1

Affine Root Systems and Abstract Buildings

In this chapter we collect some of the background material used throughout the book. We encourage the reader to skim this chapter, rather than read it linearly, and refer back to it as needed. While some important objects, like the subgroups $G(k)^0$ and $G(k)^1$, are defined in this chapter and used throughout the book, the index of notation should help the reader locate the appropriate places in this chapter as needed.

1.1 Metric Spaces

Let (X, d) be a metric space. Recall the notion of the ball $B(x, r) = \{y \in X \mid d(x, y) < r\}$ of radius r with center x.

Definition 1.1.1

(1) A subset $A \subset X$ is called *bounded* if there exist $x \in X$ and $r > 0$ such that $A \subset B(x, r)$.

(2) For any two non-empty subsets $A, B \subset X$ their *joint diameter* is

$$\mathrm{diam}(A, B) = \sup\{d(a, b) \mid a \in A, b \in B\}.$$

(3) The *diameter* of $A \subset X$ is $\mathrm{diam}(A) = \mathrm{diam}(A, A)$.

Note that $\mathrm{diam}(A)$ is finite precisely when A is bounded.

A *curve* is a continuous map $c \colon [0, 1] \to X$. The *length* of the curve is defined as

$$\ell(c) = \sup \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})), \tag{1.1.1}$$

where the supremum is taken over the set of finite sequences $0 = t_0 < t_1 <$

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$\cdots < t_n = 1$. The curve is said to *join* the points $x, y \in X$ if $c(0) = x$ and $c(1) = y$.

Definition 1.1.2 A curve c is called

(1) *rectifiable*, if $\ell(c) < \infty$,
(2) *geodesic*, if c is rectifiable and for all $0 < t_1 < t_2 < 1$ the equalities hold

$$\ell(c|_{[t_1,t_2]}) = d(c(t_1), c(t_2)) = d(c(0), c(1))|t_1 - t_2|,$$

where $c|_{[t_1,t_2]} \colon [0,1] \to X$ is defined by $s \mapsto c(t_1 + s(t_2 - t_1))$.

Remark 1.1.3 If c is rectifiable, then so is $c|_{[t_1,t_2]}$. Hence the definition of geodesic makes sense.

It is clear from the definition that if $c_1, c_2 \colon [0,1] \to X$ both join $x, y \in X$ and c_1 is geodesic, then $d(x,y) = \ell(c_1) \leqslant \ell(c_2)$. The second equality in the definition of geodesic expresses the fact that c is parameterized by arc length.

Definition 1.1.4 The space (X, d) is called *geodesic*, if any two points $x, y \in X$ are connected by a geodesic. It is called *uniquely geodesic* if that geodesic is unique.

The *length metric* d_ℓ on X is defined as

$$d_\ell(x,y) = \inf \ell(c),$$

where the infimum is taken over all rectifiable curves c joining x and y, assuming that at least one such curve exists (as is the case with a geodesic space). Since $d_\ell(x,y) \geqslant d(x,y)$ one sees easily that d_ℓ is a metric.

Definition 1.1.5 The space (X, d) is called a *length space* if $d = d_\ell$.

Fact 1.1.6 *Every geodesic space is a length space.*

Example 1.1.7 Consider the circle $\mathbf{S}^1$ as a metric space, where the metric d is the restriction of the Euclidean metric on $\mathbf{R}^2$. For $x, y \in \mathbf{S}^1$ we have $d(x,y) = ||x - y||$, where $|| - ||$ is the Euclidean norm on $\mathbf{R}^2$, while $d_\ell(x,y) = \arccos(\langle x, y \rangle) > d(x,y)$. Thus $(\mathbf{S}^1, d)$ is not a length space. Tautologically, $(\mathbf{S}^1, d_\ell)$ is a length space, and in fact a geodesic space. However, it is not uniquely geodesic, because two antipodal points are joined by two distinct geodesics.

Definition 1.1.8 The space (X, d) is called *non-positively curved*, if for every $x, y \in X$ there exists $m \in X$ such that for all $z \in X$,

$$d(x,z)^2 + d(y,z)^2 \geqslant 2d(m,z)^2 + (1/2)d(x,y)^2.$$

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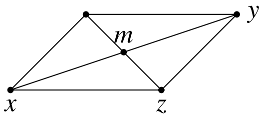
Remark 1.1.9 In a Euclidean space, if we let m be the midpoint between x, y, then the above inequality becomes an equality, and is called the parallelogram law: in a parallelogram, the sum of the squares of the lengths of the two diagonals is equal to the sum of the squares of the lengths of the four sides.



Remark 1.1.10 If (X, d) is uniquely geodesic, then for given x, y there is at most one m that satisfies the inequality of Definition 1.1.8, namely the midpoint $c(1/2)$ of the unique geodesic c joining x and y, see the following Lemma.

Furthermore, the inequality of Definition 1.1.8 has the geometric interpretation of pinching of triangles relative to Euclidean "reference" triangles, see [BH99, Figure II.1.1, Proposition II.1.7(1),(2) and Exercise II.1.9(1)(a,c)].

Lemma 1.1.11 *A non-positively curved geodesic space is uniquely geodesic. The midpoint of the unique geodesic connecting x and y is the unique point m satisfying the inequality of Definition 1.1.8.*

Proof Let (X, d) be a non-positively curved geodesic space, let $x, y \in X$ and let c be a geodesic connecting x and y. Let $m = c(1/2)$ be the midpoint. Then $d(x, m) = (1/2)d(x, y) = d(y, m)$. If m' is any point satisfying the inequality, we can take $z = m$ and see $(1/2)d(x, y)^2 \geqslant 2d(m', m)^2 + (1/2)d(x, y)^2$, thus $m' = m$.

If c' is another geodesic joining x and y we see that $c(1/2) = c'(1/2)$. Further bisections produce a dense set of $t \in [0, 1]$ such that $c(t) = c'(t)$, hence $c = c'$. □

Definition 1.1.12 Let (X, d) be a uniquely geodesic space.

(1) A subset $M \subset X$ is called *convex* if given $x, y \in M$ the (image of the) unique geodesic joining x and y lies in M.
(2) The *radius* of a convex set M is $r(M) := \inf\{\operatorname{diam}(x, M) | x \in M\}$.
(3) A *barycenter* of a convex set M is a point $x \in M$ such that $\operatorname{diam}(x, M) = r(M)$.

Lemma 1.1.13 *Let (X, d) be a non-positively curved metric space. Each closed ball $\overline{B(z, r)} = \{y \in X \mid d(x, y) \leqslant r\}$ is convex.*

Proof Let $x, y \in \overline{B(z, r)}$. Let c be the unique geodesic joining x and y and let $m = c(1/2)$ be its midpoint. Definition 1.1.8 implies $d(m, z) \leqslant r$, hence

$m \in \overline{B(z,r)}$. The two halves of c are the geodesics joining x with m and m with y. Proceeding inductively we obtain a dense set of $t \in [0,1]$ such that $c(t) \in \overline{B(z,r)}$. □

Definition 1.1.14 Let (X,d) be a non-positively curved metric space and $M \subset X$ a non-empty bounded subset. The *closed convex hull* of M is the intersection of all closed bounded convex subsets of X containing M.

We note that Lemma 1.1.13 implies the existence of a closed bounded convex subset of X containing M, therefore the closed convex hull of M exists and is by construction a closed, bounded, convex set containing M.

Theorem 1.1.15 *Let (X,d) be a complete non-positively curved geodesic space.*

(1) *A non-empty bounded closed convex subset $M \subset X$ has a unique barycenter. It is invariant under all isometries of X that map M to M.*
(2) *If $M \subset X$ is a non-empty bounded subset, the stabilizer of M in the group of isometries of X has a fixed point in X.*

Proof The second point follows by applying the first point to the closed convex hull of the bounded subset M in the sense of Definition 1.1.14. In the first point, the fact that the unique barycenter is invariant under all isometries preserving M is obvious, since the notion of barycenter is defined in terms of the metric. It remains to prove the existence and uniqueness of a barycenter.

Assume now that M is bounded, closed, and convex. To prove existence of a barycenter, let $f(x) = \operatorname{diam}(x, M)$ for $x \in M$ and let $r = \inf\{f(x) \mid x \in M\}$. Let ϵ be a positive real number and let $x, y \in M$ be such that $f(x) < r + \epsilon$ and $f(y) < r + \epsilon$. We claim that x and y are close to each other. More precisely, we claim $d(x,y)^2 < 16r\epsilon$. To see this, let m be the midpoint of the geodesic joining x and y. Since M is convex, $m \in M$. Thus $f(m) \geqslant r$. Therefore there exists $z \in M$ such that $d(m,z) > r - \epsilon$. Applying the non-positive curvature property to x, y, m, z and using that both $d(x,z)$ and $d(y,z)$ are less than $r + \epsilon$, whereas $d(m,z) > r - \epsilon$, we obtain the inequality $d(x,y)^2 < 16r\epsilon$, proving the claim.

