

SOLITON EQUATIONS EXHIBITING PFAFFIAN SOLUTIONS

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Abstract. Soliton equations whose solutions are expressed by Pfaffians are briefly discussed. Included are a discrete-time Toda equation of BKP type, a modified Toda equation of BKP type, a coupled modified KdV equation and a coupled modified KdV equation of derivative type.

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1. Introduction. A Pfaffian is the square root of an anti-symmetric determinant of order $2n$:

$$\det |a_{jk}|_{1 \leq j, k \leq 2n} = [\text{pf}(a_1, a_2, \dots, a_{2n})]^2.$$

We have, for example, for $n = 1$

$$\begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} = \text{pf}(a_1, a_2)^2.$$

Hence

$$\text{pf}(a_1, a_2) = a_{12}.$$

For $n = 2$,

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} = [a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}]^2.$$

Hence

$$\text{pf}(a_1, a_2, a_3, a_4) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

A square root of a determinant of a matrix M of order n which is a sum of a unit matrix E and a product of anti-symmetric matrices A and B is expressed by a two-component Pfaffian:

$$\det |E + A \times B| = [\text{pf}(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)]^2,$$

where

$$\begin{aligned} \text{pf}(a_j, a_k) &= a_{jk} = -a_{kj}, \\ \text{pf}(b_j, b_k) &= b_{jk} = -b_{kj}, \\ \text{pf}(a_j, b_k) &= \delta_{j,k} \end{aligned}$$

and where $\delta_{j,k}$ is Kronecker's delta symbol.

We have, for example, for $n = 3$

$$\begin{aligned} E + A \times B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 + a_{12}b_{21} + a_{13}b_{31} & a_{13}b_{32} & a_{12}b_{23} \\ a_{23}b_{31} & 1 + a_{21}b_{12} + a_{23}b_{32} & a_{21}b_{13} \\ a_{32}b_{21} & a_{31}b_{12} & 1 + a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}. \end{aligned}$$

Accordingly we find

$$\det |E + A \times B| = [-1 + a_{12}b_{12} + a_{13}b_{13} + a_{23}b_{23}]^2.$$

On the other hand using the expansion rule of the Pfaffian

$$\text{pf}(c_1, c_2, \dots, c_{2n}) = \sum_{j=1}^{2n} (-1)^{j-1} \text{pf}(c_1, c_j) \text{pf}(c_2, c_3, \dots, \hat{c}_j, \dots, c_{2n}),$$

we obtain

$$\text{pf}(a_1, a_2, a_3, b_1, b_2, b_3) = -1 + a_{12}b_{12} + a_{13}b_{13} + a_{23}b_{23}.$$

Hence

$$\det |E + A \times B| = \text{pf}(a_1, a_2, a_3, b_1, b_2, b_3)^2.$$

The two-component Pfaffian plays an important role in expressing soliton solutions of the coupled modified KdV equations and coupled modified equation of derivative type described below.

We shall show that soliton solutions of the following equations are expressed by Pfaffians.

(i) Discrete-time Toda equation of BKP type

$$\begin{aligned} W_n^{m+1} - W_n^{m-1} &= \log \frac{\delta^2[\exp(W_{n+1}^m - V_n^m) + \exp(V_n^m - W_n^m)] + (1 - 2\delta^2)}{\delta^2[\exp(W_n^m - V_{n-1}^m) + \exp(V_{n-1}^m - W_{n-1}^m)] + (1 - 2\delta^2)}, \\ V_n^{m+1} - V_n^m &= W_{n+1}^m - W_n^m. \end{aligned}$$

(ii) Modified Toda equation of BKP type

$$\frac{d}{dt} \log \frac{\beta + V_n}{\beta - \alpha(I_n + I_{n-1})} = I_n - I_{n+1},$$

$$\frac{d}{dt} I_n = V_{n-1} - V_n.$$

(iii) Coupled modified KdV equation

$$\frac{\partial}{\partial t} v_j + 6 \left[\sum_{1 \leq j < k \leq N} c_{jk} v_j v_k \right] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0, \quad j = 1, 2, \dots, N.$$

(iv) Coupled modified KdV equation of derivative type

$$\frac{\partial}{\partial t} v_j + 6 \left[\sum_{1 \leq j < k \leq N} c_{jk} D_x v_j \cdot v_k \right] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0, \quad j = 1, 2, \dots, N.$$

We have the discrete KP equation

$$[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3)]f \cdot f = 0$$

where D_1, D_2, D_3 and z_1, z_2, z_3 are bilinear operators and constants, respectively.

