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ON OUTER-COMMUTATOR WORDS

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Introduction. Let F be the group freely generated by the countably infinite set $X = \{x_1, x_2, \ldots, x_i, \ldots\}$. Let $w(x_1, x_2, \ldots, x_n)$ be a reduced word representing an element of F and let G be an arbitrary group. Then $V(w, G)$ will denote the set

$$\{w(g_1, g_2, \ldots, g_n) | g_i \in G\}$$

whose elements will be called *values of w in G.* The subgroup of G generated by $V(w, G)$ will be called the *verbal subgroup of G with respect to w* and be denoted by $w(G)$.

A conjecture attributed by Turner-Smith [**7**] to P. Hall states that if $V(w, G)$ is finite, then $w(G)$ is finite. A word w for which the conjecture holds for all groups G is called *concise.* It is an unsettled problem whether all words are concise. For a survey of present knowledge on this problem the reader is referred to D. Robinson [**5**]. In [**7**] Turner-Smith made a detailed study of conciseness for a special class of commutator words, namely the outer-commutator words (henceforth OC-words,) which we now define.

Take Γ to be the set of all commutator subgroup functions ϕ (see P. Hall [**1**]) obtainable from the identity function γ (define by $\gamma(G) = G$ for all groups G) by a finite succession of commutator operations. For $\phi, \psi \in \Gamma$, define

$$(\phi\psi)(G) = [\phi(G), \psi(G)],$$

so that Γ is a commutative groupoid generated by the single element γ. For each $\phi \in \Gamma$ we may now define the length $l(\phi)$, by taking $l(\gamma) = 1$ and $l(\alpha\beta) = l(\alpha) + l(\beta)$ for $\alpha, \beta \in \Gamma$. We now associate with each element of Γ a word as follows:

(i) with γ is associated the word x_1;

(ii) if $u(x_1, x_2, \ldots, x_r)$ and $v(x_1, \ldots, x_s)$ are associated with ϕ and $\psi \in \Gamma$ respectively, then

$$[u(x_1, x_2, \ldots, x_r), v(x_{r+1}, \ldots x_{r+s})]$$

is associated with $\phi\psi$.

The collection of all words associated with elements of Γ are called *outer-commutator words*. In future, if w is a word associated with $\phi \in \Gamma$, then $V(w, G)$ will be denoted $\phi^*(G)$. (It should be noted that, though two different words

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can be associated with the same subgroup-function, they will always give rise to the same value set.)

In this paper it will be proved that

THEOREM 1. *All outer-commutator words are concise.*

A related problem to that of conciseness is verbal ellipticity. Let w as before be an element of F and G be an arbitrary group. If x is an element of $w(G)$, then

$$x = w_1^{\epsilon_1} w_2^{\epsilon_2} \dots w_r^{\epsilon_r} \quad \text{where } w_i \in V(w, G) \text{ and } \epsilon_i = \pm 1.$$

The smallest natural number r for which such a set of w_i's exists is called the *w-length of x*. If there is a finite bound on the w-length of the elements of $w(G)$, then G is called *w-elliptic*. If a group G is w-elliptic for all words w, then it is called *verbally elliptic*. In [**6**] P. Stroud was able to prove the following.

THEOREM (Stroud). *If F is finitely-generated Abelian-by-nilpotent group, then G is verbally elliptic.*

However there are plenty of groups which fail to be verbally elliptic, as is shown by a result of A. H. Rhemtulla [**3**].

THEOREM (Rhemtulla). *Let A and B be non-trivial groups and let w be a non-trivial proper word. Then the free product $A * B$ is not w-elliptic unless A and B both have order two.*

In his thesis P. Stroud asked whether polycyclic groups are verbally elliptic, the answer to which is still unknown. However for OC-words we can prove

THEOREM 2. *A polycyclic group is w-elliptic for every* OC-*word w.*

In the case when $w = [x_1, x_2]$, a far more general result has been obtained by A. H. Rhemtulla [**4**], namely.

THEOREM (Rhemtulla). *If G is Abelian-by-*(soluble with the maximal condition on normal subgroups), *then G is w-elliptic.*

(Some of the techniques used to establish Theorem 2 are essentially generalizations of techniques used in [**4**].)

Theorems 1 and 2 are both proved by being reduced to the same question about a free commutative groupoid with one generator. Then the groupoid problem is solved.

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1. Conciseness. In this section the conciseness of OC-words is reduced to a question about a free commutative groupoid on one generator. (An element $\phi \in \Gamma$ will be called concise if an associated word is concise, i.e. if $\phi(G)$ is finite whenever $\phi^*(G)$ is finite.)

The following reduction lemma holds for an arbitrary word, but we will only prove it for OC-words.

LEMMA 1. *If $\phi \in \Gamma$ is not concise, then there exists a group G for which $\phi^*(G)$ is finite and $\phi(G)$ is non-trivial, torsion-free and Abelian.*

Proof. If $\phi \in \Gamma$ is not concise, then there exists a group H for which $\phi(H)$ is infinite and $\phi^*(H)$ finite. Let $x \in \phi^*(H)$. Then it is clear that all conjugates of x are also in $\phi^*(H)$ and since $\phi^*(H)$ is finite it follows that $C_{\phi(H)}(x)$ has finite index in $\phi(H)$. But $Z(\phi(H))$ is the intersection of a finite number of such centralizers and hence has finite index in $\phi(H)$, so that by Schur's Theorem (see for example [**2**, Theorem 8.1, p. 59]) $\phi(H)'$ is finite. Now $\phi(H)/\phi(H)'$ is finitely generated and Abelian, so there exists $T \trianglelefteq H$ such that

$$\phi(H)' \trianglelefteq T \trianglelefteq \phi(H)$$

with $T/\phi(H)'$ infinite and $\phi(H)/T$ non-trivial and torsion-free. Since $\phi(H)'$ and $T/\phi(H)'$ are finite, T is finite, and since $\phi(H)$ is infinite, $\phi(H)/T$ is infinite. Let $G = H/T$. Then $\phi^*(G)$ is finite and $\phi(G) = \phi(H)/T$, which is infinite, torsion-free and Abelian.

Before we proceed to reduce the problem to one about groupoids, we need a few preliminary definitions concerning free commutative groupoids.

When writing products in a commutative groupoid a left-normed notation will be adopted. This is to say if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are elements of the groupoid, then $\alpha_1\alpha_2 \ldots \alpha_n$ will mean $((\ldots(\alpha_1\alpha_2)\alpha_3)\ldots)\alpha_n)$.

In future $L(\gamma)$ will denote a free commutative groupoid with generator γ. We define the length function $l:L(\gamma) \rightarrow \mathbf{N}$ as for Γ.

