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A KADISON–SAKAI-TYPE THEOREM

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Abstract

Suppose that $\sigma : \mathfrak{M} \to \mathfrak{M}$ is an ultraweakly continuous surjective $*$-linear mapping and $d : \mathfrak{M} \to \mathfrak{M}$ is an ultraweakly continuous $*$-σ-derivation such that $d(I)$ is a central element of $\mathfrak{M}$. We provide a Kadison–Sakai-type theorem by proving that $\mathfrak{H}$ can be decomposed into $\mathfrak{K} \oplus \mathfrak{L}$ and d can be factored as the form $\delta \oplus 2Z\tau$, where $\delta : \mathfrak{M} \to \mathfrak{M}$ is an inner $*$-$\sigma_{\mathfrak{K}}$-derivation, Z is a central element, $2\tau = 2\sigma_{\mathfrak{L}}$ is a $*$-homomorphism, and $\sigma_{\mathfrak{K}}$ and $\sigma_{\mathfrak{L}}$ stand for compressions of σ to $\mathfrak{K}$ and $\mathfrak{L}$, respectively.

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1. Introduction

The theory of algebras of operators on Hilbert spaces started around 1930 with papers of von Neumann and Murray. The principal motivations of these authors were the theory of unitary group representations and certain aspects of the quantum mechanical formalism. The von Neumann algebras are significant for mathematical physics since the most fruitful algebraic reformulation of quantum statistical mechanics and quantum field theory was in terms of these algebras, see [1, 2]. The study of theory of derivations in operator algebras is motivated by questions in quantum physics and statical mechanics. One important question in the theory of derivations is: when are all bounded derivations inner? Forty years ago, Kadison [6] and Sakai [11] independently proved that every derivation of a von Neumann algebra $\mathfrak{M}$ into itself is inner. This was the starting point for the study of the so-called bounded cohomology groups. This nice result can be restated as saying that the first bounded cohomology group $H^1(\mathfrak{M}; \mathfrak{M})$ (that is, the vector space of derivations modulo the inner derivations) vanishes.

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two algebras, $\mathfrak{X}$ is a $\mathfrak{B}$-bimodule and $\sigma, \tau : \mathfrak{A} \to \mathfrak{B}$ are two linear mappings, which are called the ground mappings. A linear mapping $d : \mathfrak{A} \to \mathfrak{X}$ is called a (σ, τ)-derivation if $d(ab) = d(a)\sigma(b) + \tau(a)d(b)$ for all a, $b \in \mathfrak{A}$. In the case $\sigma = \tau$, it is called a σ-derivation. These maps have been investigated extensively in pure algebra. Recently, they have been treated in the Banach algebra theory (see [3, 5, 7, 8, 10, 13] and references therein).

There are some applications of σ-derivations to develop an approach to deformations of Lie algebras which have many applications in models of quantum phenomena and in analysis of complex systems; see [4]. A wide range of examples are as follows.

(i) Every ordinary derivation of an algebra $\mathfrak{A}$ into an $\mathfrak{A}$-bimodule $\mathfrak{X}$ is an $\iota_{\mathfrak{A}}$-derivation (throughout the paper, $\iota_{\mathfrak{A}}$ denotes the identity map on the algebra $\mathfrak{A}$).

(ii) Every endomorphism α on $\mathfrak{A}$ is a $\alpha/2$-derivation.

(iii) For a given homomorphism ρ on $\mathfrak{A}$ and a fixed arbitrary element X in an $\mathfrak{A}$-bimodule $\mathfrak{X}$, the linear mapping $\delta(A) = X\rho(A) - \rho(A)X$ is a ρ-derivation of $\mathfrak{A}$ into $\mathfrak{X}$ which is said to be an inner ρ-derivation.

(iv) Every point derivation $d : \mathfrak{A} \to \mathbb{C}$ at the character θ is a θ-derivation.