Now let $\{x_i\}$ be a sequence in M such that $f(x_i) = r + \epsilon_i$, where $\{\epsilon_i\}$ is a decreasing sequence of positive real numbers that converges to 0. For $i < j$ we have $d(x_i, x_j)^2 < 16r\epsilon_i$. The sequence $\{x_i\}$ is thus Cauchy and converges to a point $c \in X$ by completeness of X. Since M is closed, $c \in M$. By construction $f(c) = r$, proving existence of c.

The uniqueness of c follows at once from the claim: if c_1 and c_2 are two barycenters, then $f(c_1) = r = f(c_2)$ implies $d(c_1, c_2) < 16r\epsilon$ for all $\epsilon > 0$, thus $c_1 = c_2$. □

Remark 1.1.16 The second point of Theorem 1.1.15 is called the *Bruhat–Tits Fixed Point Lemma*. The proof presented here is due to Serre. The original statement due to Bruhat–Tits is actually more general, as it does not assume that the metric space is geodesic. This will not be relevant to us, since the Bruhat–Tits building is a geodesic space. For the proof of the more general statement we refer the reader to [BT72, Lemma 3.2.3], whose proof is self-contained as long as one takes [BT72, Lemma 3.2.1] as a definition.

Lemma 1.1.17 *For a set A of isometries of a non-empty metric space (X, d) the following are equivalent.*

(1) *For every $x \in X$ the set $\{g \cdot x | g \in A\}$ is bounded.*
(2) *There exists $x \in X$ for which the set $\{g \cdot x \mid g \in A\}$ is bounded.*

Proof Let $x \in X$ such that the set $\{g \cdot x | g \in A\}$ is of bounded diameter. For any $y \in X$, we have

$$d(gy, y) \leqslant d(gy, gx) + d(gx, x) + d(x, y) = 2d(x, y) + d(gx, x)$$

and hence $d(gy, y)$ is also bounded. □

Definition 1.1.18 A set A of isometries of (X, d) that satisfies the equivalent conditions of Lemma 1.1.17 is said to have *bounded action* on X.

Corollary 1.1.19 *Let (X, d) be a non-empty complete non-positively curved metric space. A group A of isometries of (X, d) that has bounded action fixes a point of X.*

Proof Any orbit of A in X is non-empty and bounded. □

Proposition 1.1.20 *Let (X, d) be a complete non-positively curved geodesic space. Let $Y \subset X$ be a closed convex subset.*

(1) *Given $x \in X$ there exists among all $y \in Y$ a unique one whose distance to x is minimal.*
(2) *The function $\pi \colon X \to Y$, defined so that $\pi(x) \in Y$ is the unique point of Y closest to X, is continuous.*
(3) *The function $\pi \colon X \to Y$ of (2) is equivariant with respect to any isometry of X that preserves Y.*

Proof (1) The proof is very similar to that of Theorem 1.1.15(1) so we only give a sketch. Let $r = \inf\{d(x, y) | y \in Y\}$. Existence and uniqueness are reduced to the claim that for any $\epsilon > 0$ and $y_1, y_2 \in Y$ such that $d(x, y_1) < r + \epsilon$ and

$d(x, y_2) < r + \epsilon$ we have $d(y_1, y_2)^2 < 16r\epsilon$. This in turn is proved by taking m to be the midpoint of the geodesic joining y_1, y_2, observing $m \in Y$ by convexity of Y, concluding $d(x, m) \geqslant r$, hence $d(x, m) > r - \epsilon$, and applying the non-positive curvature property to y_1, y_2, m, x.

(2) Let (x_n) be a sequence of points of X that converges to $x \in X$. We need to show that the sequence $(\pi(x_n))$ converges to $\pi(x)$. We have

$$d(x, \pi(x)) \leqslant d(x, \pi(x_n)) \leqslant d(x, x_n) + d(x_n, \pi(x_n)) \leqslant d(x, x_n) + d(x_n, \pi(x)),$$

the first inequality by definition of $\pi(x)$, the second by the triangle inequality, and the third by definition of $\pi(x_n)$. The limit for $n \to \infty$ of the right-most term equals $d(x, \pi(x))$.

(3) Let $f : X \to X$ be an isometry that preserves Y. Then

$$d(f(x), \pi(f(x))) \leqslant d(f(x), f(\pi(x))) = d(x, \pi(x)), \text{ for all } x \in X.$$

Applying the same argument to f^{-1} gives the opposite inequality. We conclude that

$$d(f(x), \pi(f(x))) = d(f(x), f(\pi(x))).$$

The uniqueness of $\pi(f(x))$ implies $\pi(f(x)) = f(\pi(x))$. □

Proposition 1.1.20 can be strengthened when X has more structure, see Remark 4.2.19.

1.2 Affine Spaces

Let W be a vector space over a field k and let $V \subset W$ be a subspace. The quotient W/V consists of the orbits in W for the action of V by translation. In some sense all orbits look the same, with the following exception. The orbit through 0, i.e. the subspace V itself, is special, because it has a distinguished element, namely 0. No other orbit has a special point.

In this section we will recall the concept of an affine space, which is a formalization of this basic example. This will be important in the development of Bruhat–Tits theory, because apartments in Bruhat–Tits buildings are affine spaces. The simplest example of this is discussed towards the end of §3.1.

Let V be a vector space over a field k.

Definition 1.2.1 An *affine space* over V is a non-empty set A equipped with a simply transitive action of the additive group of V. We declare $\dim(A) := \dim(V)$. More generally, an affine space over k is a pair (V, A) consistsing of a k-vector space V and an affine space A over V.

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In particular, every $v \in V$ gives a map $T_v \colon A \to A$, usually called *translation by* v, and one often writes $T_v(x) = x+v$. For any $x, y \in A$ one writes $y-x \in V$ for the unique $v \in V$ such that $y = x+v$. Then one has the rule $(z-y)+(y-x) = z-x$ for $x, y, z \in A$.

Example 1.2.2 The vector space V is tautologically an affine space over V. More generally, if V is a subspace in a vector space W, then every fiber of the projection map $W \to W/V$ is an affine space over V. We will see in Proposition 1.2.10 that every affine space over V arises in this way in a canonical manner.

For any $x \in A$ the map $i_x \colon V \to A$, $v \mapsto x + v$ is a bijection that translates the action of V on A to the action of V on itself by translation. Thus, one may intuitively think of A as being V, but "after one has forgotten where the origin is." The inverse $i_x^{-1} \colon A \to V$ is given by $i_x^{-1}(y) = y - x$.

Definition 1.2.3 Let A be an affine space over V. An *affine subspace* $B \subset A$ is a non-empty subset of A having the property that $W = \{y - x \mid x, y \in B\}$ is a vector subspace of V. We may call W the *derivative* of B, and write $f = \nabla F$.

Clearly B is an affine space over W. Under the bijection i_a the set of affine subspaces of A is identified with the set of subsets of V of the form $v + W$ for an element $v \in V$ and a vector subspace $W \subset V$.

Definition 1.2.4 Let A and A' be affine spaces over the vector spaces V and V' respectively. A map $F \colon A \to A'$ is called *affine* if there exists a linear map $f \colon V \to V'$ such that $F(x + v) = F(x) + f(v)$ for all $x \in A$ and $v \in V$. We call f the *derivative* of F.

Note that f is uniquely determined by F, namely via $f(v) = F(x+v) - F(x)$ for some fixed x. If $G \colon A' \to A''$ is another affine map, with derivative g, then $G \circ F \colon A \to A''$ is also affine and its derivative is $g \circ f$. If $F_1, F_2 \colon A \to A'$ have the same derivative, then the vector $v = F_2(x) - F_1(x) \in V'$ is independent of $x \in A$ and thus $F_2(x) = F_1(x) + v$ for all $x \in A$. We may write $v = F_2 - F_1$.

Example 1.2.5 (1) The constant map $F \colon A \to A'$ given by $F(x) = x'$ for a fixed $x' \in A'$ is affine. Its derivative of F is the zero linear map.
(2) The translation $T_v \colon A \to A$ for $v \in V$ is affine. Its derivative is the identity map on V.

If we fix origins $x \in A$ and $x' \in A'$, then $F \mapsto i_{x'}^{-1} \circ F \circ i_x$ identifies the set of affine maps $A \to A'$ with the set of maps $V \to V'$ of the form $v \mapsto f(v) + v'$, where $f : V \to V'$ is linear and $v' \in V'$.

Remark 1.2.6 One checks immediately that an affine map is an isomorphism

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if and only if it is bijective, or equivalently that its derivative is an isomorphism of vector spaces.

It is clear that if A and A' are affine spaces over the same vector space V, then there exists an affine isomorphism $A \to A'$. The set of affine isomorphisms whose derivative is the identity is again an affine space over V. If $A = A'$, then that affine space has a distinguished point, namely id_A, and is hence naturally identified with V.