Definition 1. Let $\alpha, \beta \in L(\gamma)$. The sentence "$\alpha$ is β-valued" is defined by induction on $l(\alpha)$. If $l(\alpha) = 1$; then α is β-valued if $\beta = \gamma$. Let $n > 1$ and suppose the sentence "α is β-valued" has been defined for $l(\alpha) < n$. Then if $l(\alpha) = n$, α is β-valued if either $\beta = \gamma$ or there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L(\gamma)$ such that α_1 and α_2 are β_1- and β_2-valued respectively, where $\alpha = \alpha_1\alpha_2$ and $\beta = \beta_1\beta_2$.

Note that "α is β-valued" is a transitive relation. We can define a groupoid homomorphism $f:L(\gamma) \rightarrow \Gamma$ by defining $f(\gamma)$ to be the identity subgroup function γ. (In future f will always refer to this homomorphism.)

Definition 2. If $\alpha, \beta \in \Gamma$, then we will say that α is β-valued if there exist $\alpha', \beta' \in L(\gamma)$ such that α' is β'-valued, $f(\alpha') = \alpha$ and $f(\beta') = \beta$.

The motivation behind the last definition is that if α is β-valued for $\alpha, \beta \in \Gamma$, then $\alpha^*(F) \subseteq \beta^*(F)$.

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We now define a set of quasi-orders (i.e. reflexive, transitive relations) on $\bar{S}(A)$, the collection of all finite subsets of A, where A is either $L(\gamma)$ or Γ.

Definition 3. Let $A = L(\gamma)$ or Γ. Then we introduce the following relations on $\bar{S}(A)$.

(I) Let $S_1, S_2 \in \bar{S}(A)$. Then we will write $S_1 < S_2$ for each $\alpha \in S_1$, if one of the following holds:

(i) there exist elements $\alpha_i (i = 1, 2, \ldots, n)$ in A such that $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ and such that $\alpha_2 \ldots \alpha_n$ is a member of S_2,

(ii) there exist elements $\alpha_i (i = 1, 2, \ldots, n)$ with $n \geqq 3$ in A such that $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ and such that $\alpha_3\alpha_1\alpha_2\alpha_4 \ldots \alpha_n$ and $\alpha_2\alpha_3\alpha_1\alpha_4 \ldots \alpha_n$ are in S_2 (i.e. the elements obtained from α by cyclically permuting the first three α_i's are in S_2),

(iii) there exists an element $\beta \in S_2$ such that α is β-valued.

(II) Let $S_1, S_2 \in \bar{S}(A)$ and $\beta \in A$. Then we will write $S_1 < \beta S_2$ if for each $\alpha \in S_1$ either

(i) $\alpha \in S_2$ or

(ii) there exist $\alpha_i (i = 1, 2, \ldots, n)$, $n \geqq 2$, such that $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$, $\gamma\alpha_2 \ldots \alpha_n$ is β-valued, and the set

$$T(\alpha_1, \alpha_2, \ldots, \alpha_n) = \{\alpha_1(\alpha_1\alpha_2)\alpha_3 \ldots \alpha_n, \alpha_1\alpha_2(\alpha_1\alpha_2\alpha_3)\alpha_4 \ldots \alpha_n, \ldots, \alpha_1 \ldots \alpha_{i-1}(\alpha_1 \ldots \alpha_i)\alpha_{i+1} \ldots \alpha_n \ldots, , \alpha_1 \ldots \alpha_{n-1}(\alpha_1\alpha_2 \ldots \alpha_n)\}$$

is contained in S_2.

If $S_1, S_2 \in \bar{S}(A)$, $\beta \in A$ and either $S_1 < S_2$ or $S_1 < \beta S_2$, then we will write $S_1 < (\beta) S_2$.

Using (I) and (II) we now define two quasi-orderings on $\bar{S}(A)$.

(III) If $S_i < S_{i+1}$ for $S_i \in \bar{S}(A) (i = 1, 2, \ldots, n - 1)$, then $S_1 \ll S_n$.

(IV) If $S_i < (\beta)S_{i+1}$ for $S_i \in \bar{S}(A) (i = 1, 2, \ldots, n - 1)$, and $\beta \in A$, then $S_1 \ll (\beta)S_n$.

Definition 4. The derived elements $\delta_r \in L(\gamma) (r \geqq 0)$ are defined by induction on r. If $r = 0$, $\delta_0 = \gamma$. If δ_r is defined for $0 \leqq r \leqq n$, then $\delta_{n+1} = \delta_n\delta_n$. The image of δ_r in Γ under the homomorphism f will also be written δ_r and be referred to as a derived element, but no confusion will occur since it will always be clear which groupoid we are working in.

The following lemma turns our problem into one about the groupoid Γ.

LEMMA 2. *Let r be a fixed positive integer such that for some $\phi \in \Gamma$ we have $\phi \ll (\phi)\delta_k$ for all $k \geqq r$. Then ϕ is concise.*

To facilitate the proof we need the following additional

LEMMA 3. *Suppose that $\phi \in \Gamma$ is not concise and G is a group such that $\phi(G)$ is torsion-free Abelina and $\phi^*(G)$ is finite. Then if $\alpha_i \in \Gamma (i = 1, 2, \ldots, r)$, $\gamma\alpha_2 \ldots \alpha_r$ is ϕ-valued and $\beta(G) = 1$ for all $\beta \in T(\alpha_1, \ldots, \alpha_r)$, it follows that $(\alpha_1\alpha_2 \ldots \alpha_r)(G) = 1$.*

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Proof. Let $a_i \in \alpha_i^*(G)$ for $i = 1, 2, \ldots, r$. It is easily verified that for all $m \geqq 1$,

$$[a_1^m, a_2] \equiv [a_1, a_2]^m \bmod (\alpha_1\alpha_2\alpha_1)(G),$$
$$[a_1^m, a_2, a_3] \equiv [[a_1, a_2]^m, a_3] \bmod (\alpha_1(\alpha_1\alpha_2)\alpha_3)(G)$$
$$\equiv [a_1, a_2, a_3]^m \bmod (\alpha_1(\alpha_1\alpha_2)\alpha_3)(G)(\alpha_1\alpha_2(\alpha_1\alpha_2\alpha_3))(G),$$

and by induction

$$[a_1^m, a_2, \ldots, a_r]$$
$$\equiv [a_1, a_2, \ldots, a_r]^m \bmod \prod_{i=2}^{r} (\alpha_1 \ldots \alpha_{i-1}(\alpha_1 \ldots \alpha_i) \ldots \alpha_r)(G)$$
$$= [a_1, a_2, \ldots, a_r]^m,$$

because $\beta(G) = 1$ for all $\beta \in T(\alpha_1, \alpha_2, , \ldots, \alpha_r)$. Now $a_1^m \in G$, so the left-hand side is always an element of $(\gamma\alpha_2 \ldots \alpha_r)^*(G)$. But $\gamma\alpha_2 \ldots \alpha_r$ is ϕ-valued, hence $(\gamma\alpha_2 \ldots \alpha_r)^*(G) \subseteq \phi^*(G)$. By hypothesis $\phi^*(G)$ is finite. Therefore the set $\{[a_1, a_2, \ldots, a_r]^m | m \geqq 1\}$ is finite. It follows that there exists an integer m such that $[a_1, a_2, \ldots, a_r]^m = 1$. Since $\phi(G)$ is torsion-free $[a_1, a_2, \ldots, a_r] = 1$. Therefore $(\alpha_1\alpha_2 \ldots \alpha_r)(G) = 1$. (This lemma was motivated by a study of Proposition 6 in [**7**].)

