J. Austral. Math. Soc. (Series A) 33 (1982), 401–410

THE LATTICE OF STABLE MARRIAGES AND PERMUTATIONS

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(Received 12 May 1981; revised 30 September 1981)

Communicated by W. D. Wallis

Abstract

Recently, we have introduced the notion of stable permutations in a Latin rectangle $L(r \times c)$ of r rows and c columns. In this note, we prove that the set of all stable permutations in $L(r \times c)$ forms a distributive lattice which is Boolean if and only if $c \leqslant 2$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 06 B 05; secondary 05 B 15.
Keywords and phrases: lattice, Latin rectangle, stable marriage, stable permutation, system of I-M preference, and duality principle.

1. Introduction

A Latin rectangle $L(r \times c)$ of r rows and c columns, where $c \leqslant r$, is an $r \times c$ matrix over r elements, say $1, 2, \ldots, r$, such that no element occurs twice within any row or column of the matrix. By bordering with the natural order $i = 1, 2, \ldots, r$, there can be found permutations Π_j, $j = 1, 2, \ldots, c$, such that, see [4],

$$L(r \times c) = \{i : a_{ij} \mid \Pi_j(i) = a_{ij}\}, \qquad i = 1, 2, \ldots, r, \; j = 1, 2, \ldots, c.$$

Each Π_j is called the jth fundamental permutation of $L(r \times c)$.

We now consider $L(r \times c)$ as being in a plane and the cells of $L(r \times c)$ as being points in that plane. If Π is a permutation over $1, 2, \ldots, r$, then $\Pi(i)$ denotes the cell (point) in row i which contains the image of the symbol i under Π. We let $\Pi(r \times c)$ denote the class of all permutations Π over $1, 2, \ldots, r$ whose

https://doi.org/10.1017/S1446788700018838 Published online by Cambridge University Press

images $\Pi(i) \in L(r \times c)$, that is,

$$\Pi(i) = \Pi_j(i) \text{ for some } j = j(i), \quad \text{where } i = 1, 2, \ldots, r.$$

For instance, each fundamental permutation Π_j belongs to $\Pi(r \times c)$.

For each $\Pi \in \Pi(r \times c)$, we consider Π as a polygon on the set $L(r \times c)$ by joining all points $\Pi(i)$ in the natural order

$$\Pi = \overline{\Pi(1)\Pi(2) \cdots \Pi(r)}.$$

This polygon Π divides the set $L(r \times c)$ into the left and right open part which are denoted by $L(\Pi)$ and $R(\Pi)$ respectively. For each i, let $\Pi'(i)$ be the vertical line passing through the point $\Pi(i)$. Again, we let $L(\Pi'(i))$ and $R(\Pi'(i))$ denote the left and right open half-plane separated by the line $\Pi'(i)$ in $L(r \times c)$. We say that a permutation $\Pi \in \Pi(r \times c)$ is left (right) stable in the Latin rectangle $L(r \times c)$ if and only if it satisfies

$$\Pi(i) \notin R(\Pi'(i)) \cap L(\Pi) \quad \text{for all } i = 1, 2, \ldots, r,$$
$$(\Pi(i) \notin L(\Pi'(i)) \cap R(\Pi) \quad \text{for all } i = 1, 2, \ldots, r).$$

In [3, Theorem 1], we have shown the following unification between the concepts of left and right stability in $L(r \times c)$.

THEOREM 1. *Let $L(r \times c)$ be a Latin rectangle and let Π be a permutation in the class $\Pi(r \times c)$. Then Π is left stable in $L(r \times c)$ if and only if it is right stable in $L(r \times c)$.*

We notice that if a matrix is not a Latin rectangle, then the concepts of left and right stability cannot be unified [3, Example 1].