Given an affine space A over V let $\mathrm{Aff}(A)$ denote the set of affine isomorphisms $A \to A$. Composition turns $\mathrm{Aff}(A)$ into a group. Taking derivative produces a group homomorphism $\nabla\colon \mathrm{Aff}(A) \to \mathrm{Aut}(V)$. This homomorphism is surjective, for given $f \in \mathrm{Aut}(V)$ we can choose $x \in A$ and then $i_x \circ f \circ i_x^{-1} \in \mathrm{Aff}(A)$ has derivative f. We thus obtain the exact sequence

$$0 \to V \to \mathrm{Aff}(A) \to \mathrm{Aut}(V) \to 1$$

where the inclusion $V \to \mathrm{Aff}(A)$ maps v to the translation T_v. This exact sequence splits non-canonically; a choice of $x \in A$ gives the splitting $f \mapsto i_x \circ f \circ i_x^{-1}$.

Definition 1.2.7 Let Γ be a group. An *affine action* of Γ on A is a group homomorphism $\tau : \Gamma \to \mathrm{Aff}(A)$.

Clearly an affine action τ of Γ on A leads to a linear action $\nabla \circ \tau$ of Γ on V. Assume now that the action τ is faithful, so we can identify Γ with its image under τ, a subgroup of $\mathrm{Aff}(A)$. There are two extreme cases that are useful to keep in mind. If Γ is finite and k is infinite, then $V \cap \Gamma = \{0\}$ and ∇ restricts to an isomorphism $\Gamma \to \nabla\Gamma$. If Γ contains V, then we obtain the exact sequence

$$0 \to V \to \Gamma \to \nabla\Gamma \to 1$$

which is again split, a splitting being given by a choice of $a \in A$ as above.

Definition 1.2.8 An *affine functional* on A is an affine map $\psi\colon A \to k$. We will write $\dot{\psi} := \nabla\psi \in V^*$ and $H_\psi = A^{\psi=0} = \{x \in A \mid \psi(x) = 0\}$.

If ψ is not constant, the subset H_ψ is a hyperplane in A. We have $H_\psi = H_\eta$ if and only if $\eta = r\psi$ with $r \in k^\times$.

If ψ is an affine functional, then so is $-\psi$. It has the properties $\nabla(-\psi) = -\nabla\psi$ and $H_{-\psi} = H_\psi$, and these properties characterize $-\psi$ uniquely.

The set A^* of all affine functionals $A \to k$ is a vector space over k with respect to pointwise addition and scalar multiplication. The map $\nabla\colon A^* \to V^*$ is linear and surjective and its kernel is the subspace of constant affine linear functionals, which is naturally identified with k. In this way we obtain the exact sequence of k-vector spaces

$$0 \to k \to A^* \to V^* \to 0. \tag{1.2.1}$$

The map $\psi \mapsto H_\psi$ is a bijection between the set of lines in A^*, without the constant line, and the set of affine hyperplanes in A.

Example 1.2.9 In the special case $A = V$, an affine functional has the form $\psi(v) = \lambda(v) + c$, where $\lambda \in V^*$ and $c \in k$. Thus $A^* = V^* \oplus k$, that is, the above exact sequence is canonically split. For general A, the above exact sequence is non-canonically split; choosing $x \in A$ and using the identification $i_x : V \to A$ induces a splitting. If $x \in A$ is replaced by $x + v$ for $v \in V$, then the splitting changes by the automorphism of $V^* \oplus k$ sending $\lambda + c$ to $\lambda - \lambda(v) + c$.

We will now show that any affine space over a vector space V arises naturally as an orbit of the action of V on a larger vector space W. For this, consider the linear dual space A^{**} of A^*. The exact sequence (1.2.1) dualizes to

$$0 \to V \to A^{**} \to k \to 0. \tag{1.2.2}$$

In concrete terms, the embedding $V \to A^{**}$ is given by the pairing $\langle\psi, v\rangle = \langle\dot\psi, v\rangle$ for $\psi \in A^*$ and $v \in V$. The map $A^{**} \to k$ in this sequence is the linear functional on A^{**} corresponding to the element of A^* that is the affine functional on A with constant value 1. Let us write $\mathbf{1}_A$ for it. We have the natural embedding $A \to A^{**}$ via the natural pairing $\langle\psi, a\rangle = \psi(a)$. This embedding identifies A with the fiber over $1 \in k$ of the linear functional $\mathbf{1}_A : A^{**} \to k$.

Proposition 1.2.10 *Consider the category* $\underline{\mathrm{Aff}}$ *whose objects are all affine spaces over k and whose morphisms are affine transformations between affine spaces. Consider the category* $\underline{\mathrm{fVect}}$ *whose objects are pairs* (W, λ) *consisting of a k-vector space W and a non-zero linear functional* $0 \neq \lambda \in W^*$, *and whose morphisms are linear maps of vector spaces that respect the given functionals. Then the functors*

$$F : \underline{\mathrm{Aff}} \to \underline{\mathrm{fVect}}, \quad (V, A) \mapsto (A^{**}, \mathbf{1}_A), f \mapsto f^{**}$$

and

$$G : \underline{\mathrm{fVect}} \to \underline{\mathrm{Aff}}, \quad (W, \lambda) \mapsto (\lambda^{-1}(0), \lambda^{-1}(1)), F \mapsto F|_{\lambda^{-1}(1)}$$

are mutually inverse equivalences of categories.

Proof We need to exhibit natural transformations $\eta : \mathrm{id}_{\underline{\mathrm{Aff}}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\underline{\mathrm{fVect}}}$. Given an affine space (V, A) set $W = A^{**}$ and $\lambda = \mathbf{1}_A$. The discussion before the statement of this proposition provides isomorphisms $V \to \lambda^{-1}(0)$ and $A \to \lambda^{-1}(1)$. These isomorphisms comprise $\eta_{(V,A)}$. Conversely,

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given a vector space W and a non-zero linear functional λ, set $V = \lambda^{-1}(0)$ and $A = \lambda^{-1}(1)$. Restriction from W to A provides a linear map $W^* \to A^*$. We will prove that it is an isomorphism. Admitting this, we obtain dually an isomorphism $A^{**} \to W$ which tautologically identifies $\mathbf{1}_A$ with λ, and hence provides the desired $\epsilon_{(W,\lambda)}$. To prove that $W^* \to A^*$ is an isomorphism we note that the kernel of any element of W^* is a hyperplane in W containing zero and hence cannot contain the affine hyperplane $\lambda^{-1}(1)$. Therefore $W^* \to A^*$ is injective. When $\dim(W) < \infty$ this is enough. In general let $\mu \in A^*$. Choose an element $\mu_1 \in V^*$ that extends $\dot{\mu} \in V^*$. The restriction $\mu_1|_A$ is an affine functional with the same derivative as μ. Therefore $\mu - \mu_1|_A$ is a constant $n \in k$, and we see $\mu = (\mu_1 + n\lambda)|_A$. □

Definition 1.2.11 For $x \in A$ define $A^*_x = \{\psi \in A^* \mid \psi(x) = 0\}$.

It is clear that restricting ∇ to A^*_x provides an isomorphism $\nabla : A^*_x \to V^*$.

Definition 1.2.12 Let A be an affine space over V and let $W \subset V$ be a subspace. The *quotient* A/W is the set of orbits in A for the action of W.

It is clear that A/W is an affine space over V/W. Pulling back affine functionals under the quotient map $A \to A/W$ gives an injection $(A/W)^* \to A^*$ which identifies $(A/W)^*$ with the subspace $\{\psi \in A^* \mid \dot{\psi} \in W^\perp\}$, where $W^\perp \subset V^*$ is the annihilator of W.

Definition 1.2.13 For a non-empty subset $\Omega \subset A$ let $\langle\Omega\rangle \subset A$ be the smallest affine space containing Ω, and let A^*_Ω be the subspace of A^* consisting of those affine functionals that vanish identically on Ω.

Note that when $\Omega = \{x\}$ we obtain $A^*_\Omega = A^*_x$.

Fact 1.2.14 *Let $\Omega \subset A$ be non-empty.*

(1) *An affine functional vanishes on Ω if and only if it vanishes on $\langle\Omega\rangle$.*
(2) *If $W \subset V$ is the derivative space of $\langle\Omega\rangle$, then the map $\nabla : A^*_\Omega \to W^\perp$ is an isomorphism.*
(3) *The bijection $H_\psi \leftrightarrow k^\times \cdot \psi$ restricts to a bijection between the set of affine hyperplanes in A containing Ω and the set of lines in A^*_Ω, hence the set of lines in $W^\perp$.*

Let A and A' be affine spaces over V and V' and let $f : A \to A'$ be an affine map. Its dual $f^* : (A')^* \to A^*$ is defined by $f^*(\psi') = \psi \circ f$. Then f^* is a linear map.

1.2.15 Given affine spaces A_1, A_2 over vector spaces V_1, V_2 we can form the affine space $A = A_1 \times A_2$ over the vector space $V = V_1 \times V_2$. The projection

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$A \to A_i$ dualizes to an injection $A_i^* \to A^*$ that identifies A_i^* with those elements of A^* whose derivative lies in $V_i^* \subset V^*$. The sum of these two injections fits into the exact sequence

$$0 \to k \to A_1^* \oplus A_2^* \to A^* \to 0, \tag{1.2.3}$$

where k is embedded anti-diagonally into $A_1^* \oplus A_2^*$. More generally, if $\{A_i\}_{i=1}^n$ is a finite collection of affine spaces over vector spaces V_i and we consider the affine space $A = \prod A_i$ over the vector space $\prod_i V_i$, we have the surjective map $\bigoplus A_i^* \to A^*$ whose kernel is the hyperplane in $k^n \subset \bigoplus A_i^*$ consisting of those tuples whose coordinates sum to 0.