Proof of Lemma 2. Suppose by way of contradiction that there exists $\phi \in \Gamma$, satisfying the conditions of the lemma yet failing to be concise. By Lemma 1 there exists a group G with $\phi(G)$ torsion-free Abelian and non-trivial and $\phi^*(G)$ finite. Since $G/\phi(G)$ is soluble and $\phi(G)$ is Abelian, G is soluble. Hence $\delta_k(G) = 1$ for all sufficiently large k.

Since $\phi \ll (\phi)\delta_k$ for all sufficiently large k, it is enough to prove that $\alpha(G) = 1$ for all $\alpha \in S_1$, whenever $S_1 < (\phi)S_2$ and $\alpha(G) = 1$ for all $\alpha \in S_2$. From this it would follow that $\phi(G) = 1$.

Let us suppose that $\alpha = \alpha_1\alpha_2 \ldots \alpha_n \in S_1$ and $\beta(G) = 1$ for all $\beta \in S_2$.

Case (a) $S_1 < S_2$ *and* I(i) *holds:* Here $\alpha_2 \ldots \alpha_n \in S_2$. Thus

$$\alpha(G) = [\alpha_1(G), \alpha_2(G), \ldots, \alpha_n(G)] \subseteq [\alpha_2(G), \ldots, \alpha_n(G)] = 1.$$

Case (b) $S_1 < S_2$ *and* I(ii) *holds:* Here $\alpha_2\alpha_3\alpha_1\alpha_4 \ldots \alpha_n$ and $\alpha_3\alpha_1\alpha_2\alpha_4 \ldots \alpha_n \in S_2$. Thus

$$\alpha(G) = [\alpha_1(G), \alpha_2(G), \ldots, \alpha_n(G)] \subseteq [\alpha_2(G), \alpha_3(G), \alpha_1(G), \alpha_4(G), \ldots, \alpha_n(G)][\alpha_3(G), \alpha_1(G), \alpha_2(G), \alpha_4(G), \ldots, \alpha_n(G)] = 1.$$

Case (c) $S_1 < S_2$ *and* I(iii) *holds:* Here $\beta \in S_2$ and α is β-valued. Thus $\alpha^*(G) \subseteq \beta^*(G) = 1$. Therefore $\alpha(G) = 1$.

Case (d) $S_1 < \phi S_2$ *and* II(i) *holds:* Here $\alpha \in S_2$. Thus $\alpha(G) = 1$.

Case (e) $S_1 < \phi S_2$ *and* II(ii) *holds*. Here $\gamma\alpha_2 \ldots \alpha_n$ is ϕ-valued and $T(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq S_2$. Thus by Lemma 3, $\alpha(G) = 1$.

COROLLARY *If* $\alpha \in L(\gamma)$ *and* $\alpha \ll (\alpha)\delta_k$ *for all* $k \geqq r$ *for some fixed integer* r, *then* $f(\alpha) \in \Gamma$ *is concise.*

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Proof. A routine check will establish that when $S_1 < (\phi)S_2$ for some $\phi \in L(\gamma)$, it follows that $f(S_1) < (f(\phi))f(S_2)$. The details are omitted.

Some of the more important properties of the relations defined in Definition 3 are now established.

LEMMA 4. *If α, β, ϕ and $\psi \in L(\gamma)$, with ϕ being ψ-valued and $\alpha \ll (\phi)\beta$, then $\alpha \ll (\psi)\beta$.*

Proof. This follows almost immediately from the definitions.

In the rest of this paper if U and $V \in S(L(\gamma))$ then UV will denote the set $\{uv | u \in U, v \in V\}$.

LEMMA 5. *If α, $\beta \in L(\gamma)$ with U, $V \in S(L(\gamma))$, and $U \ll (\alpha)V$, then $\beta \ll (\alpha\beta)V\beta \cup \alpha\beta\alpha$.*

Proof. Let $U = S_1 < (\alpha)S_2 < (\alpha) \ldots < (\alpha)S_r = V$, where $S_i \in S(L(\gamma))$ $(i = 1, 2, \ldots, r)$. It will be shown that

(*) $\quad S_i\beta \ll (\alpha\beta)S_{i+1}\beta \cup \alpha\beta\alpha \quad$ for $i = 1, 2, \ldots, r - 1$.

Since $\alpha\beta\alpha$ appears on the right-hand side, we may by either (I)(iii) or (II)(i) add $\alpha\beta\alpha$ to the left-hand side, obtaining

(**) $\quad S_i\beta \cup \alpha\beta\alpha \ll (\alpha\beta)\, S_{i+1}\beta \cup \alpha\beta\alpha \quad$ for $i = 1, 2, \ldots, r - 1$.

It follows from (*) and (**) that

$$\beta \ll (\alpha\beta)\, V\beta \cup \alpha\beta\alpha$$

and all that remains is to prove (*).

Let us suppose that $X = S_i < (\alpha)S_{i+1} = Y$. If $\phi \in X$ with $\phi = \alpha_1\alpha_2 \ldots \alpha_n$, where $\alpha_j \in L(\gamma)$ and (I)(i) or (ii) holds, then $\phi\beta \in X\beta$ and $\alpha_2 \ldots \alpha_n\beta \in Y\beta \cup \alpha\beta\alpha$ or $\alpha_2\alpha_3\alpha_1\alpha_4 \ldots \alpha_n\beta$ and $\alpha_3\alpha_1\alpha_2\alpha_4 \ldots \alpha_n\, \beta (n \geqq 3) \in Y\beta \cup \alpha\beta\alpha$. If (I)(iii) holds, then there exists $\theta \in Y$ such that ϕ is θ-valued. Hence there exists $\theta\beta \in Y\beta$ such that $\phi\beta$ is $\theta\beta$-valued. Therefore $X\beta < Y\beta \cup \alpha\beta\alpha$.