Soliton solutions to the discrete KP equation are known to be expressed by determinants [1].

While the discrete BKP equation is expressed by

$$[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3) + z_4 \exp(D_4)]f \cdot f = 0$$

where D_1, D_2, D_3, D_4 and z_1, z_2, z_3, z_4 are bilinear operators and constants, respectively, satisfying the relations

$$D_1 + D_2 + D_3 + D_4 = 0,$$

$$z_1 + z_2 + z_3 + z_4 = 0.$$

Solutions to the discrete BKP equation are well parametrized by parameters a, b and c which are the intervals of the coordinates l, m, n [2]. We rewrite the discrete BKP equation using these parameters:

$$(a + b)(a + c)(b - c)\tau(l + 1, m, n)\tau(l, m + 1, n + 1)$$

$$+ (b + c)(b + a)(c - a)\tau(l, m + 1, n)\tau(l + 1, m, n + 1)$$

$$+ (c + a)(c + b)(a - b)\tau(l, m, n + 1)\tau(l + 1, m + 1, n)$$

$$+ (a - b)(b - c)(c - a)\tau(l, m, n)\tau(l + 1, m + 1, n + 1) = 0.$$

Solutions are expressed by the Pfaffian [2,3]:

$$\tau(l, m, n) = \text{pf}(1, 2, 3, \dots, 2N),$$

where the elements of the Pfaffian are

$$\begin{aligned} \text{pf}(j, k) &\equiv c_{jk} \\ &+ \sum_{s=-\infty}^{n-1} [\phi_j(l, m, s + 1)\phi_k(l, m, s) - \phi_j(l, m, s)\phi_k(l, m, s + 1)] \end{aligned}$$

and $\phi_j(l, m, n)$ are a linear combination of exponential functions in discrete space

$$\left(\frac{1 - ap_\mu}{1 + ap_\mu}\right)^l \left(\frac{1 - bp_\mu}{1 + bp_\mu}\right)^m \left(\frac{1 - cp_\mu}{1 + cp_\mu}\right)^n,$$

namely

$$\phi_j(l, m, n) = \sum_{\mu} \alpha_{j\mu} \left(\frac{1 - ap_\mu}{1 + ap_\mu}\right)^l \left(\frac{1 - bp_\mu}{1 + bp_\mu}\right)^m \left(\frac{1 - cp_\mu}{1 + cp_\mu}\right)^n.$$

2. Soliton equations generated by the bilinear BKP equation. The following are well-known soliton equations generated by the discrete BKP equation.

(a) Sawada-Kotera equation

$$\begin{aligned} D_x(D_t + D_x^5)f \cdot f &= 0, \\ u &= 2 \frac{\partial^2}{\partial x^2} \log f, \\ u_t + 15(u^3 + uu_{xx})_x + u_{xxxxx} &= 0. \end{aligned}$$

(b) Model equation of shallow-water wave

$$\begin{aligned} D_x(D_t - D_t D_x^2 + D_x)f \cdot f &= 0, \\ u &= 2 \frac{\partial^2}{\partial x^2} \log f, \\ u_t - u_{xxt} - 3uu_t + 3u_x \int_x^\infty u_t dx' &= 0. \end{aligned}$$

Here we add two more examples.

(1) Discrete-time Toda equation of BKP type.

