

Fourth and fifth places.—For every farthing above sixpences, 4, with a unit of carriage for every 6 farthings.

All subsequent places.—For every farthing above three-halfpences, 1, with 6 for a denominator, and reduction to a decimal. Thus at $8\frac{3}{4}d.$ the sixth and following figures are as in $\frac{2}{5}$, namely, 8333 . .

The third rule may be advantageously abandoned in favour of the following:—When the fourth and fifth figures are 00, 25, 50, 75, the decimal has terminated; in every other case the complement to 5 of the fifth figure is the numerator; or, when the fifth figure is 5 or upwards, the complement to 10. That is, when the decimal is interminable, or when the fourth and fifth figures are not 00, 25, 50, 75,

The fifth figure.	Is followed by the places of	The fifth figure.	Is followed by the places of
0	5 sixths.	5	5 sixths.
1	4 „	6	4 „
2	3 „	7	3 „
3	2 „	8	2 „
4	1 „	9	1 „

And this sub-rule is convenient; a fifth figure *three* is followed by nothing but *threes*, a *six* by nothing but *sizes*.

Yours truly,
A. DE MORGAN.

ON THE FACILITY WITH WHICH THE ORDINARY ANNUITY AND ASSURANCE VALUES ARE DERIVED FROM THE VALUE OF THE ENDOWMENT.

To the Editor of the Assurance Magazine.

SIR,—The ordinary tables of life annuities and assurances which have hitherto been published, as well as the tables on the commutation method, are unquestionably of great value; but, nevertheless, are not, I submit, so extensively useful as they might be made by the introduction of certain supplemental columns of quantities required in practice, the want of which arises with sufficient frequency to call for their being tabulated. This view is, to some extent, recognised by Mr. Thomson, in his valuable work, entitled *Actuarial Tables*; and the object of the present communication is to draw attention to the fact, that the values of assurances, as well as of annuities, fixed and increasing, temporary and deferred, may be easily obtained and tabulated directly from the values of endowments.

On a previous occasion, I had the honour of addressing you on the desirableness of an extension of the D and N method, by the introduction of columns of differences (*Assurance Magazine*, vol. viii., p. 168), and endeavoured to point out the importance of tables in that form. I beg now to submit a specimen table of another kind, exhibiting various columns of values not usually given, the adoption of which would tend much to abridge or simplify certain computations, in which such values occur as functions. The table is similar in principle, as regards a portion of the annuity values, to the tables given in Mr. Thomson's valuable work before mentioned, but differing from those tables in this respect, that the whole of the assurance values, as well as the values of the annuities, are derived, as above remarked, directly from the endowments at the corresponding ages.

The relation subsisting between these various values, and their result-
 ance from the values of the endowments, are so obvious that demonstration
 is unnecessary; but I am not aware that any writer has pointed out such
 relation, in the manner presently shown (p. 57), as a means for the direct
 deduction of the assurance values. For this reason, I assume that my
 drawing attention to the subject may not be altogether without interest to
 some, at least, of the members of the Institute.

The values in the table here given coincide with those which would be
 shown by Mr. Thomson's method, as regards columns Nos. 1, 5, and 7 of
 the annuity values, and as regards Nos. 9, 11, and 12 of the assurance
 values. The remaining columns are produced by a very obvious and easy
 process, as will appear from the following equations and explanations.

Col. 1.

$$e_{x+1} = v.p_x; e_{x+2} = v^2.p_{x+1}, \&c.$$

Col. 2.

Assigning to n all values from unity to the oldest age in the table—

$$n.e_{x+1} = e_{x+1}; n.e_{x+2} = 2e_{x+2}; n.e_{x+3} = 3e_{x+3}, \&c.$$

The formation of this column is obvious, and contains the product of
 the value, against each age, in col. 1, by 1, 2, 3, &c., according to the
 number of years deferred, which is represented by n , and at which the
 benefit may commence or cease.

Col. 3.

$$\begin{aligned} a_{x|1} &= e_{x+1} = a_x - a_x|_1 \\ a_{x|2} &= a_{x|1} + e_{x+2} = e_{x+1} + e_{x+2} = a_x - a_x|_2 \\ a_{x|3} &= a_{x|2} + e_{x+3} = e_{x+1} + e_{x+2} + e_{x+3} = a_x - a_x|_3 \\ &\vdots \\ a_{x|n} &= a_{x|n-1} + e_{x+n} = e_{x+1} + e_{x+2} + \dots + e_{x+n}, \\ &= a_x - a_x|_n. \end{aligned}$$

This column may be described as the summation of col. 1, commencing
 from the youngest age.