Suppose that (II) holds. It is now shown that if $\phi = \alpha_1\alpha_2 \ldots \alpha_n \in X$ for $\alpha_i \in L(\gamma)$, and $\gamma\alpha_2 \ldots \alpha_n$ is α-valued, then $\gamma\alpha_2 \ldots \alpha_n\beta$ is $\alpha\beta$-valued, and that if $T(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq Y$, then $T(\alpha_1, \alpha_2, \ldots, \alpha_n, y) \ll Y\beta \cup \alpha\beta\alpha$. The former statement follows from our definition of valuedness. To prove the latter we calculate $T(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta)$ which is equal to

$$\begin{aligned}&\{\alpha_1(\alpha_1\alpha_2)\alpha_3 \ldots \alpha_n\beta, \ldots, \alpha_1 \ldots \alpha_i(\alpha_1 \ldots \alpha_{i+1})\alpha_{i+2} \ldots \alpha_n\beta, \ldots,\\ &\quad \alpha_1 \ldots \alpha_{n-1}(\alpha_1 \ldots \alpha_n)\beta, \alpha_1 \ldots \alpha_n(\alpha_1 \ldots \alpha_n\beta)\}\\ &= \{\alpha_1(\alpha_1\alpha_2)\alpha_3 \ldots \alpha_n, \ldots, \alpha_1 \ldots \alpha_i(\alpha_1 \ldots \alpha_{i+1})\alpha_{i+2} \ldots \alpha_n, \ldots,\\ &\quad \alpha_1 \ldots \alpha_{n-1}(\alpha_1 \ldots \alpha_n)\}\beta \cup \phi\beta\phi.\end{aligned}$$

Now ϕ is $\gamma\alpha_2 \ldots \alpha_n$-valued and $\gamma\alpha_2 \ldots \alpha_n$ is α-valued. So by the transitivity of valuedness ϕ is α-valued. Hence by (I)(iii)

$$\phi\beta \ll (\alpha\beta)T(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta) \ll Y\beta \cup \alpha\beta\alpha.$$

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Therefore $\phi\beta \ll (\alpha\beta) Y\beta \cup \alpha\beta\alpha$ for all $\phi \in X$. In other words $X\beta \ll (\alpha\beta) Y\beta \cup \alpha\beta\alpha$ and (*) is proved.

The next lemma could be thought of as a proof of the conciseness of the derived words (a fact originally proved by Turner-Smith [**7**].)

(If $\alpha \in L(\gamma)$, α^r will denote $\alpha\alpha \ldots \alpha$ with r α's.)

LEMMA 6. *For $r \geqq 1$, $\delta_r \ll (\delta_r)\delta_{r+1}$.*

Proof. Proceed by induction on r. If $r = 1$, $\delta_1 = \gamma^2$ and since $\gamma\gamma$ is γ^2-valued,

$$\gamma^2 \ll (\gamma^2)T(\gamma, \gamma) = \gamma^3 \ll (\gamma^2)T(\gamma^2, \gamma) < \gamma^2 \cdot \gamma^2.$$

Therefore $\delta_1 \ll (\delta_1)\delta_2$.

Suppose that $\delta_r \ll (\delta_r)\delta_{r+1}$ for some $r \geqq 1$. Then $\delta_r^2 \ll (\delta_{r+1}) \{\delta_r^3, \delta_{r+1}^2\}$ by Lemma 5, and by induction one can prove that

$$(*)_s \quad \delta_r^s \ll (\delta_{r+1}) \{\delta_r^{s+1}, \delta_{r+1}^2\}.$$

Next note that $\delta_{r+1} = \delta_0\delta_0\delta_1\delta_2 \ldots \delta_r$ and that $\delta_s^{r+3} = \delta_{r+1}\delta_r \ldots \delta_r$ where δ_r occurs $r + 1$ times. Since $\gamma\delta_r \ldots \delta_r$, with $r + 1$ δ_r's, is δ_{r+1}-valued,

$$\delta_r^{r+3} \ll (\delta_{r+1})T(\delta_{r+1}, \delta_r, \ldots, \delta_r).$$

Now the term on the right is equal to

$$\{\delta_{r+1}(\delta_{r+1}\delta_r)\delta_r \ldots \delta_r, \ldots, \delta_{r+1}\delta_r \ldots \delta_r(\delta_{r+1}\delta_r \ldots \delta_r)\}$$

$\ll \delta_{r+1}^2$ by (I)(i). So $\delta_r^{r+3} \ll (\delta_{r+1})\delta_{r+1}^2 = \delta_{r+2}$.

By the repeated use of $(*)_s$ for $s = 2, 3, \ldots, r + 2$, one obtains

$$\delta_{r+1} = \delta_r^2 \ll (\delta_{r+1}) \{\delta_r^{r+3}, \delta_{r+1}^2\}.$$

Therefore

$$\delta_{r+1} \ll (\delta_{r+1})\delta_{r+2}.$$

Consider the set

$$\mathscr{Y} = \{\alpha \in L(\gamma) | \alpha \ll (\alpha)\delta_r \text{ for all } r\}.$$

If $\beta \in \mathscr{Y}$, then by the corollary to Lemma 2, $f(\beta)$ is concise. It will now be shown that $\mathscr{Y}$ has a certain property $\mathscr{P}$ and in the third section of the paper we will discover that $\mathscr{P}$ is possessed only by $L(\gamma)$ and possibly $L(\gamma)\backslash\{\gamma\}$. This will establish Theorem 1. In order to define $\mathscr{P}$ we need another quasi-ordering.

Definition 5. Let $\mathscr{X}$ be a subset (not necessarily finite) of $L(\gamma)$, and let $U, V \in \bar{S}(L(\gamma))$ with $\phi \in L(\gamma)$. Then we shall write $U \prec (\phi, \mathscr{X}) V$ if for each $\alpha \in U$ one of the following holds:

(V) (i) $\alpha \in V$;

(ii) there exist $\theta, \psi \in L(\gamma)$ such that $\alpha = \theta\psi$, α is ϕ-valued, $\psi \in \mathscr{X}$ and $\theta\psi\psi = \alpha\psi \in V$.

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If there exist $S_i (i = 1, 2, \ldots, n) \in \bar{S}(L(\gamma))$ such that for each $i < n$ either $S_i \ll S_{i+1}$ or $S_i \prec (\phi, \mathscr{X})S_{i+1}$ then we will write $S_1 \lll (\phi, \mathscr{X})S_n$. (The $\mathscr{X}$'s will usually be omitted except where there might be confusion.)

Definition 6. A subset $\mathscr{X}$ of $L(\gamma)$ (not necessarily finite) will be said to possess $\mathscr{P}$ if the following conditions hold:
(i) $\delta_r \in \mathscr{X}$ for $r \geqq 1$;
(ii) if $\phi \lll (\phi, \mathscr{X})\delta_r$ for all r, then $\phi \in \mathscr{X}$.