Let

$$\begin{aligned} D_1 &= \frac{3}{2} \delta D_t, & z_1 &= 1, \\ D_2 &= -\frac{1}{2} \delta D_t, & z_2 &= -1 + 2\delta^2, \\ D_3 &= D_n - \frac{1}{2} \delta D_t, & z_3 &= -\delta^2, \\ D_4 &= -D_n - \frac{1}{2} \delta D_t, & z_4 &= -\delta^2. \end{aligned}$$

Then

$$\begin{aligned} [e^{\delta D_t + \frac{1}{2} \delta D_t} + e^{-\delta D_t + \frac{1}{2} \delta D_t} - 2e^{-\frac{1}{2} \delta D_t} \\ - \delta^2 (e^{D_n - \frac{1}{2} \delta D_t} + e^{-D_n - \frac{1}{2} \delta D_t} - 2e^{-\frac{1}{2} \delta D_t})] f \cdot f = 0, \end{aligned}$$

which is rewritten as

$$\cosh \frac{1}{2} \delta D_t [\sinh^2 \frac{1}{2} \delta D_t - \delta^2 \sinh^2 \frac{1}{2} D_n] f \cdot f = 0. \tag{1}$$

The bilinear form indicates that the 1-soliton solution is the same as that of the discrete-time Toda equation

$$[\sinh^2 \frac{1}{2} \delta D_t - \delta^2 \sinh^2 \frac{1}{2} D_n] f \cdot f = 0.$$

The bilinear equation (1) is transformed into the nonlinear difference-difference equation in the ordinary form

$$W_n^{m+1} - W_n^{m-1} = \log \frac{\delta^2 [\exp(W_{n+1}^m + V_n^m) + \exp(V_n^m - W_n^m)] + (1 - 2\delta^2)}{\delta^2 [\exp(W_n^m + V_{n-1}^m) + \exp(V_{n-1}^m - W_{n-1}^m)] + (1 - 2\delta^2)},$$

$$V_n^{m+1} - V_n^m = W_{n+1}^m - W_n^m,$$

through a series of dependent variable transformations

$$\begin{aligned} f_n^m &= e^{\phi_n^m}, \\ \Delta_m \phi_n^m &= T_n^m, \\ \Delta_n \phi_n^m &= S_n^m, \\ \Delta_m S_n^m &= W_n^m, \\ \Delta_n S_n^m &= V_n^m, \end{aligned}$$

where Δ is the forward difference operator defined by

$$\begin{aligned} \Delta_m f_n^m &= \delta^{-1} [f_n^{m+1} - f_n^m], \\ \Delta_n f_n^m &= f_{n+1}^m - f_n^m. \end{aligned}$$

(2) Modified Toda equation of BKP type.

Let

$$\begin{aligned} D_1 &= \frac{1}{2} (\delta D_x + \epsilon D_y), & z_1 &= 1, \\ D_2 &= \frac{1}{2} (\delta D_x - \epsilon D_y), & z_2 &= -1 + \beta \delta \epsilon, \\ D_3 &= -D_n - \frac{1}{2} (\delta D_x + \epsilon D_y), & z_3 &= -\alpha \epsilon - \beta \delta \epsilon, \\ D_4 &= D_n - \frac{1}{2} (\delta D_x - \epsilon D_y), & z_4 &= \alpha \epsilon. \end{aligned}$$

Then

$$\begin{aligned} & \{ e^{\frac{1}{2}(\delta D_x + \epsilon D_y)} - e^{\frac{1}{2}(\delta D_x - \epsilon D_y)} \\ & + \alpha \epsilon [e^{D_n - \frac{1}{2}(\delta D_x - \epsilon D_y)} - e^{-D_n - \frac{1}{2}(\delta D_x + \epsilon D_y)}] \\ & + \beta \delta \epsilon [e^{\frac{1}{2}(\delta D_x - \epsilon D_y)} - e^{-D_n - \frac{1}{2}(\delta D_x + \epsilon D_y)}] \} f \cdot f = 0, \end{aligned}$$

which is rewritten as

$$\begin{aligned}
 & 2\delta^{-1}\epsilon^{-1} \sinh\frac{1}{2}\delta D_x \sinh\frac{1}{2}\epsilon D_y f_n \cdot f_n \\
 & - \alpha\delta^{-1} e^{\epsilon D_y} \sinh\frac{1}{2}\delta D_x f_{n+1} \cdot f_{n-1} \\
 & - \beta[e^{\frac{1}{2}(\delta D_x + \epsilon D_y)} f_{n+1} \cdot f_{n-1} - e^{\frac{1}{2}(\delta D_x - \epsilon D_y)} f_n \cdot f_n] = 0.
 \end{aligned}$$