From the above unification theorem, we now define a permutation to be stable in a Latin rectangle if it is either left or right stable. In particular, if Π is a permutation whose images $\Pi(i)$, $i = 1, 2, \ldots, r$, lie within two adjacent columns, then we have $R(\Pi'(i)) \cap L(\Pi) = \varnothing$, $i = 1, 2, \ldots, r$. This yields

THEOREM 2. *Under the hypothesis of Theorem 1, if the images $\Pi(i)$, $i = 1, 2, \ldots, r$, lie within two adjacent columns of $L(r \times c)$, then the permutation Π is stable. In particular, all fundamental permutations are stable.*

Theorem 2 is sharp as will be seen from Example 1 at the end of next section.

2. Partially ordered set

We refer the reader to the definition in [1, page 1]. As before, we let $L(r \times c)$ be a Latin rectangle and let S be the set of all stable permutations in $L(r \times c)$.

For a permutation $\Pi \in S$, we let $\#(\Pi(i))$ be the number of the column in $L(r \times c)$ in which the element $\Pi(i)$ is located, where $i = 1, 2, \ldots, r$. Let two permutations $\Pi_1, \Pi_2 \in S$. We define $\Pi_1 \leqslant \Pi_2$ if and only if

$$(1) \qquad \#(\Pi_1(i)) \leqslant \#(\Pi_2(i)) \quad \text{for each } i = 1, 2, \ldots, r.$$

According to the above definition (1), it is easy to prove the following result.

THEOREM 3. *The set S of all stable permutations in a Latin rectangle is a partially ordered set.*

To end up this section, we give the following

EXAMPLE 1. Let $L(4 \times 3)$ be the Latin rectangle defined by

$$\begin{array}{cccc}
\#: & 1 & 2 & 3 \\
1: & 1 & 2 & 3 \\
2: & 2 & 1 & 4 \\
3: & 3 & 4 & 1 \\
4: & 4 & 3 & 2
\end{array}$$

where the three fundamental permutations are $\Pi_1 = (1)$, $\Pi_2 = (12)(34)$ and $\Pi_3 = (13)(24)$. $L(4 \times 3) = \{i : a_{ij} \mid \Pi_j(i) = a_{ij}\}$. $\#(\Pi(1)) = 1$, $\#(\Pi(2)) = 3, \ldots, \#(\Pi(4)) = 3$. The permutation $\Pi = (24)$ is denoted by the polygon which is unstable in $L(4 \times 3)$, because $\Pi(1) = 1 = a_{22} \in R(\Pi'(1)) \cap L(\Pi)$. The permutation $\Pi^* = (14)(23)$ does not belong to the class $\Pi(4 \times 3)$.

3. Lattice of stable permutations

As usual (see, for example, Birkhoff [1, page 16]), a lattice is a partially ordered set S in which any two elements have a g.l.b. or "meet" $x \wedge y$, and l.u.b. or "join" $x \vee y$. With this definition, we have

THEOREM 4. *The set S of all stable permutations in a Latin rectangle $L(r \times c)$ forms a lattice.*

PROOF. Let $\Pi_1, \Pi_2 \in S$ and $\Pi_3 = \Pi_1 \wedge \Pi_2$. Then by (1), we have

$$(2) \qquad \#(\Pi_3(i)) = \min(\#(\Pi_1(i)), \#(\Pi_2(i))) \quad \text{for } i = 1, 2, \ldots, r.$$

The assertion will be proved if we can show that Π_3 is a permutation and stable in $L(r \times c)$.

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Suppose on the contrary that Π_3 is not a permutation. Then we have

(3) $$\Pi_3(i) = \Pi_3(j) \quad \text{for some } i \neq j,\ 1 \leqslant i, j \leqslant r.$$

We may, without loss of generality, assume that

(4) $$\Pi_3(i) = \Pi_1(i), \quad \text{where } \#(\Pi_1(i)) \leqslant \#(\Pi_2(i)).$$