1.2.16 Let A be an affine space over the vector space V and assume given a direct product decomposition $V = V_1 \times V_2$. This decomposition induces a direct product decomposition $A = A_1 \times A_2$ as follows. Let $A_1 = A/V_2$ and $A_2 = A/V_1$. Then A_i is an affine space over V_i. The product of the two projections $A \to A_1 \times A_2$ is equivariant for the translation action of $V = V_1 \times V_2$ and injective, hence bijective, and therefore an isomorphism of affine spaces over V.

Assume from now on that $k = \mathbf{R}$.

Definition 1.2.17

(1) For an affine functional ψ we will write $A^{\psi>0} = \{x \in A \mid \psi(x) > 0\}$. Analogously we define $A^{\psi\geqslant 0}$, $A^{\psi<0}$, $A^{\psi\leqslant 0}$.
(2) For two affine functionals $\psi_1, \psi_2 : A \to \mathbf{R}$ we write $\psi_1 \leqslant \psi_2$ if $\psi_1(x) \leqslant \psi_2(x)$ for all $x \in A$.

Fact 1.2.18 *$\psi_1 \leqslant \psi_2$ if and only if $\dot\psi_1 = \dot\psi_2$ and $\psi_2 - \psi_1 \geqslant 0$.*

Assume further that V is finite-dimensional and equipped with a scalar product (a positive definite symmetric bilinear form) $\langle -, - \rangle$. We may then identify V with V^* and consider $\dot\psi$ as an element of V. Define a symmetric bilinear form on A^*, again denoted by $\langle -, - \rangle$, via $\langle \psi_1, \psi_2 \rangle = \langle \dot\psi_1, \dot\psi_2 \rangle$. This symmetric bilinear form is positive semi-definite and ψ is isotropic ($\langle \psi, \psi \rangle = 0$) if and only if it is constant. Define $v^\vee = 2v/\langle v, v\rangle$ and $\psi^\vee = 2\psi/\langle \psi, \psi \rangle$ for any non-zero $v \in V$ and any non-constant $\psi \in A^*$. We have $(cv)^\vee = c^{-1}v^\vee$ and $(c\psi)^\vee = c^{-1}\psi^\vee$ for $c \in \mathbf{R}^\times$. We have $\nabla(\psi^\vee) = (\nabla\psi)^\vee$.

For non-constant ψ, the orthogonal reflection along the affine hyperplane H_ψ is given by

$$r_\psi(x) = x - \psi(x) \cdot \dot\psi^\vee = x - \psi^\vee(x) \cdot \dot\psi. \tag{1.2.4}$$

Note $r_{c\psi} = r_\psi$ for any $c \in \mathbf{R}^\times$. We denote the dual of r_ψ again by r_ψ. We have $r_\psi \psi = -\psi$. If $v = \dot\psi$, then we also have the orthogonal reflection in V along the derivative hyperplane $v^\perp$ of H_ψ (cf. Definition 1.2.3). It has the analogous formula

$$r_v(w) = w - \langle w, v\rangle v^\vee$$

and we have $\nabla r_\psi = r_{\nabla\psi}$.

Definition 1.2.19 Let A_1 and A_2 be affine spaces An affine map $f\colon A_1 \to A_2$ is called an *isometry* if its derivative $\nabla f\colon V_1 \to V_2$ is an isometry, that is, a linear isomorphism preserving given scalar products on V_1 and V_2.

The affine space A is a metric space with metric $d(x, y) = ||x - y||$, where $||v|| = \langle v, v\rangle^{1/2}$. The translation action of V on A is via isometries for this metric. The topology on A induced by this metric coincides with the topology transported from the topology on V coming from V being a finite-dimensional **R**-vector space.

1.3 Affine Root Systems

We review here the definition and basic properties of affine root systems following [Mac72, §§2-6]. Note however that we are using Greek letters for affine roots, and roman letters for their derivatives, which is the opposite of the convention of [Mac72].

Let V be a finite-dimensional **R**-vector space. Let A be an affine space over V. We refer to §1.2 for relevant notation. We denote by V^* the dual of V, by A^* the dual of A, and by A^{**} the vector space dual of A^*. Recall the exact sequence (1.2.2)

$$0 \to V \to A^{**} \to \mathbf{R} \to 0.$$

Definition 1.3.1 An *affine root system* is a subset $\Psi \subset A^*$ subject to the following axioms.

AR 1 Ψ spans A^*, consists of non-constant functionals, and its image under $\nabla\colon \Psi \to V^*$, to be denoted by Φ, is finite.

AR 2 For every $\psi \in \Psi$ there exists $\dot\psi^\vee \in V$ such that $\dot\psi(\dot\psi^\vee) = 2$ and the reflection $r_{\psi,\dot\psi^\vee}$ on A^*, defined by

$$r_{\psi,\dot\psi^\vee}(x) = x - \dot x(\dot\psi^\vee)\psi \quad \text{for } x \in A^*,$$

preserves Ψ.

AR 3 For every $\psi, \eta \in \Psi$ we have $\dot\psi(\dot\eta^\vee) \in \mathbf{Z}$.

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AR 4 For each $a \in \Phi$, the set $\{\psi \in \Psi \,|\, \dot{\psi} = a\}$ does not have an accummulation point.

The group $W(\Psi)$ generated by $\{r_{\psi,\dot{\psi}^\vee} \mid \psi \in \Psi\}$ is called the *affine Weyl group* of Ψ. The number $\dim V$ is called the *rank* of Ψ (which is positive precisely when $\Psi \neq \varnothing$).

Remark 1.3.2 (1) We emphasize that Ψ is a subset of the vector space A^* of affine functionals on A, and not a subset of the affine space A.

(2) Since the derivative of $r_{\psi,\dot{\psi}^\vee}$ is the reflection $r_{\dot{\psi},\dot{\psi}^\vee}$ on V^*, it follows as in the case of finite root systems from [Bou02, Chapter VI, §1, no. 1, Lemma 1] that the element $\dot{\psi}^\vee \in V$ of AR 2 is unique, and is uniquely determined by $\dot{\psi}$. Hence AR 3 makes sense. We will write r_ψ in place of $r_{\psi,\dot{\psi}^\vee}$ in what follows.

(3) For $\psi \in \Psi$, we also have the dual reflection on A, to be denoted again by r_ψ, defined by $x \mapsto x - \psi(x)\dot{\psi}^\vee$ for $x \in A$.

(4) Proposition 1.3.11 below shows that both Ψ and $W(\Psi)$ are necessarily infinite, provided $\Psi \neq \varnothing$.

Throughout this section, unless specifically stated otherwise, A will denote an affine space over a $\mathbf{R}$-vector space V and $\Psi \subset A^*$ will denote an affine root system.

The following proposition shows that affine root systems are closely related to finite root systems. As before, we will often denote the derivative $\nabla\psi$ $(\in V^*)$ of a real valued affine function ψ on A by $\dot{\psi}$.

Proposition 1.3.3

(1) $\Phi = \{\dot{\psi} \mid \psi \in \Psi\} \subset V^*$ *is a finite root system.*

(2) $\dot{\psi}^\vee$ *is the coroot of* $\dot{\psi}$. *Hence,* $\{\dot{\psi}^\vee \mid \psi \in \Psi\}$ *span* V.

(3) *Given* $\psi \in \Psi$, *and* $c, r \in \mathbf{R}$, *if* $\eta := r\psi + c$ *is in* Ψ, *then* $r \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$ *and* $\dot{\psi}^\vee = r\dot{\eta}^\vee$.

(4) *The mapping* $\nabla\colon W(\Psi) \to W(\Phi)$ *is surjective and its kernel consists of the subgroup of translations contained in* $W(\Psi)$.

We call Φ the *derivative root system* of Ψ.

Proof (1) The axioms for Φ follow at once from those for Ψ.

(2) As $\dot{\psi}^\vee$ lies in V, the reflection $r_{\psi,\dot{\psi}^\vee}$ is the identity on $\mathbf{R} \subset A^*$ and induces on V^* the reflection $r_{\dot{\psi},\dot{\psi}^\vee}$.

(3) Since $\dot{\eta} = r\dot{\psi}$, it follows that the finite root system Φ contains both $\dot{\psi}$ and $r\dot{\psi}$. This implies the third assertion.

(4) Immediate. □

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We now introduce the concept of a reduced affine root system.

Definition 1.3.4 The affine root system Ψ is called *reduced* if $\psi \in \Psi$ implies $2\psi \notin \Psi$.