Col. 4.

$$\begin{aligned} i_{x|1} &= e_{x+1} = n.e_{x+1} \\ i_{x|2} &= e_{x+1} + 2e_{x+2} = i_{x|1} + n.e_{x+2} \\ i_{x|3} &= e_{x+1} + 2e_{x+2} + 3e_{x+3} = i_{x|2} + n.e_{x+3} \\ &\vdots \\ i_{x|n} &= e_{x+1} + 2e_{x+2} + 3e_{x+3} + \dots + n.e_{x+n} = \\ &= i_{x|n-1} + n.e_{x+n} = i_x - i_x|_n - n.a_n|_n. \end{aligned}$$

This column is formed from col. 2, as col. 3 is from col. 1.

* The final value in this column, which is obviously that of an annuity for the whole
 of life.

† The final value of this column, which is manifestly that of an annuity increasing
 £1 per annum for the whole of life.

Col. 5.

Put $x+z$ = the oldest age in the table.

$x+n$ = any age intermediate between x and $x+z$.

$$\begin{aligned} a_{x-z} &= 0 \\ a_x \gamma_{x-1} &= e_{x+z} \\ a_x \gamma_{x-2} &= e_{x+z} + e_{x+z-1} \\ &\cdot \\ a_x \gamma_{x-n} &= e_{x+z} + e_{x+z-1} + \dots + e_{x+z-(n-1)} \\ &\cdot \\ a_x \gamma_1 &= e_{x+z} + e_{x+z-1} + \dots + e_{x+2} \\ a_x \gamma_0 &= e_{x+z} + e_{x+z-1} + \dots + e_{x+1} = a_x \end{aligned}$$

By subtraction,

$$\begin{aligned} a_x \gamma_0 &= a_x \\ a_x \gamma_1 &= a_x \gamma_0 - e_{x+1} = a_x - a_{x \gamma_1} \\ a_x \gamma_2 &= a_x \gamma_1 - e_{x+2} = a_x - a_{x \gamma_2} \\ &\cdot \\ a_x \gamma_n &= a_x \gamma_{n-1} - e_{x+n} = a_x - a_{x \gamma_n} \end{aligned}$$

This column is formed by summing column 1, commencing at the oldest age, and placing the results at each age opposite to the age one year younger; or, it may be formed by placing the value of the whole-life annuity opposite to "0 years deferred," and subtracting successively the quantities in column 1 downwards.

Col. 6.

Putting, as before, $n=1, 2, 3, \&c.$,

$$n \cdot a_x \gamma_1 \pm a_x \gamma_1; n \cdot a_x \gamma_2 = 2a_x \gamma_2; n \cdot a_{x+3} = 3a_{x+3}, \&c.$$

This column is formed from col. 5 as col. 2 is from col. 1.

Col. 7.

Putting $x+z$, as before, $\&c.$,

$$\begin{aligned} i_x \gamma_x &= 0 \\ i_x \gamma_{x-1} &= a_x \gamma_{x-1} = e_{x+z} \\ i_x \gamma_{x-2} &= i_x \gamma_{x-1} + a_{x+z-2} = a_x \gamma_{x-1} + a_x \gamma_{x-2} = 2e_{x+z} + e_{x+z-1} \\ &\cdot \\ i_x \gamma_{x-n} &= i_x \gamma_{x-n-1} + a_x \gamma_{x-n} = (z-n)e_{x+z} + (z-1-n)e_{x+z-1} \\ &\quad + (z-2-n)e_{x+z-2} + \dots + (z-n-1)e_{x+z-n-1} \\ &\cdot \\ i_x \gamma_1 &= i_x \gamma_2 + a_x \gamma_1 = a_x \gamma_x + a_x \gamma_{x-1} + \dots + a_x \gamma_1 = \\ &= (z-1)e_{x+z} + (z-2)e_{x+z-1} + \dots + e_{x+2} \\ i_x \gamma_0 &= i_x \gamma_1 + a_x \gamma_0 = a_x \gamma_x + a_x \gamma_{x-1} + \dots + a_x \gamma_0 \end{aligned}$$

And generally,

$$i_x \gamma_n = i_x \gamma_{n+1} + a_x \gamma_n = e_{x+n+1} + 2e_{x+n+2} + 3e_{x+n+3} + \dots \&c.$$

Or, by subtraction, commencing at the earliest age,

$$\begin{aligned} i_x \gamma_1 &= i_x - a_x \\ i_x \gamma_2 &= i_x \gamma_1 - a_x \gamma_1 \\ i_x \gamma_3 &= i_x \gamma_2 - a_x \gamma_2 \\ &\vdots \\ i_x \gamma_n &= i_x \gamma_{n-1} - a_x \gamma_{n-1} = i_x - i_x \gamma_{n-1} - n \cdot a_x \gamma_n \end{aligned}$$

This column is the summation of col. 5, beginning with the oldest age; or it may be found by placing the value of a whole term increasing annuity opposite "0 years deferred," subtracting first the whole term annuity and then the successive values of the deferred annuities from the value last found.