LEMMA 7. *$\mathscr{Y} = \{\alpha \in L(\gamma) | \alpha \ll (\alpha)\delta_r$ for all $r\}$ possesses $\mathscr{P}$.*

Proof. Certainly $\delta_r \ll (\delta_r)\delta_r$. By Lemma 6, $\delta_r \ll (\delta_r)\delta_{r+1}$ and by induction one can show that $\delta_r \ll (\delta_r)\delta_s$ for all $s \geqq r$. If $r > s$, then δ_r is δ_s-valued and $\delta_r \ll \delta_s$. So $\delta_r \in \mathscr{Y}$ for all $r \geqq 1$.

Now suppose that $U, V \in S(L(\gamma))$, $\phi \in L(\gamma)$ and $U \lll (\phi) V$. We prove that $U \ll (\phi) V$. Now if $\phi \lll (\phi)\delta_r$ for all r, then we have a sequence

$$\{S_i | S_i \in \bar{S}(L(\gamma)), i = 1, 2, \ldots, n\}$$

such that $S_1 = \phi$ and $S_n = \delta_r$, and for $i = 1, 2, \ldots, n-1$ either $S_i \ll S_{i+1}$ or $S_i \prec (\phi)S_{i+1}$. Hence either $S_i \ll S_{i+1}$ or $S_i \ll (\phi)S_{i+1}$, from which it follows that $\phi \ll (\phi)\delta_r$ for all r, i.e., $\phi \in \mathscr{Y}$.

Suppose that $U \prec (\phi) V$. Let $\alpha \in U$. Then either $\alpha \in V$ or there exist θ, $\psi \in L(\gamma)$ such that α is ϕ-valued, $\psi \in \mathscr{Y}$ and $\alpha\psi = \theta\dot{\psi}\psi \in V$. Since $\psi \in \mathscr{Y}$, it follows that $\psi \ll (\psi)\delta_t$ for all t. Let s be an integer such that δ_s is ψ-valued. Then since $\psi \ll (\psi)\delta_{s+1} = \delta_s^2$ it follows, from the fact that $\delta_s^2 \ll \psi^2$, that $\psi \ll (\psi)\psi^2$. Hence by Lemma 5,

$$\psi\theta \ll (\psi\theta)\{\psi^2\theta, \psi\theta\psi\}.$$

By (I)(ii), $\{\psi^2\theta, \psi\theta\psi\} < \theta\psi\psi$. Therefore $\alpha = \theta\psi \ll (\alpha)\alpha\psi$, and since α is ϕ-valued, $\alpha \ll (\phi)\alpha\psi$ by Lemma 4. So $U \ll (\phi) V$ and our lemma is proved.

2. Ellipticity in polycyclic groups. If $\phi \in \Gamma$ then a group G will be called ϕ-elliptic if G is w-elliptic for some word w associated with ϕ. It will be shown that the set

$$\mathscr{X} = \{\alpha \in L(\gamma) | \text{ every polycyclic group is } f(\alpha)\text{-elliptic}\}$$

has the property $\mathscr{P}$. Applying the result of the final section we will obtain Theorem 2.

Given a group G and $\phi \in \Gamma$, $l_\phi(x, G)$ will denote the ϕ-length of $x \in \phi(G)$ (i.e. the w-length of x where w is some word associated with ϕ), and $l_\phi(N, G)$ will denote the maximum of the set $\{l_\phi(x, G) | x \in N\}$ where $N \subseteq \phi(G)$.

The next result is probably well-known though I can find no reference for it.

LEMMA 8. *If G is a group, $\phi \in \Gamma$, $N \subseteq S \subseteq \phi(G)$ and $N \trianglelefteq G$, then*

$$l_\phi(N, G) + l_\phi(S/N, G/N) \geqq l_\phi(S, G).$$

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Proof. Let $x \in S$, $l_\phi(S/N, G/N) = r$ and $l_\phi(N, G) = s$. Then $l_\phi(xN, G/N) \leqq r$. Hence $xN = x_1^{\epsilon_1}Nx_2^{\epsilon_2}N \ldots x_j^{\epsilon_j}N$, where $x_i \in \phi^*(G)$, $j \leqq r$ and $\epsilon_i = \pm 1$. Since x and $x_i \in \phi^*(G)$ for $i = 1, 2, \ldots, j$, $x = x_1^{\epsilon_1}x_2^{\epsilon_2} \ldots x_j^{\epsilon_j}y$, where $y \in N$. Since $l_\phi(N, G) = s$, $y = y_1^{\xi_1}y_2^{\xi_2} \ldots y_k^{\xi_k}$, where $y_i \in \phi^*(G)$, $\xi_i = \pm 1$ and $k \leqq s$, and it follows that $l_\phi(x, G) \leqq j + k \leqq r + s$.

LEMMA 9. *If G is polycyclic, α, β, α' and $\beta' \in \Gamma$, G is β'-elliptic, β is β'-valued, $\alpha\beta'$ is α'-valued and $(\alpha\beta\beta)(G) = 1$, then $l_{\alpha'}((\alpha\beta)(G), G)$ is finite.*

Proof. Let $l_{\beta'}(\beta(G), G) = r$. Since $(\alpha\beta\beta)(G) = 1$ and $(\alpha\beta)(G) \subseteq (G)$, $(\alpha\beta)(G)$ is Abelian, and since G is polycyclic $(\alpha\beta)(G)$ is finitely generated. Therefore $(\alpha\beta)(G)$ is generated by elements of the form $[a_i, b_i]$, where $a_i \in \alpha^*(G)$ and $b_i \in \beta^*(G)$ for $i = 1, 2, \ldots, n$. Now $[a_i, b_i]^m = [a_i, b_i^m]$ because $[(\alpha\beta)(G), \beta(G)] = 1$, but $b_i^m \in \beta(G)$ and $l_{\beta'}(\beta(G), G) = r$. Therefore, since $\beta(G) \subseteq \beta'(G)$, $b_i^m = c_1c_2 \ldots c_t$ for $c_i \in \beta'^*(G)$ and $t \leqq r$. It will now be shown by induction on s that $l_{\alpha'}([a_i, c_1c_2 \ldots c_s], G) \leqq s$. If $s = 1$, $[a_i, c_1] \in (\alpha\beta')^*(G)$ and hence $[a_i, c_1] \in \alpha'^*(G)$. So $l_{\alpha'}([a_i, c_1], G) \leqq 1$. Suppose the statement is true for all s less than some fixed s_1. Now

$$[a_i, c_1 \ldots c_{s_1}] = [a_i, c_{s_1}][a_i, c_1 \ldots c_{s_1-1}]^{c_{s_1}}.$$

By the induction hypothesis, $[a_i, c_1c_2 \ldots c_{s_1-1}] = u_1u_2 \ldots u_h$ where $u_k \in \alpha'^*(G)$ $(k = 1, \ldots, h)$ and $h \leqq s_1 - 1$. Thus $l_{\alpha'}([a_i, c_1c_2 \ldots c_s], G) \leqq s_1$. We now see that $l_{\alpha'}([a_i, b_i^m], G) \leqq r$ and hence $l_{\alpha'}((\alpha\beta)(G), G) \leqq nr$.

