is at least two-dimensional. Laplaza [**15**] offers yet another proof that E is infinite dimensional. Other proofs have been obtained by Alling [**1**] and Banaschewski [**3**].

We prove that E is infinite dimensional by embedding in $G(E)$ a cardinal product of countably many copies of $\mathbf{Z}$. The factorization theorem of Weirstrass [**19**, p. 298] is the clue to this embedding.

If $f \in E$, let

$$Z(f) = \{(z, k) | z \text{ is a zero of } f \text{ of multiplicity } k\}.$$

PROPOSITION 3.3. *The group of divisibility of the ring of entire functions contains a cardinal product of countably many copies of* $\mathbf{Z}$. *In particular, the ring of entire functions is infinite dimensional.*

Proof. Choose a countable set of points $\{w_n\}_{n=1}^{\infty}$ in $\mathbf{C}$ that satisfy the hypothesis of Weirstrass' Factorization Theorem, that is, $\{w_n\}$ is closed, discrete subset of $\mathbf{C}$. Then, Weirstrass' Theorem implies that for any countable set of non-negative integers $\{k_n\}_{n=1}^{\infty}$, there is an entire function f such that $Z(f) = \{(w_n, k_n)\}_{n=1}^{\infty}$. Furthermore, for $g \in E$, $Z(f) = Z(g)$ if and only if f and g are associates in E.

Now, if $p = (p_1, \ldots, p_n, \ldots)$ is an element of the cardinal product P of countably many copies of $\mathbf{Z}$, let $p_i = k_i - l_i$, where k_i and l_i are non-negative integers. Then, let f_1 and f_2 be entire functions such that $Z(f_1) = \{(w_n, k_n)\}_{n=1}^{\infty}$ and $Z(f_2) = \{(w_n, l_n)\}_{n=1}^{\infty}$. Define $\sigma : P \rightarrow G(E)$ by $\sigma(p) = f/gE$. Then, σ is an l-isomorphism of P into $G(E)$, and the proposition is proved.

In [**10**, p. 349], Helmer showed that $G(E)$ is a complete l-group. For an arbitrary integral domain D, it is easy to see that $G(D)$ is complete if and only if each non-zero v-ideal of D is principal, and, in this case, D must be completely integrally closed.

In a complete l-group each prime filter contains a unique ultrafilter [**2**, p. 121]. Thus, Corollary 2.4 leads immediately to Proposition 3.4 and subsequently to a result of Henriksen [**12**].

PROPOSITION 3.4. *If D is a Bezout domain with complete group of divisibility, then each non-zero prime ideal of D is contained in a unique maximal ideal.*

COROLLARY 3.5 (Henriksen). *Each non-zero prime ideal of the ring of entire functions is contained in a unique maximal ideal.*

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