In this paper, we employ our methods from [8] to provide a version of the celebrated Kadison–Sakai theorem. We divide our work into three sections. The first section is devoted to prove a Kadison–Sakai-type theorem for ρ-derivations on von Neumann algebras when ρ is an ultraweakly continuous homomorphism. In the next section, we briefly discuss an extension of the main result of [8]. In the last section, we decompose a σ-derivation into a sum of an inner σ-derivation and a multiple of a homomorphism. The importance of our approach is that σ is a linear mapping in general, not necessarily a homomorphism.

A von Neumann algebra $\mathfrak{M}$ is an ultraweakly closed $*$-subalgebra of $B(\mathfrak{H})$ containing the identity operator I, where $\mathfrak{H}$ is a Hilbert space. By the weak (operator) topology on $B(\mathfrak{H})$ we mean the topology generated by the semi-norms $T \mapsto |\langle T\xi, \eta\rangle| (\xi, \eta \in \mathfrak{H})$. We also use the terminology ultraweak (operator) topology for the weak*-topology on $B(\mathfrak{H})$ considered as the dual space of the nuclear operators on $\mathfrak{H}$. When we speak of ultraweak continuity (weak continuity, respectively) of a mapping between von Neumann algebras $\mathfrak{M}$ and $\mathfrak{N}$ we mean that we equipped both $\mathfrak{M}$ and $\mathfrak{N}$ with the ultraweak topology (the weak topology, respectively). We refer the reader to [12] for undefined phrases and notation.

2. Toward a Kadison–Sakai-type theorem for σ-derivations

We start our work with the following lemma due to Sakai which can be found in [9]. We state its proof for the sake of convenience. Recall that a linear mapping T between two $*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is called a $*$-mapping if $T(a^*) = (Ta)^*$ for all $a \in \mathfrak{A}$.

Lemma 2.1. *Let $\rho : \mathfrak{M} \to \mathfrak{N}$ be an ultraweakly continuous $*$-epimorphism and $\delta : \mathfrak{M} \to \mathfrak{N}$ be a $*$-ρ-derivation between von Neumann algebras. Then there is a central projection $P \in \mathfrak{M}$ and an $*$-isomorphism $\widetilde{\rho} : \mathfrak{M}P \to \mathfrak{N}$ such that $\delta : \mathfrak{M}P \to \mathfrak{N}$ is a $*$-$\widetilde{\rho}$-derivation. Moreover, $\delta(A) = 0$ for each $A \in \mathfrak{M}(I - P)$. Also, $\widetilde{\rho}^{-1} \circ \delta$ is an ordinary derivation on $\mathfrak{M}P$.*

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PROOF. Since ρ is an ultraweakly continuous $*$-homomorphism, its kernel is an ultraweakly closed ideal of $\mathfrak{M}$. Hence, there is a central projection $Q \in \mathfrak{M}$ such that $\ker \rho = \mathfrak{M}Q$. Set $P = I - Q$. Then $\widetilde{\rho} = \rho|_{\mathfrak{M}P} : \mathfrak{M}P \to \mathfrak{N}$ is a $*$-isomorphism. We have

$$\delta(ABP) = \delta(APBP) = \delta(AP)\rho(AP) + \rho(AP)\delta(BP).$$

Hence, δ is a $*$-$\widetilde{\rho}$-derivation on $\mathfrak{M}P$. Moreover, if $A = B(I - P) \in \mathfrak{M}(I - P)$ is a projection, then

$$\delta(A) = \delta((BQ)^2) = \delta(BQ)\rho(BQ) + \rho(BQ)\delta(BQ) = 0,$$

since $BQ \in \mathfrak{M}Q = \ker \rho$. The space $\mathfrak{M}(I - P)$ is a von Neumann algebra, because it is $\ker \rho = \rho^{-1}(\{0\})$ and ρ is ultraweakly continuous. Thus, $\mathfrak{M}(I - P)$ is generated by its projection and so $\delta(A) = 0$ for each $A \in \mathfrak{M}(I - P)$. The last assertion is now obvious. □

THEOREM 2.2. *If $\rho : \mathfrak{M} \to \mathfrak{N}$ is an ultraweakly continuous $*$-epimorphism and $\delta : \mathfrak{M} \to \mathfrak{N}$ is an ultraweakly continuous $*$-ρ-derivation between von Neumann algebras, then there is an element $U \in \mathfrak{M}$ with $\|U\| \le \|\delta\|$ such that $\delta(A) = U\rho(A) - \rho(A)U$ for each $A \in \mathfrak{M}$. In other words, δ is an inner ρ-derivation.*