Col. 8.

$v \cdot e_{x+n}$ = the several values in column 1, each multiplied into the present value of £1 to be received at the end of 1 year.

Col. 9.

$$\begin{aligned} E_{x+1} &= v - e_{x+1} \\ E_{x+2} &= v \cdot e_{x+1} - e_{x+2} \\ &\vdots \\ E_{x+n} &= v \cdot e_{x+(n-1)} - e_{x+n} \end{aligned}$$

These values are obtained by subtracting those in col. 1 from those in col. 8 respectively opposite to an age 1 year younger.

Cols. 10, 11, 12, 13, 14 and 15.

The equations given for the annuity values in cols. 2, 3, 4, 5, 6, and 7, will also express the equations subsisting between the assurance columns by substituting E, A, I, for e, a, i; and the mode of construction is in all respects similar to that of the corresponding annuity columns before described.

Col. 12.

$$eA_{x-1} = e_{x+1} + A_{x-1}, eA_{x-2} = e_{x+2} + A_{x-2}, \&c.,$$

and is formed by adding at each age the value in col. 1 to that in col. 11 at the corresponding age.

With regard to col. 8, which is introduced for the purpose of showing the manner in which we pass from the values of the endowment to the assurance values, it may be remarked that it consists of the several values of col. 1 each discounted one year. For it will be obvious that the present value of £1 to be received if death occur within the first year, is the difference between the discounted value of £1 for one year and the value of £1 to be received if the life survive that term; the value of £1 to be received if death take place within the second year will be the difference between the discounted value of £1 to be payable on surviving one year and the present value of £1 to be received on surviving two years; and so

forth. Owing to this relation between the endowment and the endowment-assurance (the latter term being used in the sense applied to it by Professor de Morgan in his paper in the *Companion to the Almanack*, and by Mr. Peter Gray in his *Tables and Formulæ*), the values in col. 9 are deduced, viz., $E_{x+n} = v \cdot e_x - e_{x+1}$; $E_{x+2} = v \cdot e_{x+1} - e_{x+2}$, &c.

The benefits, the present values of which are contained in the several columns of the table, will, I think, be sufficiently obvious without verbal explanation. It may be interesting, however, to compare these columnar results with the equivalent formulæ of the commutation method, both as ordinarily exhibited, and in the form they would assume, did we possess the supplementary columns of differences to which attention was directed in the communication above referred to, viz.:—

		By Commutation Method.	By Commutation Method, with Col. of Differences.
1.	e_{x+n}	$= \frac{D_{x+n}}{D_x}$	
2.	$a_{x n}$	$= \frac{N_x - N_{x+n}}{D_x}$	$= \frac{N_{x n}}{D_x}$
3.	$i_{x n}$	$= \frac{S_x - S_{x+n} - nN_{x+n}}{D_x}$	$= \frac{S_{x n}}{D_x}$
4.	$a_x \overline{]n}$	$= \frac{N_{x+n}}{D_x}$. . .
5.	$i_x \overline{]n}$	$= \frac{S_{x+n}}{D_x}$. . .
6.	$v \cdot e_{x+n}$	$= \frac{vD_{x+n}}{D_x}$. . .
7.	E_{x+n}	$= \frac{vD_{x+n} - D_{x+n+1}}{D_x}$. . .
8.	$A_x \overline{]n}$	$= \frac{M_x - M_{x+n}}{D_x}$	$= \frac{M_{x n}}{D_x}$
9.	$I_x \overline{]n}$	$= \frac{R_x - R_{x+n} - nM_{x+n}}{D_x}$	$= \frac{R_{x n}}{D_x}$
10.	$A_x \overline{]n}$	$= \frac{M_{x+n}}{D_x}$. . .
11.	$I_x \overline{]n}$	$= \frac{R_{x+n}}{D_x}$. . .
12.	$eA_x \overline{]n}$	$= \frac{D_{x+n} + M_x - M_{x+n}}{D_x}$	$= \frac{D_{x+n} + M_{x x+n}}{D_x}$

Numerous combinations of these terms are of frequent occurrence, many of which are interesting as regards the comparative merits of the different methods; but having already trespassed upon your valuable space, I will refrain from introducing them in the present letter. I may, at a future opportunity, request the favour of being allowed again to refer to the subject.