Since Π_1 is a permutation, it follows from (2), (3), and (4) that

(5) $$\Pi_3(j) = \Pi_2(j) \quad \text{and} \quad \#(\Pi_2(j)) < \#(\Pi_1(j)).$$

From the Latinness of $L(r \times c)$, we can see that the same element in (3) cannot occur in the same column of $L(r \times c)$ and therefore by (4) and (5), we must have

(6) $$\text{either} \quad \#(\Pi_1(i)) < \#(\Pi_2(j)) \quad \text{or} \quad \#(\Pi_1(i)) > \#(\Pi_2(j)).$$

We now consider the first case of (6) which means that the element $\Pi_2(j)$ is located on the right hand side of the vertical line $\Pi_1'(i)$ passing through the element $\Pi_1(i)$, that is

(7) $$\Pi_2(j) \in R(\Pi_1'(i)).$$

In view of (5), the same element $\Pi_2(j)$ is located on the left hand side of $\Pi_1(j)$ and therefore we have

(8) $$\Pi_2(j) \in L(\Pi_1).$$

Since $\Pi_2(j)$ and $\Pi_1(i)$ denote the same element, hence by (7) and (8) we obtain

$$\Pi_1(i) \in R(\Pi_1'(i)) \cap L(\Pi_1).$$

This contradicts the stability of Π_1.

By the same argument, we can see that the second case of (6) will contradict the stability of Π_2. Thus Π_3 is a permutation.

It remains to prove that Π_3 is stable in $L(r \times c)$. Again, suppose not. Then we have

$$\Pi_3(i) \in R(\Pi_3'(i)) \cap L(\Pi_3) \quad \text{for some } i,\ 1 \leqslant i \leqslant r.$$

Again, assume that $\Pi_3(i) = \Pi_1(i)$. Then the above relation becomes

(9) $$\Pi_1(i) \in R(\Pi_1'(i)) \cap L(\Pi_3).$$

Moreover, from (2), we can see that

(10) $$L(\Pi_3) \subseteq L(\Pi_1).$$

Combining (9) and (10), we obtain

$$\Pi_1(i) \in R(\Pi_1'(i)) \cap L(\Pi_1).$$

This contradicts the stability of Π_1 and shows that Π_3 is stable in $L(r \times c)$, so that the meet $\Pi_1 \wedge \Pi_2 \in S$ for $\Pi_1, \Pi_2 \in S$.

Finally, according to Theorem 1, we know that the concepts of left and right stability are equivalent. Therefore, by the same argument, we can obtain the join

$\Pi_1 \vee \Pi_2 \in S$ for $\Pi_1, \Pi_2 \in S$. This shows that the set S is a lattice and the proof is complete.

4. Boolean algebra

Recall (Birkhoff [1, page 152]) that a distributive lattice B is a Boolean algebra if it is complemented, that is if for every $x \in B$, there is an element $x' \in B$ such that

$$x \wedge x' = 0 \quad \text{and} \quad x \vee x' = I,$$

and x' is unique.

As before, let $L(r \times c)$ be a Latin rectangle and let S be the set of all stable permutations in $L(r \times c)$. Let F_1 and F_c denote respectively the first and last fundamental permutation in $L(r \times c)$. By Theorem 2 we can see that both of them are stable in $L(r \times c)$. In view of (1), we have

$$(11) \qquad F_1 = 0 \quad \text{and} \quad F_c = I.$$

With the observation of (11), we shall now prove the following theorem.

THEOREM 5. *Let S_c be the set of all stable permutations in a Latin rectangle $L(r \times c)$. Then S_c is a Boolean algebra if and only if $c \leqslant 2$.*

PROOF. We shall first prove that if $c \leqslant 2$ then the set S_c is a Boolean algebra. We need only verify the case $c = 2$. In this case, we let F_1 and F_2 be the first and last column of $L(r \times 2)$. Relation (11) is then clearly satisfied.