PROOF. The mapping $\widetilde{\rho}^{-1} \circ \delta$ is an ordinary derivation on $\mathfrak{M}P$, where P is as in Lemma 2.1. Thus, by [11, Theorem 2.5.3], there is a $V \in \mathfrak{M}P$ with $\|V\| \le \|\widetilde{\rho}^{-1} \circ \delta\| \le \|\delta\|$ such that $(\widetilde{\rho}^{-1} \circ \delta)(A) = VA - AV$ for all $A \in \mathfrak{M}P$. Thus, $\delta(A) = \widetilde{\rho}(V)\widetilde{\rho}(A) - \widetilde{\rho}(A)\widetilde{\rho}(V)$, for all $A \in \mathfrak{M}P$. Putting $U = \widetilde{\rho}(V)$ we have $\delta(A) = U\rho(A) - \rho(A)U$ for all $A \in \mathfrak{M}P$. The latter equality is also valid for $A \in \mathfrak{M}(I - P)$, since both sides are zero for these elements. Finally, $\|U\| = \|\rho(V)\| \le \|V\| \le \|\delta\|$. □

The following result can be found in [8] and we omit its proof.

PROPOSITION 2.3. *Let $\mathfrak{A}$, $\mathfrak{B}$ be $*$-algebras and σ, $\tau : \mathfrak{A} \to \mathfrak{B}$ be $*$-linear mappings. Then any $*$-(σ, τ)-derivation $d : \mathfrak{A} \to \mathfrak{B}$ is a $*$-$(\sigma + \tau)/2$-derivation.*

The previous proposition enables us to focus on $*$-σ-derivations while we deal with $*$-algebras. In particular, we obtain the following generalization of Theorem 2.2.

THEOREM 2.4. *Let $\mathfrak{M}, \mathfrak{N}$ be von Neumann algebras, let $\rho_1, \rho_2 : \mathfrak{M} \to \mathfrak{N}$ be ultraweakly continuous and norm continuous $*$-homomorphisms such that $\rho_1 + \rho_2$ be surjective, and let $\delta : \mathfrak{M} \to \mathfrak{N}$ be an ultraweakly continuous $*$-(ρ_1, ρ_2)-derivation. Then there is an element U in $\mathfrak{M}$ with $\|U\| \le ((\|\rho_1\| + \|\rho_2\|)/2)\|\delta\|$ such that $\delta(A) = U(\rho_1(A) + \rho_2(A))/2 - ((\rho_1(A) + \rho_2(A))/2)U$ for each $A \in \mathfrak{M}$.*

3. A Kadison–Sakai-type theorem for σ-derivations

In this section, we aim to remove the condition of being a homomorphism from the ground mapping and provide our main result on the generalization of the

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Kadison–Sakai theorem. Indeed, we would like to decompose d as a direct sum of an inner $*$-σ-derivation and a $*$-homomorphism.

Recall that if we have a Hilbert space direct sum $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{L}$, P is the projection corresponding to $\mathfrak{K}$ and $T \in \mathcal{B}(\mathfrak{H})$, then the compression $T_{\mathfrak{K}}$ to $\mathfrak{K}$ is the operator $T_{\mathfrak{K}} \in \mathcal{B}(\mathfrak{K})$ defined by $T_{\mathfrak{K}}(k) = PT(k)$ $(k \in \mathfrak{K})$. Obviously, $PTP \leftrightarrow T_{\mathfrak{K}}$ is an isometric $*$-isomorphism between von Neumann algebras $P\mathcal{B}(\mathfrak{H})P$ and $\mathcal{B}(\mathfrak{K})$. If $\sigma : \mathfrak{M} \to \mathfrak{M}$ is a linear mapping such that $\sigma(A)P = P\sigma(A)$ for all $A \in \mathfrak{M}$, then $\sigma_{\mathfrak{K}} : \mathfrak{M} \to P\mathfrak{M}P$, $\sigma_{\mathfrak{L}} : \mathfrak{M} \to (I - P)\mathfrak{M}(I - P)$ can be defined by $\sigma_{\mathfrak{K}}(A) = P\sigma(A)P$, $\sigma_{\mathfrak{L}}(A) = (I - P)\sigma(A)(I - P)$. Furthermore, $\sigma = \sigma_{\mathfrak{K}} \oplus \sigma_{\mathfrak{L}}$. Note that $P\mathfrak{M}P$ is a von Neumann subalgebra of $B(\mathfrak{K})$, and $(I - P)\mathfrak{M}(I - P)$ is a von Neumann subalgebra of $B(\mathfrak{L})$.