I am, Sir,

Your obedient servant,

S. L. LAUNDY.

March 1863.

Values of Annuities (Experience 3 per Cent.)

Age 60.

Deferred Age.	Years Deferred.	Endowment.	Endowment of $\frac{1}{2}$ th.	Temporary Annuity.	Temporary Increasing Annuity.	Deferred Annuity.	Deferred Annuity of $\frac{1}{2}$ th.	Deferred Increasing Annuity.	Endowment Discounted one Year.	Years Deferred.	Deferred Age.
$x+n$.	n .	e_{x+n} .	$n.e_{x+n}$.	$a_{\overline{n} }$.	$i_{\overline{n} }$.	$a_{x \overline{n} }$.	$n.a_{x \overline{n} }$.	$i_{x \overline{n} }$.	$v.e_{x+n}$.	n .	$x+n$.
60	0	10-18782	..	85-03695	-97087	0	60
1	1	.94142	.94142	.94142	.94142	9-24640	9-24640	74-84913	-91400	1	1
2	2	.88419	1-76838	1-82561	2-70980	8-36221	16-72442	65-60273	-85844	2	2
3	3	.82829	2-48487	2-65390	5-19467	7-53392	22-60176	57-24052	-80417	3	3
4	4	.77374	3-09496	3-42764	8-28963	6-76018	27-04072	49-70660	-75120	4	4
65	5	.72053	3-60265	4-14817	11-89223	6-03965	30-19825	42-94642	-69954	5	65
6	6	.66871	4-01226	4-81688	15-90454	5-37094	32-22564	36-90677	-64923	6	6
7	7	.61832	4-32824	5-43520	20-23278	4-75262	33-26834	31-53583	-60031	7	7
8	8	.56941	4-55528	6-00461	24-78806	4-18321	33-46568	26-78321	-55283	8	8
9	9	.52207	4-69863	6-52668	29-48669	3-66114	32-95026	22-60000	-50686	9	9
70	10	.47641	4-76410	7-00309	34-25079	3-18473	31-84730	18-93886	-46253	10	70
1	1	.43250	4-75750	7-43559	39-00829	2-75223	30-27453	15-75413	-41990	1	1
2	2	.39044	4-68528	7-82603	43-69357	2-36179	28-34148	13-00190	-37907	2	2
3	3	.35034	4-55442	8-17637	48-24799	2-01145	26-14885	10-64011	-34014	3	3
4	4	.31228	4-37192	8-48865	52-61991	1-69917	23-78838	8-62866	-30318	4	4
75	15	.27636	4-14540	8-76501	56-76531	1-42281	21-34215	6-92949	-26831	15	75
6	6	.24267	3-88272	9-00768	60-64803	1-18014	18-88224	5-50668	-23560	6	6
7	7	.21130	3-59210	9-21898	64-24013	.96884	16-47028	4-32654	-20515	7	7
8	8	.18227	3-28086	9-40125	67-52099	.78657	14-15827	3-35770	-17696	8	8
9	9	.15565	2-95735	9-55690	70-47834	.63092	11-98748	2-57113	-15112	9	9
80	20	.13146	2-62920	9-68836	73-10754	.49946	9-98920	1-94021	-12763	20	80
1	1	.10971	2-30391	9-79807	75-41145	.38975	8-18475	1-44075	-10652	1	1
2	2	.09039	1-98858	9-88846	77-40003	.29936	6-58592	1-05100	-08776	2	2
3	3	.07343	1-68889	9-96189	79-08892	.22593	5-19639	.75164	-07129	3	3
4	4	.05875	1-41000	10-02064	80-49892	.16718	4-01232	.52571	-05704	4	4
85	25	.04622	1-15550	10-06686	81-65442	.12096	3-02490	.35853	-04487	25	85
6	6	.03567	.92742	10-10253	82-58184	.08529	2-21794	.23757	-03463	6	6
7	7	.02693	.72711	10-12946	83-30895	.05836	1-57572	.15228	-02614	7	7
8	8	.01981	.55468	10-14927	83-86363	.03855	1-07940	.09392	-01923	8	8
9	9	.01413	.40977	10-16340	84-27340	.02442	.70818	.05537	-01372	9	9
90	30	.00971	.29130	10-17311	84-56470	.01471	.44130	.03095	-00943	30	90
1	1	.00637	.19747	10-17948	84-76217	.00834	.25854	.01624	-00618	1	1
2	2	.00395	.12640	10-18343	84-88857	.00439	.14048	.00790	-00384	2	2
3	3	.00228	.07524	10-18571	84-96381	.00211	.06963	.00351	-00221	3	3
4	4	.00120	.04080	10-18691	85-00461	.00091	.03094	.00140	-00117	4	4
95	35	.00057	.01995	10-18748	85-02456	.00034	.01190	.00049	-00055	35	95
6	6	.00023	.00828	10-18771	85-03284	.00011	.00396	.00015	-00022	6	6
7	7	.00008	.00296	10-18779	85-03580	.00003	.00111	.00004	-00008	7	7
8	8	.00002	.00076	10-18781	85-03656	.00001	.00038	.00001	-00002	8	8
9	9	.00001	.00039	10-18782	85-03695	9	9
		1.	2.	3.	4.	5.	6.	7.	8.		