In the limit of small δ, ϵ we have

$$D_x D_y f_n \cdot f_n = \alpha D_x f_{n+1} \cdot f_{n-1} + \beta(f_{n+1} f_{n-1} - f_n^2).$$

The bilinear equation is transformed into the ordinary form

$$\begin{aligned}
 \frac{\partial}{\partial x} \log \frac{\beta + V_n}{\beta - \alpha(I_n + I_{n+1})} &= I_n - I_{n+1}, \\
 \frac{\partial}{\partial y} I_n &= V_{n-1} - V_n,
 \end{aligned}$$

through the transformation

$$\begin{aligned}
 f_n &= e^{S_n}, \\
 V_n &= \frac{\partial^2}{\partial x \partial y} S_n, \\
 I_n &= \frac{\partial}{\partial x} (S_{n-1} - S_n),
 \end{aligned}$$

which we call the modified Toda equation of BKP type.

The bilinear form of the modified Toda equation of BKP type

$$D_x D_y f_n \cdot f_n = \alpha D_x f_{n+1} \cdot f_{n-1} + \beta(f_{n+1} f_{n-1} - f_n^2),$$

suggests that a new equation may be obtained by replacing the term

$$f_{n+1} f_{n-1} - f_n^2$$

by

$$D_x f_{n+1} \cdot f_{n-1}.$$

We shall replace the term $g_j g_k$ by $D_x g_j \cdot g_k$ in order to obtain a coupled modified KdV equation of derivative type.

The well-known modified KdV equation

$$\frac{\partial}{\partial t} v + 6v^2 \frac{\partial}{\partial x} v + \frac{\partial^3}{\partial x^3} v = 0,$$

is transformed into the bilinear form

$$\begin{aligned}
 (D_t + D_x^3) f \cdot g &= 0, \\
 D_x^2 f \cdot f &= 2g^2,
 \end{aligned}$$

through the dependent variable transformation

$$v = \frac{g}{f}.$$

A coupled modified KdV equation is obtained by considering the following coupled form of the modified KdV equations [4].

$$\begin{cases} (D_t + D_x^3)f \cdot g_j = 0, & \text{for } j = 1, 2, \dots, N, \\ D_x^2 f \cdot f = 2 \sum_{1 \leq j < k \leq N} c_{jk} g_j g_k, \end{cases} \tag{2}$$

which is transformed into the ordinary form

$$\frac{\partial}{\partial t} v_j + 6 \left[\sum_{1 \leq j < k \leq N} c_{jk} v_j v_k \right] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0,$$

through the transformation

$$v_j = \frac{g_j}{f}.$$

The multisoliton solution to equation (2) is expressed by two-component Pfaffians [4]:

$$\begin{aligned} f &= \text{pf}(a_1, a_2, \dots, a_L, b_1, b_2, \dots, b_L), \\ g_j &= \text{pf}(d_0, a_1, a_2, \dots, a_L, b_1, b_2, \dots, b_L, c_j), \text{ for } j = 1, 2, \dots, N, \end{aligned}$$

where the elements of Pfaffians are defined as follows

$$\begin{aligned} \text{pf}(d_n, a_\nu) &\equiv \frac{\partial^n}{\partial x^n} \exp(\eta_\nu), \text{ for } n = 0, 1, 2, 3, \dots, \\ \text{pf}(a_\mu, a_\nu) &\equiv \frac{p_\mu - p_\nu}{p_\mu + p_\nu} \exp(\eta_\mu + \eta_\nu), \\ \text{pf}(a_\mu, b_\nu) &\equiv \delta_{\mu,\nu}, \\ \text{pf}(b_\mu, b_\nu) &\equiv -\frac{c_{jk}}{p_\mu^2 - p_\nu^2}, \quad \left\{ \begin{array}{l} b_\mu \in B_j \\ b_\nu \in B_k \end{array} \right\}, \\ \text{pf}(b_\mu, c_j) &\equiv \begin{cases} 1, & \text{if } b_\mu \in B_j, \\ 0, & \text{if } b_\mu \notin B_k \end{cases} \\ \text{pf}(\text{otherwise}) &\equiv 0. \end{aligned}$$