EXAMPLE 3.1. Let $\mathfrak{H}$ be the separable Hilbert space with orthonormal basis $\{e_n\}$. Define $\sigma : \mathcal{B}(\mathfrak{H}) \to \mathcal{B}(\mathfrak{H})$ by

$$\langle \sigma(A)e_n, e_m\rangle = \begin{cases} \exp\left(\frac{\mathrm{i}}{s} - \frac{\mathrm{i}}{r}\right)\langle Ae_s, e_r\rangle & \text{if } n = 2s - 1,\ m = 2r - 1; \\ \frac{1}{2}\exp\left(\frac{\mathrm{i}}{s} - \frac{\mathrm{i}}{r}\right)\langle Ae_s, e_r\rangle & \text{if } n = 2s,\ m = 2r; \\ 0 & \text{otherwise} \end{cases}$$

and $d : \mathcal{B}(\mathfrak{H}) \to \mathcal{B}(\mathfrak{H})$ by

$$\langle d(A)e_n, e_m\rangle = \begin{cases} \left(\frac{1}{s} - \frac{1}{r}\right)\exp\left(\frac{\mathrm{i}}{s} - \frac{\mathrm{i}}{r}\right)\langle Ae_s, e_r\rangle & \text{if } n = 2s - 1,\ m = 2r - 1; \\ \exp\left(\frac{\mathrm{i}}{s} - \frac{\mathrm{i}}{r}\right)\langle Ae_s, e_r\rangle & \text{if } n = 2s,\ m = 2r; \\ 0 & \text{otherwise.} \end{cases}$$

Then d is a σ-derivation. To show this let E_{rs} be the operator defined on $\mathfrak{H}$ by

$$\langle E_{rs}e_n, e_m\rangle = \delta_{rm}\delta_{sn},$$

where δ is the δ-Kroneker function. Then

$$\langle d(E_{rs})e_n, e_m\rangle = \begin{cases} \left(\frac{1}{s} - \frac{1}{r}\right)\exp\left(\frac{\mathrm{i}}{s} - \frac{\mathrm{i}}{r}\right) & \text{if } n = 2s - 1,\ m = 2r - 1; \\ \exp\left(\frac{\mathrm{i}}{s} - \frac{\mathrm{i}}{r}\right) & \text{if } n = 2s,\ m = 2r; \\ 0 & \text{otherwise} \end{cases}$$

and so

$$\langle d(E_{tu}E_{rs})e_n, e_m\rangle = \delta_{ur} \begin{cases} \left(\frac{1}{s}-\frac{1}{t}\right)\exp\left(\frac{\mathrm{i}}{s}-\frac{\mathrm{i}}{t}\right) & \text{if } n=2s-1,\ m=2t-1;\\ \exp\left(\frac{\mathrm{i}}{s}-\frac{\mathrm{i}}{t}\right) & \text{if } n=2s,\ m=2t;\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand,

$$\langle \sigma(E_{rs})e_n, e_m\rangle = \begin{cases} \exp\left(\frac{\mathrm{i}}{s}-\frac{\mathrm{i}}{r}\right) & \text{if } n=2s-1,\ m=2r-1;\\ \frac{1}{2}\exp\left(\frac{\mathrm{i}}{s}-\frac{\mathrm{i}}{r}\right) & \text{if } n=2s,\ m=2r;\\ 0 & \text{otherwise}\end{cases}$$