Value of Assurances (Experience 3 per Cent.).

Age 60.

Deferred Age.	Years Deferred.	Value of Assurance in each Year. (Endowment Assurance.)	Value of $\frac{1}{2}\%$ Assurance in the $x+n$ -th year.	Temporary Assurance.	Temporary Increasing Assurance.	Deferred Assurance.	Deferred Assurance of $\frac{1}{2}\%$.	Deferred Increasing Assurance.	Endowment with Temporary Assurance.	Years Deferred.	Deferred Age.
$x+n$.	n .	E_{x+n} .	$\frac{1}{2}E_{x+n}$.	A_{x+n} .	I_{x+n} .	A_{x+n} .	nA_{x+n} .	I_{x+n} .	eA_{x+n} .	n .	$x+n$.
60	0	67412	838461	0	60
1	1	02945	02945	02945	02945	64467	64467	771049	97087	1	61
2	2	02981	05962	05926	08907	61486	122972	706582	94345	2	62
3	3	03015	09045	08941	17952	58471	175413	645096	91770	3	63
4	4	03043	12172	11984	30124	55428	221712	586625	89358	4	64
65	5	03067	15335	15051	45459	52361	261805	531197	87104	5	65
6	6	03083	18498	18134	63957	49278	295668	478836	85005	6	66
7	7	03091	21637	21225	85594	46187	323309	429558	83057	7	67
8	8	03090	24720	24315	110314	43097	344776	383371	81256	8	68
9	9	03076	27684	27391	137998	40021	360189	340274	79598	9	69
70	10	03045	30450	30436	168448	36976	369760	300253	78077	10	70
1	1	03003	33033	33439	201481	33973	373703	263277	76689	1	71
2	2	02946	35352	36385	236833	31027	372324	229304	75429	2	72
3	3	02873	37349	39258	274182	28154	366002	198277	74292	3	73
4	4	02786	39004	42044	313186	25368	355152	170123	73272	4	74
75	15	02682	40230	44726	353416	22686	340290	144755	72362	15	75
6	6	02564	41024	47290	394440	20122	312952	122069	71557	6	76
7	7	02430	41310	49720	435750	17692	300764	101947	70850	7	77
8	8	02288	41184	52008	476934	15404	272722	84255	70235	8	78
9	9	02131	40489	54139	517423	13273	252187	68851	69704	9	79
80	20	01966	39320	56105	556743	11307	226140	55578	69251	20	80
1	1	01792	37632	57897	594375	09515	199815	44271	68868	1	81
2	2	01613	35486	59510	629861	07102	173844	34756	68549	2	82
3	3	01493	32959	60943	662820	06469	148787	26854	68286	3	83
4	4	01254	30096	62197	692916	05215	125160	20385	68072	4	84
85	25	01082	27050	63279	719966	04133	103325	15170	67901	25	85
6	6	00920	23920	64199	743886	03213	083538	11037	67766	6	86
7	7	00770	20790	64969	764676	02443	65961	07824	67662	7	87
8	8	00633	17724	65602	782400	01810	50680	05381	67583	8	88
9	9	00510	14790	66112	797190	01300	37700	03571	67525	9	89
90	30	00401	12030	66513	809220	00899	26970	02371	67484	30	90
1	1	00306	09486	66819	818706	00593	18743	01372	67456	1	91
2	2	00223	07136	67042	825842	00370	11840	00779	67437	2	92
3	3	00156	05148	67198	830990	00214	07062	00409	67426	3	93
4	4	00101	03434	67299	834424	00113	03842	00195	67419	4	94
95	35	00060	02100	67359	836524	00053	01855	00182	67416	35	95
6	6	00032	01152	67391	837676	00021	00756	00029	67414	6	96
7	7	00014	00518	67405	838194	00007	00259	00008	67413	7	97
8	8	00006	00228	67411	838422	00001	00058	00001	69413	8	98
9	9	00001	00039	67412	838461	67412	9	99
		9.	10.	11.	12.	13.	14.	15.	16.		