Remark 1.3.5 To Ψ we associate the subsets

$$\Psi^{\mathrm{nd}} = \{\psi \in \Psi \mid \psi/2 \notin \Psi\},$$
$$\Psi^{\mathrm{nm}} = \{\psi \in \Psi \mid 2\psi \notin \Psi\}.$$

These subsets are reduced affine subroot systems of Ψ. The three affine root systems Ψ, Ψ^{nd}, Ψ^{nm} share the same set of affine hyperplanes and the same Weyl group. We have $\Psi^{\mathrm{nd}} = \Psi = \Psi^{\mathrm{nm}}$ if and only if Ψ is reduced. We have the inclusions

$$\Phi^{\mathrm{nd}} \subset \nabla(\Psi^{\mathrm{nd}}) \subset \nabla\Psi,$$

where for a finite root system Φ we define analogously $\Phi^{\mathrm{nd}} = \{a \in \Phi | a/2 \notin \Phi\}$.

Recall that V carries a canonical topology, namely the topology transported from the identification $V \to \mathbf{R}^n$ obtained from an arbitrary choice of basis of V. In the same way, A carries a canonical topology, namely the topology transported from the identification $A \to V$ obtained by choosing an arbitrary point $a \in A$.

Lemma 1.3.6 *The group $W(\Psi)$ acts properly on A. That is, if $K_1, K_2 \subset A$ are compact subsets, the set $\{w \in W(\Psi) \mid wK_1 \cap K_2 \neq \varnothing\}$ is finite.*

Proof According to Proposition 1.3.3 and the finiteness of the Weyl group of a finite root system, the subgroup $T \subset W(\Psi)$ of translations is of finite index. It is therefore enough to prove that T acts properly on A. Identifying each element of T with the vector in V by which it translates, we obtain an identification of T with a subgroup of the additive group of V. The action of T on A is proper if and only if T, as a subgroup of V, is discrete. Assume by way of contradiction that there exists a sequence (x_n) in T that converges to 0 and $x_n \neq 0$ for all n. Since $\Phi = \nabla\Psi$ generates V^* there exists $a \in \Phi$ such that, after possibly passing to a subsequence of (x_n), we have $a(x_n) \neq 0$ for all n. Let $\eta \in \Psi$ be such that $\dot{\eta} = a$. If $t_n \in T$ is the translation by x_n, then $\eta_n := t_n\eta = \eta - a(x_n)$ is a sequence of elements of $\{\psi \in \Psi \mid \dot{\psi} = a\}$ that converges to η, but $\eta_n \neq \eta$ for all n. This contradicts AR 4. $\square$

Definition 1.3.7 For any affine root $\psi \in \Psi$, the hyperplane $H_\psi = \{a \in A \mid \psi(a) = 0\}$ in A is called the *affine root hyperplane associated to* ψ, or just an affine root hyperplane.

The derivative $\nabla H_\psi \subset V$ is the root hyperplane $H_{\dot{\psi}} = \{v \in V \mid \dot{\psi}(v) = 0\}$.

Remark 1.3.8 Let $\langle -, - \rangle$ be a scalar product on V^* invariant under the finite group $\nabla W(\Psi) = W(\Phi)$. Identifiy V with V^* via this scalar product and transport the scalar product to V via this identification. Then $r_{\dot\psi}$ becomes the orthogonal reflection in V with respect to the hyperplane $H_{\dot\psi} = \{v \in V \mid \dot\psi(v) = 0\}$. Since the derivative of r_ψ is $r_{\dot\psi}$, r_ψ becomes the orthogonal reflection in A with respect to the affine hyperplane H_ψ.

Remark 1.3.9 Definition 1.3.1 is slightly different from the definition of an affine root system given in [Mac72]. The latter definition requires that V be equipped with a scalar product. It does not require that Φ be finite. In place of AR 4, it requires that the action of $W(\Psi)$ on A be proper.

If Ψ is an affine root system in the sense of Definition 1.3.1 and we equip V with a scalar product invariant under $W(\Phi)$ as in Remark 1.3.8, then Ψ becomes an affine root system in the sense of Macdonald, the properness of the action of $W(\Psi)$ being guaranteed by Lemma 1.3.6.

Conversely, given an affine root system Ψ in the sense of Macdonald it follows from [Mac72, Proposition 6.1] that Φ is finite, and AR 4 follows from the assumed properness of the action of $W(\Psi)$ by the same argument as in the proof of Lemma 1.3.6.

For the study of affine root systems we will use the material in [Bou02, Chapter V]. This material requires that the vector space V be equipped with a scalar product so that the reflection r_ψ becomes the orthogonal reflections with respect to the affine hyperplane H_ψ. We can choose such a scalar product as in Remark 1.3.8. This material further requires that the set of hyperplanes $\{H_\psi \mid \psi \in \Psi\}$ satisfies conditions D1 and D2 of [Bou02, Chapter V, §3]. We recall that these conditions require that the set of hyperplanes be invariant under $W(\Psi)$ and that $W(\Psi)$ acts properly on A.

Lemma 1.3.10 *The set of hyperplanes $\{H_\psi \mid \psi \in \Psi\}$ satisfies conditions D1 and D2. In particular, it is locally finite.*

Proof Condition D2 is Lemma 1.3.6. Condition D1 follows from $wH_\psi = H_{w\psi}$ for all $w \in W(\Psi)$ and Axiom AR 2. The local finiteness is [Bou02, Chapter V, §3, Lemma 1]. □

Proposition 1.3.11 *Assume $\Psi \neq \emptyset$. The Weyl group $W(\Psi)$ is infinite and does not fix any non-zero vector in V. The set Ψ is infinite.*

Proof Any $v \in V$ fixed by $W(\Psi)$ must be orthogonal to $\dot\psi \in V^*$ for all $\psi \in \Psi$. By assumption Ψ generates A^*, so $v = 0$.

Assume by way of contradiction that $W(\Psi)$ is finite. We claim that then its action on A must have a fixed point. To see this, choose a scalar product on V

invariant under $W(\Psi)$ (say, by averaging an arbitrary scalar product on V) and endow A with the resulting Euclidean metric, which is then also invariant under $W(\Psi)$. The orbit under $W(\Psi)$ of an arbitrarily chosen point of A is finite, so the center of mass of its convex hull, with respect to the metric on A, is fixed by $W(\Psi)$. Let $x \in A$ be a point fixed by $W(\Psi)$. Then $\psi(x) = 0$ for all $\psi \in \Psi$. This implies $\Psi \subset A_x^*$ (Definition 1.2.11). But A_x^* is a proper subspace of A^* and this contradicts Axiom AR 1.

We conclude that the map $\nabla \colon W(\Psi) \to W(\Phi)$ must have a non-trivial kernel. Thus $W(\Psi)$ contains a non-trivial translation. Axiom AR 2 implies that Ψ is infinite. □

The following proposition gives a reformulation of the definition of an affine root system that is sometimes more convenient.

Proposition 1.3.12 *A subset $\Psi \subset A^*$ is an affine root system if and only if it satisfies the following conditions.*

(1) $\Phi(= \nabla\Psi) \subset V^*$ *is a finite root system.*
(2) *For each $\psi \in \Psi$, the affine endomorphism r_ψ of A that fixes the hyperplane $H_\psi \subset A$ and whose derivative is the reflection $r_{\dot\psi}$, preserves the set Ψ.*
(3) *For each $a \in \Phi$ the set $\{\psi \in \Psi \mid \dot\psi = a\}$ contains at least two elements and does not have an accumulation point in A^*.*

Proof Assume that Ψ is an affine root system. Then Proposition 1.3.3 shows that Φ is a finite root system. The endomorphism r_ψ is equal to the reflection $r_{\psi,\dot\psi^\vee}$ and preserves Ψ by AR 2. If for each $a \in \Phi$ the set $\{\psi \in \Psi | \dot\psi = a\}$ had just one element, then Ψ would be finite, contradicting Proposition 1.3.11.

Assume conversely that Ψ is a set satisfying the assumptions of the proposition. We will show that it satisfies the axioms of Definition 1.3.1. As Φ is a root system, it spans V^* and does not contain $0 \in V^*$. Therefore Ψ consists of non-constant elements, and since we can find two distinct $\psi_1, \psi_2 \in \Psi$ with the same derivative, the set Ψ spans A^*, proving AR 1. Given $\psi \in \Psi$ with derivative $a \in V^*$, let $a^\vee \in V$ be the coroot for a. Then $\dot\psi(a^\vee) = a(a^\vee) = 2$. The endomorphism $r_{\psi,a^\vee}$ has derivative r_a and fixes the hyperplane H_ψ. By assumption it preserves Ψ, proving AR 2. It further identifies, for any given $\psi \in \Psi$, the element $\dot\psi^\vee \in V$ as the coroot associated to the root $\dot\psi$ of the finite root system Φ. For $\psi, \eta \in \Psi$ this means $\dot\psi(\dot\eta^\vee) = \dot\psi(\dot\eta^\vee)$ which is an integer since Φ is a finite root system. This proves AR 3, and AR 4 is just assumption (3). □

Definition 1.3.13 A non-empty affine root system Ψ is called *reducible* if there exist non-empty subsets $\Psi_1, \Psi_2 \subset \Psi$ that are orthogonal in the sense that $\dot\psi(\dot\eta^\vee) = 0 = \dot\eta(\dot\psi^\vee)$ for every every $\psi \in \Psi_1$ and $\eta \in \Psi_2$ (in particular, Ψ_1 and Ψ_2 are disjoint) and $\Psi = \Psi_1 \cup \Psi_2$. Otherwise Ψ is said to be *irreducible*.