A coupled modified KdV equation of derivative type is obtained by replacing the coupling terms of the product form

$$c_{jk} g_j g_k, \quad (c_{jk} = c_{kj})$$

by coupling terms of derivative type

$$c_{jk} D_x g_j \cdot g_k, \quad (c_{jk} = -c_{kj}).$$

We obtain

$$\begin{cases} (D_t + D_x^3)f \cdot g_j = 0, \\ D_x^2 f \cdot f = 2 \sum_{1 \leq j < k \leq N} c_{jk} D_x g_j \cdot g_k, \end{cases} \quad \text{for } j = 1, 2, \dots, N, \tag{3}$$

which is transformed into the ordinary form

$$\frac{\partial}{\partial t} v_j + 6 \left[\sum_{1 \leq j < k \leq N} c_{jk} (D_x v_j \cdot v_k) \right] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0,$$

through the transformation

$$v_j = \frac{g_j}{f}.$$

The multisoliton solution to equation (3) is expressed in the same form as that of the coupled modified KdV equation [5]:

$$\begin{aligned} f &= \text{pf}(a_1, a_2, \dots, a_L, b_1, b_2, \dots, b_L), \\ g_j &= \text{pf}(d_0, a_1, a_2, \dots, a_L, b_1, b_2, \dots, b_L, c_j), \quad \text{for } j = 1, 2, \dots, N, \end{aligned}$$

where the elements of Pfaffians are the same as those of the modified KdV equation except the element

$$\text{pf}(b_\mu, b_\nu) \equiv -\frac{c_{jk}}{p_\mu^2 - p_\nu^2}, \quad \left\{ \begin{array}{l} b_\mu \in B_j \\ b_\nu \in B_k \end{array} \right\}, \quad (c_{jk} = c_{kj}),$$

is replaced by

$$\text{pf}(b_\mu, b_\nu) \equiv -\frac{c_{jk}}{p_\mu + p_\nu}, \quad \left\{ \begin{array}{l} b_\mu \in B_j \\ b_\nu \in B_k \end{array} \right\}, \quad (c_{jk} = -c_{kj}).$$

In conclusion we have shown that soliton-solutions of the following equations are expressed by Pfaffians.

(i) Discrete-time Toda equation of BKP type

$$\begin{aligned} W_n^{m+1} - W_n^{m-1} &= \log \frac{\delta^2 [\exp(W_{n+1}^m - V_n^m) + \exp(V_n^m - W_n^m)] + (1 - 2\delta^2)}{\delta^2 [\exp(W_n^m - V_{n-1}^m) + \exp(V_{n-1}^m - W_{n-1}^m)] + (1 - 2\delta^2)}, \\ V_n^{m+1} - V_n^m &= W_{n+1}^m - W_n^m. \end{aligned}$$

(ii) Modified Toda equation of BKP type

$$\begin{aligned} \frac{d}{dt} \log \frac{\beta + V_n}{\beta - \alpha(I_n + I_{n-1})} &= I_n - I_{n+1}, \\ \frac{d}{dt} I_n &= V_{n-1} - V_n. \end{aligned}$$

(iii) Coupled modified KdV equation

$$\frac{\partial}{\partial t} v_j + 6 \left[\sum_{1 \leq j < k \leq N} c_{jk} v_j v_k \right] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0, \quad j = 1, 2, \dots, N.$$

(iv) Coupled modified KdV equations of derivative type

$$\frac{\partial}{\partial t} v_j + 6 \left[\sum_{1 \leq j < k \leq N} c_{jk} (D_x v_j \cdot v_k) \right] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0, \quad j = 1, 2, \dots, N.$$

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