According to Theorem 2, any permutation lying within two adjacent columns is stable, so if $\Pi \in S_2$, then Π is just a permutation in $L(r \times 2)$. We then define the complement of Π by

$$(12) \qquad \Pi' = L(r \times 2) - \Pi.$$

It is easy to see that Π' denotes a permutation and therefore $\Pi' \in S_2$. In view of (1) and (11), we have

$$(13) \qquad \Pi \wedge \Pi' = F_1 = 0 \quad \text{and} \quad \Pi \vee \Pi' = F_2 = I.$$

It remains to prove that Π' is unique. For this, we let $\Pi_1 = \Pi \cap F_1$ and $\Pi_2 = \Pi \cap F_2$. Then by (12), we obtain

$$(14) \qquad \Pi_1' = \Pi' \cap F_1 = F_1 - \Pi_1 \quad \text{and} \quad \Pi_2' = \Pi' \cap F_2 = F_2 - \Pi_2.$$

If Π'' is an element of S_2 for which (13) holds, then we must have

$$\Pi_1'' = \Pi'' \cap F_1 = F_1 - \Pi_1 \quad \text{and} \quad \Pi_2'' = \Pi'' \cap F_2 = F_2 - \Pi_2.$$

It follows from (14) that $\Pi'' = \Pi'$. Applying a result of Birkhoff [1, page 134, Corollary 1], we conclude that the set S_2 forms a distributive lattice. Since S_2 is finite and complementary, it is a Boolean algebra (see [1, page 153, Exercise 2]).

Conversely, if $c > 2$, then it is easy to see that the second fundamental permutation has no complement. Hence the set S_c is not a Boolean algebra. This completes the proof.

The following result was suggested by D. E. Knuth.

THEOREM 6. *The lattice S of all stable permutations in a Latin rectangle $L(r \times c)$ is distributive.*

PROOF. According to a theorem of J. Bowden (see [1, page 134, Exercise 3]), we know that a lattice is distributive if and only if

$$\Pi = (\Pi_1 \vee \Pi_2) \wedge \Pi_3 \leqslant \Pi_1 \vee (\Pi_2 \wedge \Pi_3) = \Pi^*, \quad \text{for all } \Pi_1, \Pi_2, \Pi_3.$$

There are two cases to be considered; either $\Pi_1 \vee \Pi_2 \leqslant \Pi_3$ or not. The first case gives, for $j = 1, 2, \ldots, r$,

$$\#\Pi(j) = \#(\Pi_1 \vee \Pi_2)(j) = \max(\#\Pi_1(j), \#\Pi_2(j)).$$

For each j, if $\#\Pi_1(j) \geqslant \#\Pi_2(j)$, then $\#\Pi(j) = \#\Pi_1(j) \leqslant \#\Pi^*(j)$.

On the other hand, if $\#\Pi_1(j) < \#\Pi_2(j)$, then $\#\Pi(j) = \#\Pi_2(j)$. By the first case, we have $\#\Pi_2(j) \leqslant \#\Pi_3(j)$, so that

$$\#\Pi^*(j) = \#\Pi_2(j) = \#\Pi(j).$$

This yields

$$\#\Pi(j) \leqslant \#\Pi^*(j) \quad \text{for each } j = 1, 2, \ldots, r.$$

By the same argument, the above inequalities hold for the second case. This shows that $\Pi \leqslant \Pi^*$.

5. Stable marriages

The notion of stable marriages was introduced by D. Gale and L. S. Shapley [2]. And the notion of complete system of preferences was introduced by D. E. Knuth [6]. Recently, we have also introduced the notion of I-M systems [5]. We refer the reader to all of the above definitions in [5].

From the definition, it has been shown by John Conway and Knuth (see [6]) that the set of all stable marriages forms a distributive lattice. By comparing this result with Theorem 6, we can see that there are some equivalent relations between stable marriages and permutations. For this, we shall first introduce the binary relation $m_1 \leqslant m_2$ of two marriages

$$m_k = \begin{pmatrix} A_1 A_2 \cdots A_n \\ a_1^k a_2^k \cdots a_n^k \end{pmatrix} \quad \text{for } k = 1, 2.$$

Following Knuth [6], we define $m_1 \leqslant m_2$ if and only if

$$(15) \qquad A_i(a_i^1) \leqslant A_i(a_i^2) \quad \text{for each } i = 1, 2, \ldots, n.$$

Notice that if the first matrix (a_{ij}) in $P(n)$ is a Latin square and if A_i is replaced by a_i for each $i = 1, 2, \ldots, n$, then the above definition (15) is the same as that of (1).