and so

$$\langle d(E_{tu})\sigma(E_{rs})e_n, e_m\rangle = \delta_{ur} \begin{cases} \left(\frac{1}{u}-\frac{1}{t}\right)\exp\left(\frac{\mathrm{i}}{s}-\frac{\mathrm{i}}{t}\right) & \text{if } n=2s-1,\ m=2t-1;\\ \frac{1}{2}\exp\left(\frac{\mathrm{i}}{s}-\frac{\mathrm{i}}{t}\right) & \text{if } n=2s,\ m=2t;\\ 0 & \text{otherwise}\end{cases}$$

and

$$\langle \sigma(E_{tu})d(E_{rs})e_n, e_m\rangle = \delta_{ur} \begin{cases} \left(\frac{1}{s}-\frac{1}{u}\right)\exp\left(\frac{\mathrm{i}}{s}-\frac{\mathrm{i}}{t}\right) & \text{if } n=2s-1,\ m=2t-1;\\ \frac{1}{2}\exp\left(\frac{\mathrm{i}}{s}-\frac{\mathrm{i}}{t}\right) & \text{if } n=2s,\ m=2t;\\ 0 & \text{otherwise.}\end{cases}$$

This shows that

$$d(E_{tu}E_{rs}) = d(E_{tu})\sigma(E_{rs}) + \sigma(E_{tu})d(E_{rs})$$

and so d is a σ-derivation.

Now let $\mathfrak{K}$ be the Hilbert space with orthonormal basis $\{e_{2n-1}\}$ and $\mathfrak{L}$ be the Hilbert space with orthonormal basis $\{e_{2n}\}$. Then $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{L}$. If $\sigma_{\mathfrak{K}} : \mathcal{B}(\mathfrak{H}) \to \mathcal{B}(\mathfrak{K})$ and $\sigma_{\mathfrak{L}} : \mathcal{B}(\mathfrak{H}) \to \mathcal{B}(\mathfrak{L})$ are defined as above, then $\sigma = \sigma_{\mathfrak{K}} \oplus \sigma_{\mathfrak{L}}$. In the same fashion we can write $d = d_{\mathfrak{K}} \oplus d_{\mathfrak{L}}$. One can easily verify that $d_{\mathfrak{K}}$ is a $\sigma_{\mathfrak{K}}$-derivation, $d_{\mathfrak{L}}$ is a $\sigma_{\mathfrak{L}}$-derivation and $\sigma_{\mathfrak{K}}$ is a $*$-homomorphism, $\sigma_{\mathfrak{L}}$ is $\frac{1}{2}$ times a $*$-homomorphism, and $d_{\mathfrak{K}}(A) = U\sigma_{\mathfrak{K}}(A) - \sigma_{\mathfrak{K}}(A)U$, where U is the operator defined by $\langle Ue_{2n-1}, e_{2m-1}\rangle = (1/n)\delta_{nm}$. Moreover, $d_{\mathfrak{L}} = 2\sigma_{\mathfrak{L}}$. Note also that we have $\sigma_{\mathfrak{K}}(I) = I_{\mathfrak{K}}$, $\sigma_{\mathfrak{L}}(I) = \frac{1}{2}I_{\mathfrak{L}}$, $d_{\mathfrak{K}}(I) = 0$ and $d_{\mathfrak{L}}(I) = I_{\mathfrak{L}}$.

We show that the situation described in the above example is typical of the general situation.

PROPOSITION 3.2. *Suppose that $\sigma : \mathfrak{M} \to \mathfrak{M}$ is an ultraweakly continuous surjective $*$-linear mapping, $d : \mathfrak{M} \to \mathfrak{M}$ is an ultraweakly continuous $*$-σ-derivation such that $d(I)$ is a central element of $\mathfrak{M}$, $\mathfrak{L}_0 = \bigcup_{B,C\in\mathfrak{M}}(\sigma(BC) - \sigma(B)\sigma(C))(\mathfrak{H})$, $\mathfrak{L}$ is the closed linear span of $\mathfrak{L}_0$ and $\mathfrak{K} = \mathfrak{L}^{\perp}$. Then:*