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Lemma 1.3.14 *The following are equivalent.*

(1) *The affine root system Ψ is irreducible.*
(2) *The finite root system Φ is irreducible.*
(3) *The representation of $W(\Psi)$ on V is irreducible.*

Proof The equivalence of (1) and (2) is obvious from the definitions. The equivalence of (2) and (3) is [Bou02, Chapter 6, §1, no. 2, Corollary to Proposition 5]. □

Construction 1.3.15 Consider for $i = 1, 2$ affine spaces A_i over vector spaces V_i and affine root systems $\Psi_i \subset A_i^*$. The map (1.2.3) transports $\Psi_1 \cup \Psi_2 \subset A_1^* \oplus A_2^*$ bijectively to a subset of A^*, which we denote by $\Psi_1 \oplus \Psi_2$. It is an affine root system, called the *direct sum* of Ψ_1 and Ψ_2.

We will see in Proposition 1.3.21 below that every affine root system decomposes naturally as the direct sum of irreducible root systems. Before we can do this, we need to establish some basic structure.

Remark 1.3.16 The hyperplanes H_ψ endow the affine space A with important additional structure. The connected components of $A - \bigcup_{\psi \in \Psi} H_\psi$ are open in A by the local finiteness of $\{H_\psi \mid \psi \in \Psi\}$. They are called *chambers*. The connected components of

$$\bigcup_{\psi \in \Psi} H_\psi - \bigcup_{\substack{\psi_1, \psi_2 \in \Psi \\ H_{\psi_1} \neq H_{\psi_2}}} (H_{\psi_1} \cap H_{\psi_2}),$$

are open in $\bigcup_{\psi \in \Psi} H_\psi$ for the same reason. They are called *facets of codimension* 1, that is, of dimension $\dim(A) - 1$. Continuing in this manner, one expresses A as the disjoint union of facets of various dimensions. The facets of smallest dimension are called *vertices*. A *face* of a facet $\mathcal{F}$ is a facet $\mathcal{F}'$ contained in the closure $\overline{\mathcal{F}}$ of $\mathcal{F}$. A hyperplane H_ψ is called a *wall* of a chamber $\mathcal{C}$, if $H_\psi \cap \overline{\mathcal{C}}$ is a face of $\mathcal{C}$ of codimension 1.

If Ψ is irreducible, then this decomposition is a *simplicial decomposition*, that is, each facet is a simplex. That each chamber is a simplex is stated in [Bou02, Chapter V, §3, no. 9, Proposition 8], using Proposition 1.3.11 and Lemma 1.3.14. The remaining facets, being faces of chambers, are also simplices. This makes A into (the geometric realization of) a simplicial complex. The simplicial complexes for the four 2-dimensional irreducible reduced affine root systems (cf. Theorem 1.3.63) are displayed in Figure. 1.3.1. In Figure 1.3.2, the hyperplanes for affine roots with non-divisible derivative are displayed solid, while the hyperplanes for affine roots with divisible derivative (which only appear when the derivative root system is non-reduced) are displayed dotted.

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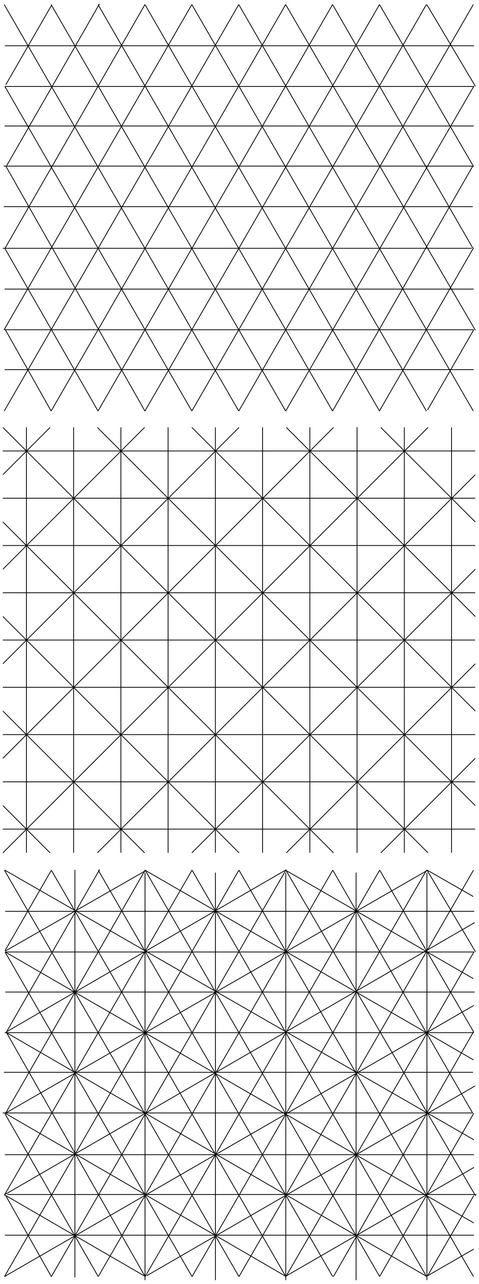

Figure 1.3.1 The hyperplane arrangements of the irreducible reduced affine root systems A_2 (top), C_2 (middle), and G_2 (bottom).

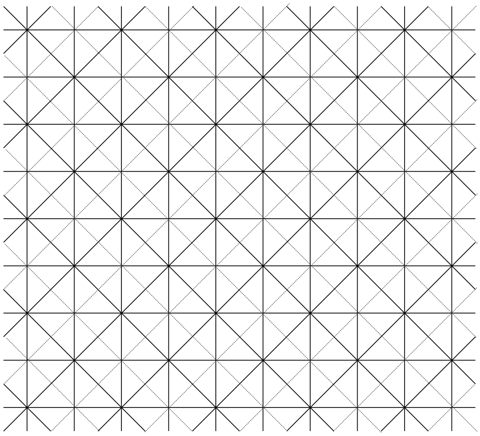

Figure 1.3.2 The hyperplane arrangement of the irreducible reduced affine root system BC_2.

When Ψ is no longer irreducible, Proposition 1.3.21 below expresses Ψ canonically as the direct sum $\Psi_1 \oplus \cdots \oplus \Psi_n$ of irreducible affine root systems, and A canonically as the product $A_1 \times \cdots \times A_n$ of affine spaces. As shown in [Bou02, Chapter V, §3, no. 8, Proposition 6], each affine root hyperplane is of the form $A_1 \times \cdots \times A_{i-1} \times H_i \times A_{i+1} \times \cdots \times A_n$ for an affine root hyperplane $H_i \subset A_i$ associated to some member of Ψ_i, and each chamber is a product of chambers $C = C_1 \times \cdots \times C_n$. More generally, each facet is a product of facets. Consequently, each facet is a product of simplices, that is, a *polysimplex*. This makes A into (the geometric realization of) a polysimplicial complex, a notion discussed in §1.5.

Lemma 1.3.17 *Let $\mathcal{C}$ be a chamber. Its closure $\overline{\mathcal{C}}$ is a fundamental domain for the action of $W(\Psi)$ on A. For $w \in W(\Psi)$ and a facet $\mathcal{F}$ the following are equivalent.*

(1) *w fixes a point of $\mathcal{F}$.*
(2) *w fixes all points of $\mathcal{F}$.*
(3) *w fixes all points of the closure of $\mathcal{F}$.*
(4) *w leaves $\mathcal{F}$ invariant.*
(5) *w is a product of reflections along hyperplanes containing $\mathcal{F}$.*

In particular, $W(\Psi)$ acts simply transitively on the set of chambers.

Proof This is [Bou02, Chapter V, §3, no. 3, Proposition 1, Theorem 2]. □

Definition 1.3.18 Let $\mathcal{C}$ be a chamber. Then $\psi \in \Psi$ is called *positive* (respectively *negative*) if $\psi(x) > 0$ (respectively $\psi(x) < 0$) for all $x \in \mathcal{C}$. We denote by $\Psi(\mathcal{C})^+$ and $\Psi(\mathcal{C})^-$ the set of positive and negative affine roots, respectively. The subsets $\Psi(\mathcal{C})^+ \cap \Psi^{\mathrm{nd}}$ and $\Psi(\mathcal{C})^- \cap \Psi^{\mathrm{nd}}$ will be denoted by $\Psi(\mathcal{C})^{\mathrm{nd},+}$ and $\Psi(\mathcal{C})^{\mathrm{nd},-}$ respectively.

Since no H_ψ meets $\mathcal{C}$, every $\psi \in \Psi$ is either positive or negative, and $-\psi$ is negative if and only if ψ is positive. Thus we have the disjoint union $\Psi = \Psi(\mathcal{C})^+ \cup \Psi(\mathcal{C})^-$.

Definition 1.3.19 Let $\mathcal{C}$ be a chamber. The set $\Psi(\mathcal{C})^0$ consisting of those indivisible $\psi \in \Psi(\mathcal{C})^+$ for which H_ψ is a wall of $\mathcal{C}$ is called a *basis* of Ψ, and its elements are called *simple affine roots*.