In view of the paper [5], we can see that if $P(n)$ is an I-M preference, then both pair matrices (a_{ij}) and (A_{ij}) are Latin squares [5, Theorem 1].

With the above two remarks, we are now able to prove the following equivalent relation between stable marriages and permutations in a system of I-M preference.

THEOREM 7. *If $P(n)$ is an I-M preference, defined in* [5, (4)], *then the lattice structure of all stable marriages in $P(n)$ is the same as that of all stable permutations in the first Latin square (a_{ij}) of $P(n)$.*

PROOF. Let m be a marriage and let Π_m be the permutation induced from m by replacing A_i by a_i for each $i = 1, 2, \ldots, n$. Owing to a previous result [5, Theorem 8], we find that m is a stable marriage in $P(n)$ if and only if Π_m is a stable permutation in (a_{ij}). Since the order of marriages is defined in the same way as that of permutations, then the lattice structure of all stable marriages in $P(n)$ is the same as that of stable permutations in (a_{ij}). This completes the proof.

Notice that if $r = c = n$, then Theorems 4 and 6 follow from Theorem 7 and the aforementioned result of Conway and Knuth. More precisely, we shall prove that the set of all stable permutations in a Latin square $L(n)$ forms a distributive lattice. To see this, we first apply condition in [5] to construct the dual Latin square $L^*(n)$ such that the resulting system $P(n) = \{L(n); L^*(n)\}$ forms an I-M preference (see [5, Theorem 1]). It follows from Conway-Knuth's theorem that the set of all stable marriages in $P(n)$ is a distributive lattice. This together with Theorem 7 yields the assertion.

6. The duality of I-M preferences

Finally, we shall study the relation between the lattice structures of stable permutations in the pair matrices of an I-M preference.

As usual (see Birkhoff [1, page 3]), an isomorphism θ between two partially ordered sets S and S^* is a one to one mapping between S and S^* such that

$$x \leqslant y \text{ implies } \theta(x) \leqslant \theta(y) \quad \text{and} \quad \theta(x) \leqslant \theta(y) \text{ implies } x \leqslant y.$$

By the converse of a relation R is meant the relation R^* such that xR^*y if and only if yRx. The well-known duality principle asserts that the converse of any partially ordered set is itself a partially ordered set (see [1, page 3, Theorem 2]). By the dual S^* of a partially ordered set S is meant that partially ordered system defined by the converse relation on the same elements. According to the duality principle, we obtain immediately the following duality of stable permutations.

THEOREM 8. *Let L be a Latin square, then the lattice $\{S, \leqslant\}$ of all stable permutations in L is dual to its converse $\{S^*, \geqslant\}$.*

We shall now consider the duality between the pair matrices in an I-M preference. For this, we say that two Latin squares L_1 and L_2 are dual if the lattices of stable permutations in L_1 and L_2 are dual.

THEOREM 9. *Let $P(n)$ be an I-M preference defined in* [5, (4)], *then the pair matrices (a_{ij}) and (A_{ij}) are dual.*

PROOF. As was mentioned before, the pair matrices denote two Latin squares. Let m be a stable marriage in $P(n)$ and let Π_m be the permutation induced from m by replacing A_i by a_i, then Π_m is stable in (a_{ij}) due to Theorem 7. Similarly, by replacing a_i by A_i in m, we obtain another permutation Π_m^* stable in (A_{ij}).