(i) $\mathfrak{K} = \bigcap_{B,C\in\mathfrak{M}} \ker(\sigma(BC) - \sigma(B)\sigma(C))$;
(ii) *if $P = P_{\mathfrak{K}}$ is the projection corresponding to $\mathfrak{K}$, then $\sigma(A)P = P\sigma(A)$ and $d(A)P = Pd(A)$ for all $A \in \mathfrak{M}$;*
(iii) *if we define $\delta : \mathfrak{M} \to P\mathfrak{M}P$ by $\delta(A) = Pd(A)P$, $\rho : \mathfrak{M} \to P\mathfrak{M}P$ by $\rho(A) = \sigma(A)P$, $\alpha : \mathfrak{M} \to (I-P)\mathfrak{M}(I-P)$ by $\alpha(A) = (I-P)d(A)(I-P)$ and $\tau : \mathfrak{M} \to (I-P)\mathfrak{M}(I-P)$ by $\tau(A) = (I-P)\sigma(A)(I-P)$, then δ is an ultraweakly continuous $*$-ρ-derivation, α is an ultraweakly continuous $*$-τ-derivation, $d = \delta \oplus \alpha$ and $\sigma = \rho \oplus \tau$; moreover ρ is an ultraweakly continuous $*$-epimorphism;*
(iv) $\mathfrak{K} = \ker \delta(I)$ *and* $\mathfrak{L} = \overline{\alpha(I)(\mathfrak{L})}$;
(v) *δ is an inner ρ-derivation;*
(vi) $\tau(I) = \frac{1}{2} I_{\mathfrak{L}}$.

PROOF. (i) For each $B, C \in \mathfrak{M}$, $h \in \mathfrak{H}$ and $k \in \mathfrak{K}$ we have

$$\begin{aligned} 0 &= \langle (\sigma(BC) - \sigma(B)\sigma(C))(h), k\rangle \\ &= \langle h, (\sigma(BC) - \sigma(B)\sigma(C))^*(k)\rangle \\ &= \langle h, (\sigma(C^*B^*) - \sigma(C^*)\sigma(B^*))(k)\rangle. \end{aligned}$$

Since $\mathfrak{M}$ is a $*$-subalgebra of $\mathcal{B}(\mathfrak{H})$, we infer that $(\sigma(BC) - \sigma(B)\sigma(C))(k) = 0$ for each $B, C \in \mathfrak{M}$ and $k \in \mathfrak{K}$. This shows that $\mathfrak{K} = \bigcap_{B,C\in\mathfrak{M}} \ker(\sigma(BC) - \sigma(B)\sigma(C))$.

(ii) Let $P = P_{\mathfrak{K}}$ be the projection corresponding to $\mathfrak{K}$. For each $B, C, A \in \mathfrak{M}$ and $k \in \mathfrak{K}$ we have

$$\begin{aligned} (\sigma(BC) - \sigma(B)\sigma(C))\sigma(A)(k) &= (\sigma(BC)\sigma(A) - \sigma(B)\sigma(C)\sigma(A))(k) \\ &= (\sigma(BCA) - \sigma(BCA))(k) \\ &= 0. \end{aligned}$$

This shows that $\sigma(A)(\mathfrak{K}) \subseteq \mathfrak{K}$ for each $A \in \mathfrak{M}$. Since σ is a $*$-linear mapping, we have $\sigma(A)P = P\sigma(A)$ for all $A \in \mathfrak{M}$.

By using [8, Lemma 2.2] we obtain

$$0 = d(B)(\sigma(CA) - \sigma(C)\sigma(A))(k) = (\sigma(BC) - \sigma(B)\sigma(C))d(A)(k)$$

for all $k \in \mathfrak{K}$. This implies that $d(A)(k) \in \mathfrak{K}$ for all $k \in \mathfrak{K}$. Hence, $d(A)(\mathfrak{K}) \subseteq \mathfrak{K}$ for all $A \in \mathfrak{M}$. Since d is a $*$-linear mapping we conclude that $d(A)P = Pd(A)$ for all $A \in \mathfrak{M}$.