COROLLARY. *If G is β-elliptic and polycyclic, α is α'-valued and $(\alpha\beta\beta)(G) = 1$, then $l_{\alpha'}((\alpha\beta)(G), G)$ is finite.*

The next lemma shows that all the derived elements are in $\mathscr{X}$.

LEMMA 10. *Let $\mathscr{X}$ be as defined above. Then $\delta_r \in \mathscr{X}$ for $r \geqq 1$.*

Proof. We have to show that every polycyclic group is δ_r-elliptic for $r \geqq 1$. The proof proceeds by double induction on r and the derived length d of a polycyclic group G. By Corollary 1 of A. H. Rhemtulla [**4**] the group G is δ_1-elliptic. Suppose that all polycyclic groups are δ_{r_1}-elliptic for $r_1 < r$, for some fixed $r > 1$. If G is Abelian, then $\delta_r(G) = 1$ and there is nothing to prove. Suppose that all polycyclic groups of derived length less than $s > 1$ are δ_r-elliptic, and that G has derived length s. Then $l_{\delta_r}(G^{(r+1)}, G')$ is finite, since G' has derived length less than s and $l_{\delta_r}(G^{(r+1)}, G)$ is finite. If it can be shown that $l_{\delta_r}(G^{(r)}/G^{(r+1)}, G/G^{(r+1)})$ is finite, then by Lemma 8, the length $l_{\delta_r}(G^{(r)}, G)$ is finite and G is δ_r-elliptic. If $G^{(r+1)} \neq 1$, then $s > r + 1$ and the required result follows from the induction hypothesis, so we may assume that $G^{(r+1)} = 1$. Therefore

$$(\delta_{r-1}\delta_r\delta_r)(G) \subseteq (\delta_r\delta_r)(G) = \delta_{r+1}(G) = 1.$$

Let $\alpha = \delta_{r-1}$, $\beta = \delta_r$, $\alpha' = \delta_r$, $\beta' = \delta_{r-1}$. Then by Lemma 9, $l_{\delta_r}((\delta_{r-1}\delta_r)(G), G)$ is finite. Let $H = (\delta_{r-1}\delta_r)(G)$. By Lemma 8 it is sufficient to show that

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$l_{\delta_r}(G^{(r)}/H, G/H)$ is finite, and without loss of generality we may assume that $H = 1$. Therefore

$$(\delta_{r-1}\delta_{r-1}\delta_{r-1})(G) = [G^{(r-1)}, G^{(r)}] = H = 1.$$

Now in the Corollary to Lemma 9 put $\alpha = \beta = \delta_{r-1}$ and $\alpha' = \delta_r$. The conditions of the lemma are satisfied. Hence $l_{\delta_r}(G^{(r)}, G)$ is finite. Therefore G is δ_r-elliptic.

We can now prove the main result of this section.

LEMMA 11. *The set $\mathscr{X}$ as defined above has the property $\mathscr{P}$.*

Proof. By Lemma 10, $\delta_r \in \mathscr{X}$ for all $r \geqq 1$. Let $\phi \in L(\gamma)$ such that $\phi \ll (\phi)\delta_r$ for all r. Then we have to prove that $\phi \in \mathscr{X}$.

Let $f(\phi) = \alpha$ and let G be a polycyclic group which is not α-elliptic. Let Ω be the set of all normal subgroups $N \subseteq \alpha(G)$ such that $l_\alpha(N, G)$ is finite. Since G is polycyclic, Ω will have maximal elements. Let N_1 and N_2 be two such maximal elements. If $x \in N_1N_2$, $x = yz$ for $y \in N_1$, $z \in N_2$. Let $l_\alpha(N_i, G) = r_i (i = 1, 2)$. Then it follows that

$$l_\alpha(x, G) \leqq l_\alpha(y, G) + l_\alpha(z, G) \leqq r_1 + r_2.$$

Hence

$$l_\alpha(N_1N_2, G) \leqq l_\alpha(N_1, G) + l_\alpha(N_2, G).$$

So $N_1 = N_2$. Thus Ω has a unique maximal element N.

Now if $M \trianglelefteq G$, $M \subseteq \alpha(G)$ and $l_\alpha(M/N, G/N)$ is finite, by Lemma 8, $l_\alpha(M, G)$ is finite so that $M \subseteq N$. Since G is not α-elliptic, $l_\alpha(\alpha(G), G)$ is infinite and by the above argument we see that $l_\alpha(\alpha(G/N), G/N)$ is infinite. Therefore G/N is not α-elliptic. Thus we may assume without loss of generality that if $N_0 \trianglelefteq G$, $N_0 \subseteq \alpha(G)$ and $l_\alpha(N, G)$ is finite, then $N = 1$.

Since G is soluble, there exists an s such that $\delta_s(G) = 1$. Now $\phi \ll (\phi)\delta_s$, so we can pick $S_i \in S(L(\gamma)) (i = 1, 2, \ldots, n)$ such that $\phi = S_1$, $S_n = \delta_s$ and $S_i \prec \phi S_{i+1}$ or $S_i < S_{i+1}$ for $i < n$.

The remainder of the proof is divided into two parts. Let $U, V \in \bar{S}(L(\gamma))$.

Part 1. If $U \prec \phi V$ and $f(\beta)(G) = 1$ for all $\beta \in V$, then $f(\beta)(G) = 1$ for all $\beta \in U$.

Part 2. If $U < V$ and $f(\beta)(G) = 1$ for all $\beta \in V$, then $f(\beta)(G) = 1$ for all $\beta \in U$.

From parts 1 and 2 it will follow that, since $\delta_s(G) = 1$, $\alpha(G) = f(\phi)(G) = 1$. Thus we will have a contradiction and G must be α-elliptic.

Proof of Part 1. Suppose that, for $U, V \in \bar{S}(L(\gamma))$, $U \prec (\phi)V$ and $f(\beta)(G) = 1$ for all $\beta \in V$. Let $\beta \in U$. Then either $\beta \in V$, in which case $f(\beta)(G) = 1$, or there exist $\theta, \psi \in L(\gamma)$ with $\psi \in \mathscr{X}$ such that β is ϕ-valued, $\beta = \theta\psi$ and $\beta\psi = \theta\psi\psi \in V$. Since $\theta\psi\psi \in V$, $f(\theta\psi\psi)(G) = 1$. Furthermore G is polycyclic and $f(\psi)$-elliptic, because $\psi \in \mathscr{X}$. Also $f(\beta)$ is $f(\phi)$-valued and $f(\beta) = f(\theta)f(\psi)$. So applying the corollary to Lemma 9, we see that $l_\alpha(f(\beta)(G), G)$ is finite. Hence $f(\beta)(G) = 1$.