Note that $\mathcal{C}$ is uniquely determined by $\Psi(\mathcal{C})^0$, namely as the intersection of the half-spaces $A^{\psi>0}$ for $\psi \in \Psi(\mathcal{C})^0$.

Proposition 1.3.20 *Let $\mathcal{C} \subset A$ be a chamber and let $\Psi(\mathcal{C})^0 \subset \Psi$ be the corresponding set of simple affine roots. Let $S \subset W(\Psi)$ be the set of reflections along the elements of $\Psi(\mathcal{C})^0$. Then $(W(\Psi), S)$ is a Coxeter system.*

Proof This is [Bou02, Chapter V, §3, no.2, Theorem 1(i)]. □

Proposition 1.3.21 *Every affine root system is in a natural way the direct sum of irreducible affine root systems.*

Proof Let $W = W(\Psi)$. Choose a chamber $\mathcal{C}$ and let $S \subset W$ be the set of reflection along the associated simple affine roots. Then (W, S) is a Coxeter system by Proposition 1.3.20. Write $S = S_1 \cup \cdots \cup S_n$ such that the S_i are pairwise orthogonal as in Definition 1.3.13 and n is maximal with this property. Let $W_i \subset W$ be the subgroup generated by S_i, let $V_i^\perp$ be the subspace of V^* fixed by ∇W_i, and let V_i be the annihilator of $V_i^\perp$ in V. According to [Bou02, Chapter V, §3, no. 7, Proposition 5] and the discussion preceding it, the subgroups W_i commute with each other, $W = W_1 \times \cdots \times W_n$, the set of subgroups $\{W_1, \ldots, W_n\}$ is independent of the choice of chamber $\mathcal{C}$, and $V = V_1 \oplus \cdots \oplus V_n$. Each V_i is stable under ∇W.

As discussed in 1.2.16, this leads to the decomposition $A = A_1 \times \cdots \times A_n$, where $A_i = A/V_i'$ and $V_i' = V_1 \oplus \cdots \oplus V_{i-1} \oplus \{0\} \oplus V_{i+1} \oplus \cdots \oplus V_n$. Let $\Psi_i \subset \Psi$ consist of those $\psi \in \Psi$ such that $\dot\psi \in V_i^*$. Then $\Psi_i \subset A_i^*$ (cf. 1.2.15) and according to [Bou02, Chapter VI, §1, no. 2, Proposition 5] $\nabla\Psi_i$ is an irreducible root system and $\nabla\Psi = \nabla\Psi_1 \oplus \cdots \oplus \nabla\Psi_n$. In particular, $\Psi = \Psi_1 \cup \cdots \cup \Psi_n$. Proposition 1.3.12 applied to both Ψ and Ψ_i implies that Ψ_i is an affine root system in A_i^*, and Lemma 1.3.14 shows that it is irreducible. Its Weyl group is

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W_i by construction, and the isomorphism $A \to A_1 \times \cdots \times A_n$ identifies Ψ with the direct sum $\Psi_1 \oplus \cdots \oplus \Psi_n$. □

Proposition 1.3.22 *Let $\mathcal{C}$ be a chamber and $\Delta = \Psi(\mathcal{C})^0$.*

1. *The group $W(\Psi)$ is generated by $S = \{r_\psi | \psi \in \Delta\}$.*
2. *$W(\Psi)$ acts simply transitively on the set of bases of Ψ.*
3. *$\Psi^{\mathrm{nd}} = W(\Psi) \cdot \Delta$.*
4. *Every $\psi \in \Psi(\mathcal{C})^+$ is a non-negative integral linear combination of elements of Δ.*

Assume that Ψ is irreducible.

5. *Δ is a basis of A^*. In particular the combination in (4) is unique.*
6. *The vertices $\{x_0, \ldots, x_\ell\}$ of $\mathcal{C}$ are in bijection with Δ specified by $\psi_i(x_j) = 0$ if $i \neq j$ and $\psi_i(x_i) > 0$.*

Proof (1) is [Bou02, Chapter V, §3, no. 1, Lemma 2].

(2) follows from Lemma 1.3.17 and the bijection between bases and chambers.

(3) Let $\psi \in \Psi^{\mathrm{nd}}$. Choose a chamber $\mathcal{C}'$ that has H_ψ as a wall. Choose $w \in W(\Psi)$ such that $w\mathcal{C}' = \mathcal{C}$ by (2). Then $H_{w\psi}$ is a wall of $\mathcal{C}$, so $w\psi \in \Delta$.

Now assume that Ψ is irreducible.

(5), (6) $\mathcal{C}$ is a simplex in A as discussed in Remark 1.3.16, therefore Δ is a basis of A^*. Moreover, a vertex of $\overline{\mathcal{C}}$ is the intersection of all walls of $\mathcal{C}$ except the one opposite to the vertex.

(4) While this point is stated without assuming that Ψ is irreducible, it reduces to this case by Proposition 1.3.21. Let L denote the **Z**-lattice in A^* spanned by Δ. For any $\psi \in \Delta$ and $\eta \in L$ we have $r_\psi(\eta) = \eta - \dot\eta(\dot\psi^\vee)\psi$ and AR 3 implies that this lies in L. Now (1) implies that L is stable under $W(\Psi)$ and (3) implies that L coincides with the lattice spanned by Ψ. This shows that every $\psi \in \Psi(\mathcal{C})^+$ is an integral linear combination of elements of Δ. To show that it is non-negative, we evaluate ψ at each vertex of $\mathcal{C}$ and apply (6). □

The possible non-uniqueness in Proposition 1.3.22(4) is explained as follows. Recall from Remark 1.3.16 that when $\Psi = \Psi_1 \oplus \Psi_2$ and chamber $\mathcal{C}$ decomposes as $\mathcal{C}_1 \times \mathcal{C}_2$.

Lemma 1.3.23 *Let $\Psi = \bigoplus \Psi_i$ and $\mathcal{C} = \prod \mathcal{C}_i$. Then $\Psi(\mathcal{C})^0 = \bigcup \Psi_i(\mathcal{C}_i)^0$. Then $\sum_{\psi \in \Psi(\mathcal{C})^0} n_\psi \psi = 0$ if and only if there exist $c_1, \ldots, c_n \in \mathbf{R}$ such that $\sum_{\psi \in \Psi_i(\mathcal{C}_i)^0} n_\psi \psi = c_i$ and $\sum c_i = 0$.*

Proof The "if" statement is clear. For the converse, taking the derivative of

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$\sum_{\psi\in\Psi(\mathcal{C})^0} n_\psi\psi = 0$ and using the direct sum decomposition $V^* = \bigoplus V_i^*$ we see $\sum_{\psi\in\Psi_i(\mathcal{C}_i)^0} n_\psi\dot{\psi} = 0$, whence the claim. □

Remark 1.3.24 Consider two irreducible affine root systems Ψ_1, Ψ_2 with bases Δ_1, Δ_2. Then the constant functional 1 in A_i^* has the unique expressions $\sum_{\psi_i\in\Delta_i} a_{\psi_i}\psi_i = 1$. Therefore, $1 \in (A_1 \times A_2)^*$ has the two distinct expressions $\sum_{\psi_i\in\Delta_i} a_{\psi_i}\psi_i$ for $i = 1, 2$ in terms of the basis $\Delta = \Delta_1 \cup \Delta_2$.

In particular, we see that $\Psi(\mathcal{C})^0$ is a basis of the vector space A^* when Ψ is irreducible, but when Ψ is not irreducible then $\Psi(\mathcal{C})^0$ is only generating, but not linearly independent.

Next is the affine analog of [Bou02, Chapter VI, §1, no. 6, Corollary 1].

Proposition 1.3.25 *Let $\mathcal{C} \subset A$ a chamber and $\alpha \in \Psi(\mathcal{C})^0$. The affine reflection r_α permutes the elements of $\Psi(\mathcal{C})^+$ which are not proportional to α.*

Proof We apply Proposition 1.3.21 and see that we can replace Ψ by the irreducible factor that contains α, because all other irreducible factors are fixed by r_α. Therefore we may assume that Ψ is irreducible.