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(iii) First, ρ is a $*$-homomorphism. In fact, if $B, C \in \mathfrak{M}$, then

$$\rho(BC)(k) = \sigma(BC)P(k) = \sigma(B)\sigma(C)(k) = \sigma(B)P\sigma(C)P(k) = \rho(B)\rho(C)(k)$$

for all $k \in \mathfrak{K}$, and

$$\rho(BC)(\ell) = \sigma(BC)P(\ell) = 0 = \sigma(B)P\sigma(C)P(\ell) = \rho(B)\rho(C)(\ell).$$

for all $\ell \in \mathfrak{L}$. Hence, ρ is a homomorphism on $\mathfrak{M}$. Moreover, $\rho(A^*) = \sigma(A^*)P = \sigma(A)^*P = (P\sigma(A))^* = (\sigma(A)P)^* = \rho(A)^*$, $(A \in \mathfrak{M})$.

Second, δ is a ρ-derivation, since if $A, B \in \mathfrak{M}$, then

$$\begin{aligned}
\delta(AB)(k) &= d(AB)P(k) \\
&= Pd(AB)(k) \\
&= Pd(A)\sigma(B)(k) + P\sigma(A)d(B)(k) \\
&= P^2d(A)\sigma(B)P^2(k) + P^2\sigma(A)d(B)P^2(k) \\
&= Pd(A)PP\sigma(B)P(k) + P\sigma(A)PPd(B)P(k) \\
&= (\delta(A)\rho(B) + \rho(A)d(B))(k),
\end{aligned}$$

for all $k \in \mathfrak{K}$, and

$$\begin{aligned}
\delta(AB)(\ell) &= Pd(AB)P(\ell) \\
&= 0 \\
&= Pd(A)PP\sigma(B)P(\ell) + P\sigma(A)PPd(B)P(\ell) \\
&= (\delta(A)\rho(B) + \rho(A)d(B))(\ell).
\end{aligned}$$

for all $\ell \in \mathfrak{L}$.

Similarly one can show that α is a τ-derivation. It is obvious that $d = \delta \oplus \alpha$ and $\sigma = \rho \oplus \tau$.

(iv) We have

$$d(I) = d(I^2) = d(I)\sigma(I) + \sigma(I)d(I) = 2\sigma(I)d(I).$$

Since σ is surjective, $d(I)$ is of the form $\sigma(E)$ for some $E \in \mathfrak{M}$. Now for each $k \in \mathfrak{K}$ we have

$$d(I)(k) = \sigma(E)(k) = \sigma(EI)(k) = \sigma(E)\sigma(I)(k) = d(I)\sigma(I)(k)$$

and so $d(I)(k) = d(I)(2\sigma(I) - I)(k) = 0$ for each $k \in \mathfrak{K}$. Hence, $\delta(I)(k) = d(I)P(k) = 0$ for each $k \in \mathfrak{K}$. Thus, $k \subseteq \ker \delta(I)$. On the other hand, the compression operator $\delta(I)$ belongs to $\mathcal{B}(\mathfrak{K})$. Hence, $\overline{\ker \delta(I)} \subseteq \mathfrak{K}$. Thus, $\mathfrak{K} = \ker \delta(I)$. Similarly, one can show that $\mathfrak{L} = \overline{\alpha(I)(\mathfrak{L})}$.

(v) By Theorem 2.2 there is an element $U \in \mathfrak{M}$ such that $\delta(A) = U\rho(A) - \rho(A)U$ for all $A \in \mathfrak{M}$. Hence, δ is an inner ρ-derivation.

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(vi) Since $d(I)$ is a central element of $\mathfrak{M}$, one easily deduce that $\alpha(I) = d(I)$ $(I - P)$ is in the center of $\mathfrak{M}$. Hence,

$$\alpha(I) = \alpha(I^2) = \alpha(I)\tau(I) + \tau(I)\alpha(I) = 2\tau(I)\alpha(I).$$

Thus, $(2\tau(I) - I)\alpha(I) = 0$. It follows from $\overline{\alpha(I)(\mathfrak{L})} = \mathfrak{L}$ that $2\tau(I) = I_{\mathfrak{L}}$. □

We can now establish a version of the Kadison–Sakai theorem as our main theorem.