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Proof of Part 2. Suppose that, for some $U, V \in \bar{S}(L(\gamma))$, $U < V$ and $f(\beta)(G) = 1$ for all $\beta \in V$. We have to consider cases (I)(i), (ii) and (iii). Let $\beta = \beta_1\beta_2 \ldots \beta_n \in U$.

Case I(i). Here $\beta_2 \ldots \beta_n \in V$. Hence

$$f(\beta_1\beta_2 \ldots \beta_n)(G) = [f(\beta_1)(G), f(\beta_2)(G), \ldots, f(\beta_n)(G)]$$

$$[f(\beta_2)(G), \ldots, f(\beta_n)(G)] = f(\beta_2 \ldots \beta_n)(G) = 1.$$

Case I(ii). Here $\{\beta_2\beta_3\beta_1\beta_4 \ldots \beta_n, \beta_3\beta_1\beta_2\beta_4 \ldots \beta_n\} \subseteq V$.

$$[f(\beta_1)(G), f(\beta_2)(G), f(\beta_3)(G)] \subseteq [f(\beta_2)(G), f(\beta_3)(G), f(\beta_1)(G)]$$
$$[f(\beta_3)(G), f(\beta_1)(G), f(\beta_2)(G)]$$

by the three-subgroup lemma (see for example corollary to Lemma 3.2 of [**2**]). Hence $f(\beta_1\beta_2\beta_3 \ldots \beta_n)(G) \subseteq f(\beta_2\beta_3\beta_1 \ldots \beta_n)(G) f(\beta_3\beta_1\beta_2 \ldots \beta_n)(G) = 1$. Therefor $f(\beta)(G) = 1$.

Case I (iii). Here there exists $\theta \in V$ such that β is θ-valued. So $f(\beta)$ is $f(\theta)$-valued and $f(\beta)(G) \subseteq f(\theta)(G) = 1$.

Hence $\mathscr{X}$ has the property $\mathscr{P}$.

3. Sets which possess $\mathscr{P}$. Here we prove the key result that if a subset $L(\gamma)$ has $\mathscr{P}$ it is either $L(\gamma)$ or $L(\gamma)\backslash\{\gamma\}$. Hence Theorems 1 and 2 follow as corollaries.

Definition 7. Define functions λ, $\Delta: L(\gamma) \to \mathbf{N}$ (the natural numbers) as follows:

if $\phi \in L(\gamma)$ is δ_r for some r, then $\lambda(\delta_r) = r$ and $\Delta(\delta_r) = 1$;
if ϕ is not derived and $\phi = \alpha\beta$, then $\lambda(\phi) = \max\{\lambda(\alpha), \lambda(\beta)\}$ and $\Delta(\phi) = \Delta(\alpha) + \Delta(\beta)$. (Note that $\Delta(\phi) = 1$ if and only if ϕ is derived.)

LEMMA 12. *If $\phi \in L(\gamma)$ and $\lambda(\phi) = m$, then $\phi \ll \delta_m$.*

Proof. If $l(\phi) = 1$, then $\phi = \gamma$ and $\lambda(\phi) = 0$. Thus $\phi \ll \delta_0$. Suppose that, for $\theta \in L(\gamma)$ such that $l(\theta) < m$, $\theta \ll \delta_{\lambda(\theta)}$. Let $\phi \in L(\gamma)$ be chosen so that $l(\phi) = m$. Then there are $\alpha, \beta \in L(\gamma)$ such that $\phi = \alpha\beta$. Since $l(\phi) = l(\alpha) + l(\beta)$, $l(\alpha)$ and $l(\beta) < m$. It then follows that $\alpha \ll \delta_{\lambda(\alpha)}$ and $\beta \ll \delta_{\lambda(\beta)}$. Therefore $\alpha\beta \ll \delta_{\lambda(\alpha)}\beta \ll \delta_{\lambda(\alpha)}\delta_{\lambda(\beta)}$. Hence $\phi \ll \delta_{\lambda(\alpha)}\delta_{\lambda(\beta)}$. If $\phi = \delta_r$ for some r, then $\phi \ll \delta_r$. If ϕ is not derived, $\lambda(\phi) = \max\{\lambda(\alpha), \lambda(\beta)\}$ and $\delta_{\lambda(\alpha)}\delta_{\lambda(\beta)} \ll \delta_{\lambda(\phi)}$. Therefore $\phi \ll \delta_{\lambda(\phi)}$.

LEMMA 13. *If $\lambda(\alpha) \leqq \lambda(\beta)$, where $\alpha, \beta \in L(\gamma)$, then $\alpha\beta \ll V$ for some $V \in \bar{S}(L(\gamma))$, where, for each $\theta \in V$, θ is α-valued, $\Delta(\theta) \leqq \Delta(\alpha)$ and $l(\theta) > l(\alpha)$.*

Proof. Choose $\alpha, \beta \in L(\gamma)$ such that $\lambda(\alpha) \leqq \lambda(\beta)$. If $\Delta(\alpha) = 1$, then $\alpha = \delta_r$ for some r. Hence $\alpha\beta \ll \alpha\delta_{\lambda(\beta)} \ll \delta_r\delta_{\lambda(\beta)}$. If $r < \lambda(\beta)$, then $\alpha\beta \ll \delta_{\lambda(\beta)}$ and $\delta_{\lambda(\beta)}$ is $\delta_r = \alpha$-valued, $\Delta(\delta_{\lambda(\beta)}) = 1 = \Delta(\alpha)$ and $l(\delta_{\lambda(\beta)}) > l(\alpha)$. If $r = \lambda(\beta)$, $\alpha\beta \ll \delta_{r+1}$, and δ_{r+1} is $\delta_r = \alpha$-valued, $\Delta(\delta_{r+1}) = 1 = \Delta(\alpha)$ and $l(\delta_{r+1}) > l(\alpha)$.

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Let $n = \Delta(\alpha) > 1$ and suppose that the lemma holds for α_1 when $\Delta(\alpha_1) < n$. Then there exist $\theta_1, \theta_2 \in L(\gamma)$ such that $\alpha = \theta_1\theta_2$, and $\Delta(\alpha) = \Delta(\theta_1) + \Delta(\theta_2)$, because α is not derived. Hence there exist $V_i \in \bar{S}(L(\gamma))$ such that $\theta_i\beta \ll V_i$ (for $i = 1, 2$) and such that if $\psi \in V_i$, then ψ is θ_i-valued, $\Delta(\psi) \leqq \Delta(\theta_i)$ and $l(\psi) > l(\theta_i)$. So

$$\alpha\beta \ll \{(\theta_1\beta)\theta_2, \theta_1(\theta_2\beta)\}.$$

Hence $\alpha\beta \ll \{V_1\theta_2, \theta_1V_2\}$. If $\psi_1 \in V_1$, then

$$\Delta(\psi_1\theta_2) \leqq \Delta(\psi_1) + \Delta(\theta_2) \leqq \Delta(\theta_1) + \Delta(\theta_2) = \Delta(\alpha),$$

$\psi_1\theta_2$ is $\theta_1\theta_2 = \alpha$-valued, and

$$l(\psi_1\theta_2) = l(\psi_1) + l(\theta_2) > l(\theta_1) + l(\theta_2) = l(\alpha).$$

Similarly we see that if $\psi_2 \in V_2$, $\theta_1\psi_2$ is α-valued, $\Delta(\theta_1\psi_2) \leqq \Delta(\alpha)$ and $l(\theta_1\psi_2) > l(\alpha)$. So, putting $V = \{V_1\theta_2, \theta_1V_2\}$, the lemma is proved.