In the first and second Latin squares we have

$$\#(\Pi_m(a_i)) = A_i(\Pi_m(a_i)) \quad \text{for } i = 1, 2, \ldots, n, \text{ and} \tag{16}$$

$$\#\left(\Pi_m^*(A_j)\right) = a_j\left(\Pi_m^*(A_j)\right) \quad \text{for } j = 1, 2, \ldots, n, \text{ respectively.} \tag{17}$$

Since both $\{A_i, \Pi_m(a_i)\}$ and $\{a_j, \Pi_m^*(A_j)\}$ denote spouse in the marriage m, it follows that

$$\Pi_m(a_i) = a_j \quad \text{if and only if} \quad \Pi_m^*(A_j) = A_i. \tag{18}$$

Combining (16), (17), and (18), and viewing [5, (4)], we obtain

$$\#(\Pi_m(a_i)) + \#\left(\Pi_m^*(A_j)\right) = n + 1. \tag{19}$$

where $j = j(i)$, $i = 1, 2, \ldots, n$.

Now, consider two marriages m_1 and m_2 stable in $P(n)$. We then have two permutations Π_{m_1} and Π_{m_2} stable in (a_{ij}), and another two $\Pi^*_{m_1}$ and $\Pi^*_{m_2}$ stable in (A_{ij}). It follows from (19) that

$$\#\big(\Pi_{m_1}(a_i)\big) \leqslant \#\big(\Pi_{m_2}(a_i)\big) \quad \text{if and only if} \quad \#\big(\Pi^*_{m_1}(A_j)\big) \geqslant \#\big(\Pi^*_{m_2}(A_j)\big).$$

According to (1), we obtain

$$\Pi_{m_1} \leqslant \Pi_{m_2} \quad \text{if and only if} \quad \Pi^*_{m_1} \geqslant \Pi^*_{m_2}.$$

We thus conclude that the lattice of stable permutations in the pair matrices are dual.

We remark that from Theorem 9 we can see that the system of an I-M preference is a dual system, but the converse is not true, as will be seen from the following

EXAMPLE 2. Let $P(3)$ be a system of Latin preference defined by

$$\begin{array}{llll} \& : 1\,2\,3 & \qquad & \# : 1\,2\,3 \\ A : a\,b\,c & & c : C\,B\,A \\ B : b\,c\,a & & a : A\,C\,B \\ C : c\,a\,b & & b : B\,A\,C \end{array}$$

Then the lattices of stable permutations in both matrices denote the same chain with the reverse order, so that they are dual. However, $P(3)$ is not an I-M preference.

In general, if (a_{ij}) is a Latin square, then its dual is determined by (A_{ij}), where $A_{ij} = a_{i(n+1-j)}$ for $i, j = 1, 2, \ldots, n$. By suitably ordering the n men A_i and n women a_i on the borders of the pair matrices, the resulting system $P(n)$ can be arranged as a non-I-M preference. For instance, if we arrange the men first and then determine the women such that the first column in both matrices denotes the same marriage m_1, then this marriage m_1 can be easily shown to be the only one stable in $P(n)$. If we represent this marriage by

$$m_1 = \begin{pmatrix} A_1 A_2 \cdots A_n \\ a_1^* a_2^* \cdots a_n^* \end{pmatrix},$$

then we have $A_i(a_i^*) + a_i^*(A_i) = 2 \neq n + 1$, $i = 1, 2, \ldots, n$.Thus the system $P(n)$ is not an I-M preference.

This shows that the duality of a pair matrices cannot be used to define the notion of I-M preference. For this, we do not define such a preference in terms of "dual preference".

The fundamental property of I-M preferences is that (see [5, Theorem 2]) a system $P(n)$ is an I-M preference if and only if all fundamental marriages (all fundamental permutations in the pair matrices) are stable in $P(n)$.

https://doi.org/10.1017/S1446788700018838 Published online by Cambridge University Press

ACKNOWLEDGEMENT. I am indebted to Professor Knuth for his many valuable comments in this research and in particular, Theorem 6 was suggested by him.

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