THEOREM 3.3. *Suppose that $\sigma : \mathfrak{M} \to \mathfrak{M}$ is an ultraweakly continuous surjective $*$-linear mapping and $d : \mathfrak{M} \to \mathfrak{M}$ is an ultraweakly continuous $*$-σ-derivation such that $d(I)$ is a central element of $\mathfrak{M}$. Then $\mathfrak{H}$ can be decomposed into $\mathfrak{K} \oplus \mathfrak{L}$ and d can be factored as the form*

$$\delta \oplus 2Z\tau,$$

where $\delta : \mathfrak{M} \to \mathfrak{M}$ is an inner $*$-$\sigma_{\mathfrak{K}}$-derivation, Z is a central element, and $2\tau = 2\sigma_{\mathfrak{L}}$ is a $*$-homomorphism.

PROOF. It remains to show the result concerning $\mathfrak{L}$. For $\ell \in \mathfrak{L}$ we have

$$\alpha(A)(\ell) = \alpha(I)\tau(A)(\ell) + \tau(I)\alpha(A)(\ell) = \alpha(I)\tau(A)(\ell) + \tfrac{1}{2}\alpha(A)(\ell).$$

Thus,

$$\alpha(A)(\ell) = 2\alpha(I)\tau(A)(\ell).$$

Putting $Z = \alpha(I)$ we obtain the result. □

References

[1] O. Bratteli and D. W. Robinson, 'Operator algebras and quantum statistical mechanics. 1. C^*- and W^*-algebras, symmetry groups, decomposition of states', in: *Texts and Monographs in Physics*, 2nd edn (Springer, New York, 1987).

[2] ———, 'Operator algebras and quantum statistical mechanics. 2. Equilibrium states. Models in quantum statistical mechanics', in: *Texts and Monographs in Physics*, 2nd edn (Springer, Berlin, 1997).

[3] M. Brešar and A. R. Villena, 'The noncommutative Singer–Wermer conjecture and ϕ-derivations', *J. London Math. Soc.* (2) **66**(3) (2002), 710–720.

[4] J. Hartwig, D. Larsson and S. D. Silvestrov, 'Deformations of Lie algebras using σ-derivations', Preprints in Mathematical Science 2003:32, LUTFMA-5036-2003, Centre for Mathematical Science, Department of Mathematics, Lund Institute of Technology, Lund University, 2003.

[5] S. Hejazian, A. R. Janfada, M. Mirzavaziri and M. S. Moslehian, 'Achievement of continuity of (φ, ψ)-derivations without linearity', *Bull. Belg. Math. Soc.-Simon Stevn.* **14**(4) (2007), 641–652.

[6] R. V. Kadison, 'Derivations of operator algebras', *Ann. Math.* **83** (1966), 280–293.

[7] M. Mirzavaziri and M. S. Moslehian, 'σ-derivations in Banach algebras', *Bull. Iranian Math. Soc.* **32**(1) (2006), 65–78.

[8] ———, 'Automatic continuity of σ-derivations in C^*-algebras', *Proc. Amer. Math. Soc.* **134**(11) (2006), 3319–3327.

[9] ———, 'Ultraweak continuity of σ-derivations on von Neumann algebras', *Math. Phys. Anal. Geom.* to appear.

[10] M. S. Moslehian, 'Approximate $(\sigma - \tau)$-contractibility', *Nonlinear Funct. Anal. Appl.* **11**(5) (2006), 805–813.

[11] S. Sakai, 'Derivations of W^*-algebras', *Ann. Math.* **83** (1966), 273–279.

[12] M. Takesaki, *Theory of Operator Algebras. I, Encyclopaedia of Mathematical Sciences, 124*, Operator Algebras and Non-commutative Geometry, 5 (Springer, Berlin, 2002), Reprint of the first (1979) edition.

[13] J. M. Zhan and Z. S. Tan, 'T-local derivations of von Neumann algebras', *Northeast. Math. J.* **20**(2) (2004), 145–152.

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