LEMMA 14. *If $\Delta(\theta) = m$ and $l(\theta) \geqq 2^r m$, then $\theta \ll \delta_r$.*

Proof. If $\Delta(\theta) = 1$ and $l(\theta) \geqq 2^r$, then $\theta = \delta_s$ for some s. Now $l(\theta) = 2^s \geqq 2^r$, so $r \leqq s$, δ_s is δ_r-valued and $\theta \ll \delta_r$.

Assume that $\psi \ll \delta_r$ for $l(\psi) \geqq \Delta(\psi)2^r$ and $1 \leqq \Delta(\psi) < m$. If $\Delta(\theta) = m > 1$ there exist $\theta_1, \theta_2 \in L(\gamma)$ such that $\theta = \theta_1\theta_2$. Suppose that $l(\theta) \geqq 2^r m$. Let $\Delta(\theta_1) = m_1$, $\Delta(\theta_2) = m_2$, $m_1, m_2 < m$. Then $l(\theta_1) + l(\theta_2) \geqq 2^r m_1 + 2^r m_2$ and either $l(\theta_1) \geqq 2^r m_1$ or $l(\theta_2) \geqq 2^r m_2$. Therefore either $\theta_1 \ll \delta_r$ or $\theta_2 \ll \delta_r$ by the induction hypothesis. Since $\theta \ll \theta_1$ and $\theta \ll \theta_2$, it follows that $\theta \ll \delta_r$.

We now prove our result on sets with the property $\mathscr{P}$.

LEMMA 15. *If $\mathscr{X}$ is a subset of $L(\gamma)$ with $\mathscr{P}$, then $L(\gamma)\backslash\{\gamma\} \subseteq \mathscr{X}$.*

Proof. Let $\theta \in L(\gamma)\backslash\{\gamma\}$, The proof proceeds by induction on $\Delta(\theta)$.

If $\Delta(\theta) = 1$, then θ is derived and $\theta \in \mathscr{X}$.

Let $n > 1$. Suppose that $\psi \in \mathscr{X}$ whenever $\Delta(\psi) < n$. Let $\Delta(\theta) = n$. Then there exist α and β such that $\theta = \alpha\beta$. Since $\Delta(\theta) > 1$, θ is not derived. Hence $\Delta(\theta) = \Delta(\alpha) + \Delta(\beta)$, $\Delta(\alpha)$ and $\Delta(\beta) < n$, and therefore α and β are elements of $\mathscr{X}$.

We claim that there exists a sequence of sets S_i in $\bar{S}(L(\gamma))$ such that $S_1 = \theta$, $S_i \lll (\theta)S_{i+1}$ and if ψ is an element of S_i, then $\psi \neq \gamma$, ψ is θ-valued and $\psi = \psi_1\psi_2$ where $\Delta(\psi_1) + \Delta(\psi_2) \leqq n$ and $l(\psi) > i$.

If this can be shown, then the rest of the lemma will follow, for if $\psi \in S_{n2^s}$, $l(\psi) > n2^s$ and, by Lemma 14, $\psi \ll \delta_s$. Therefore $S_{n2^s} \ll \delta_s$ and we see that $\theta \lll (\theta)\delta_s$ for all s. Hence $\theta \in \mathscr{X}$.

Suppose that S_i has been defined Let $\psi \in S_i$. Then $\psi = \psi_1\psi_2$, where $\Delta(\psi_1) + \Delta(\psi_2) \leqq n$. Therefore $\Delta(\psi_1)$, $\Delta(\psi_2) < n$. If follows that $\psi_1, \psi_2 \in \mathscr{X}$ by the induction hypothesis. Take $\lambda(\psi_1) \leqq \lambda(\psi_2)$ without loss of generality. Then by Definition 5, $\psi \lll (\phi)\psi_1\psi_2\psi_2$. By Lemma 13 there exists a set $V(\psi) \in \bar{S}(L(\gamma))$

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such that $\psi_1\psi_2 \ll V(\psi)$, where $V(\psi)$ consists of ψ_1-valued elements and, for each $\sigma \in V(\psi)$, $\Delta(\sigma) \leqq \Delta(\psi_1)$ and $l(\sigma) > l(\psi_1)$. Since $\psi_1\psi_2\psi_2 \ll V(\psi)\psi_2$, we can write $\psi \ll (\phi) V(\psi)\psi_2$. If $\sigma \in V(\psi)$, then $\sigma\psi_2$ is $\psi_1\psi_2 = \psi$-valued and hence ϕ-valued. Furthermore

$$\Delta(\sigma) + \Delta(\psi_2) \leqq \Delta(\psi_1) + \Delta(\psi_2) \leqq n$$

and

$$l(\sigma\psi_2) = l(\sigma) + l(\psi_2) > l(\psi_1) + l(\psi_2) = l(\psi).$$

Now $l(\psi) > i$, so $l(\sigma\psi_2) > i + 1$. Putting

$$S_{i+1} = \bigcup \{ V(\psi)\psi_2 | \psi \in S_i\},$$

we see that $S_i \ll (\phi)S_{i+1}$ and S_{i+1} has all the required properties. Hence the lemma is proved.

References

1. P. Hall, *Finiteness conditions for soluble groups*, Proc. London Math. Soc. *4* (1954), 419–436.
2. ——— *The Edmonton notes on nilpotent groups*, Queen Mary College Mathematics Notes, 1969 (originally published in 1957 by the Can. Math. Congress).
3. A. H. Rhemtulla, *A problem in bounded expressibility in free products*, Proc. Cambridge Philos. Soc. *64* (1968), 573–584.
4. ——— *On commutators of certain finitely-generated soluble groups*, Can. J. Math. *21* (1969), 1160–1164.
5. Derek J. S. Robinson, *Finiteness conditions and generalized soluble groups, I*, Ergebnisse der Mathematik, Volume *62*, 119–121 (Springer, Berlin, 1972).
6. P. W. Stroud, Thesis, Cambridge (1966).
7. R. F. Turner-Smith, *Finiteness conditions for verbal subgroups*, J. London Math. Soc. *41* (1966), 166–176.

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