For any $\beta \in \Psi(\mathcal{C})^0$, $\beta \neq \alpha$, $r_\alpha(\beta) = \beta - \dot{\beta}(\dot{\alpha}^\vee)\alpha$. Now since by Proposition 1.3.22(5), every root in $\Psi(\mathcal{C})^+$ (respectively, $\Psi(\mathcal{C})^- = -\Psi(\mathcal{C})^+$) is a unique non-negative (respectively, non-positive) integral linear combination of roots in $\Psi(\mathcal{C})^0$, the proposition is obvious. □

Proposition 1.3.26 *Let $\mathcal{C} \subset A$ be a chamber and let $\Psi(\mathcal{C})^0 \subset \Psi$ be the corresponding set of simple affine roots. Let $S \subset W(\Psi)$ be the set of reflections along the elements of $\Psi(\mathcal{C})^0$. Denote the length function of $(W(\Psi), S)$ by ℓ. Then*

(1) *Let $w \in W(\Psi)$ and let $s \in S$ be the reflection along $\alpha \in \Psi(\mathcal{C})^0$. Then $\ell(sw) > \ell(w)$ is equivalent to $w^{-1}\alpha \in \Psi(\mathcal{C})^+$.*
(2) $\ell(w) = \#(w\Psi(\mathcal{C})^{\mathrm{nd},+} \cap \Psi(\mathcal{C})^{\mathrm{nd},-})$ *for all $w \in W(\Psi)$.*
(3) *If $w = s_1 \cdots s_q$ is a reduced expression, with s_i the reflection along $\alpha_i \in \Phi(\mathcal{C})^0$, then the set of roots of $\Psi(\mathcal{C})^{\mathrm{nd},+}$ that are mapped onto negative roots by w is $\{s_q \cdots s_{i+1}(\alpha_i) \mid i = 1, \ldots, q\}$.*

Proof (1) According to [Bou02, Chapter V, §3, no.2, Theorem 1(ii)] the condition $\ell(sw) > \ell(w)$ is equivalent to the condition that the chambers $\mathcal{C}$ and $w\mathcal{C}$ are on the same side of the vanishing hyperplane H_α of α. This condition is in turn equivalent to the condition that $\alpha|_{w\mathcal{C}} > 0$. Thus $\ell(sw) > \ell(w)$ if and only if $w^{-1}\alpha \in \Psi(\mathcal{C})^+$.

(2) We induct on $\ell(w)$. When $\ell(w) = 1$ then w is the reflection along some $\alpha \in \Psi(\mathcal{C})^0$ and the claim follows form Proposition 1.3.25. For $\ell(w) > 1$ write

$w = sw'$ with $\ell(w') = \ell(w) - 1$ and $s \in S$ the reflection along some $\alpha \in \Psi(\mathcal{C})^0$. Proposition 1.3.25 implies

$$\begin{aligned}\#\big(w\Psi(\mathcal{C})^{\mathrm{nd},+} \cap \Psi(\mathcal{C})^{\mathrm{nd},-}\big) &= \#\big(w'\Psi(\mathcal{C})^{\mathrm{nd},+} \cap s\Psi(\mathcal{C})^{\mathrm{nd},-}\big)\\ &= \#\big(w'\Psi(\mathcal{C})^{\mathrm{nd},+} \cap (\{\alpha\} \cup (\Psi(\mathcal{C})^{\mathrm{nd},-} - \{-\alpha\}))\big)\\ &= \#\big(w'\Psi(\mathcal{C})^{\mathrm{nd},+} \cap \Psi(\mathcal{C})^{\mathrm{nd},-}\big) + 1\end{aligned}$$

where the final equality comes from (1).

(3) is an immediate consequence of (1) and (2). □

The following is the main construction of reduced affine root systems.

Construction 1.3.27 Let Φ be a (possibly non-reduced) finite root system in V^*. Take $A = V$ and define

$$\Psi_\Phi = \{a + n \mid a \in \Phi, n \in I_a\},$$

where $I_a = \mathbf{Z}$ if a is non-divisible, and $I_a = 2\mathbf{Z} + 1$ if a is divisible. Then Ψ is an affine root system according to Proposition 1.3.12, reduced by construction.

It is clear that $\nabla\Psi_\Phi = \Phi$. In particular, $\nabla\Psi$ need not be reduced even if Ψ is reduced. In other words, the inclusion $(\nabla\Psi)^{\mathrm{nd}} \subset \nabla(\Psi^{\mathrm{nd}})$ above can be proper.

Example 1.3.28 We now give an example of a non-reduced affine root system. Let $V = \mathbf{R}^n$ with the standard scalar product. Let $\{e_1, \ldots, e_n\}$ be the standard basis. Identifying V^* with V via this scalar product we consider the non-reduced root system BC_n. It is the set of elements

$$\Phi = \{\pm e_i \mid i = 1, \ldots, n\} \cup \{\pm 2e_i \mid i = 1, \ldots, n\} \cup \{\pm e_i \pm e_j \mid 1 \leqslant i \neq j \leqslant n\}.$$

Let $\Psi = \{a + k \mid a \in \Phi,\ k \in \mathbf{Z}\}$. Then Ψ is an affine root system by Proposition 1.3.12 and is obviously non-reduced.

The following rather innocuous construction will turn out to be quite useful when discussing isomorphisms.

Construction 1.3.29 Let $\Psi \subset A^*$ be an affine root system and let $s \in \mathbf{R}^\times$. Then $s\Psi \subset A^*$ defined by $s\Psi = \{s\psi \mid \psi \in \Psi\}$ and $\nabla(s\psi)^\vee = s^{-1}\dot{\psi}^\vee$, is also an affine root system. We say that $s\Psi$ is obtained from Ψ by *rescaling*.

Note that $H_\psi = H_{s\psi}$ and $r_\psi = r_{s\psi}$. Thus Ψ and $s\Psi$ share the same hyperplane arrangement and the same affine and extended affine Weyl groups.

Construction 1.3.30 Let $\langle -, - \rangle$ be a scalar product on V that is invariant under $W(\nabla\Psi)$. For example, we can take the canonical one [Bou02, Chapter VI, §1, no. 1, Proposition 3]. Identifying V with V^* via this scalar product, we obtain a symmetric bilinear form on A^* by $\langle \psi, \eta \rangle = \langle \dot{\psi}, \dot{\eta} \rangle$. It is degenerate. A

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vector $\psi \in A^*$ is isotropic if and only if it is a constant functional. In particular, each $\psi \in \Psi$ is anisotropic, so we can define $\psi^\vee = 2\psi/\langle\psi,\psi\rangle$. Then $\nabla(\psi^\vee)$ is equal to the element $\dot{\psi}^\vee$ of Axiom AR 2. Note that $H_{\psi^\vee} = H_\psi$. The reflection $r_\psi|_A$ is the orthogonal reflection along the hyperplane H_ψ.

Proposition 1.3.31 *If Ψ is an affine root system, then so is $\Psi^\vee = \{\psi^\vee | \psi \in \Psi\}$, called the* dual *affine root system. We have $r_{\psi^\vee} = r_\psi$, hence $W(\Psi) = W(\Psi^\vee)$, and also $\nabla(\Psi^\vee) = (\nabla\Psi)^\vee$.*

Proof Immediate from Proposition 1.3.12. □

Remark 1.3.32 We alert the reader that $(\Psi_\Phi)^\vee$ and $\Psi_{\Phi^\vee}$ are distinct affine root systems when Φ is reduced and not simply laced. More precisely, assuming that Φ is reduced and irreducible, we have

$$(\Psi_{\Phi^\vee})^\vee = \{a + n \mid a \in \Phi,\ n \in I_a\},$$

where $I_a = \mathbf{Z}$ when a is long, and $I_a = \ell^{-1}\mathbf{Z}$ when a is short, where ℓ is the integer ratio of the squares of the two different root lengths in Φ.

The definition of $\Psi^\vee$ involves a scalar product because it is not clear how to interpret A^{**} as the dual of an affine space in a natural way. This is different from the case of finite root systems. One can canonify $\Psi^\vee$ by using the canonical scalar product of [Bou02, Chapter VI, §1, no. 1, Proposition 3]. If one uses an arbitrary scalar product, then $\Psi^\vee$ is independent of that choice up to rescaling (Construction 1.3.29) when Ψ is irreducible. When Ψ is reducible, the dual of each irreducible component would be well-defined up to rescaling. We will see below that the isomorphism class of $\Psi^\vee$ is independent of the choice of scalar product.

Remark 1.3.33 Consider an affine root system Ψ and its dual $\Psi^\vee$. The set of vanishing hyperplanes for $\Psi^\vee$ is the same as that for Ψ. In particular, a chamber $\mathcal{C}$ for Ψ is also a chamber for $\Psi^\vee$. If Ψ is reduced and $\Psi(\mathcal{C})^0$ is the corresponding basis, then $\Psi^\vee(\mathcal{C})^0 = \{\psi^\vee \mid \psi \in \Psi(\mathcal{C})^0\}$.

Proposition 1.3.3 gave one way of obtaining a finite root system from an affine root system, namely by taking the derivative. The following proposition gives a different way, by looking at a neighborhood of a point in A.

Notation 1.3.34 If Ω and Ω' are two subsets of a topological space and Ω is contained in the closure of Ω', we will write $\Omega \prec \Omega'$.

Proposition 1.3.35 *Let $x \in A$. We set $\Psi_x = \{\psi \in \Psi \mid \psi(x) = 0\}$.*

(1) *Ψ_x is a finite root system in the subspace of A_x^* that it generates.*

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(2) *The map ∇ restricted to Ψ_x is injective.*
(3) Ψ_x *depends only on the facet $\mathcal{F}$ containing x and may thus be denoted $\Psi_{\mathcal{F}}$.*
(4) *The subset $\{\psi^\vee \mid \psi \in \Psi_x\} \subset \Psi^\vee$ is identified with the root system dual to Ψ_x.*
(5) *Let $W(\Psi)_x$ be the stabilizer of x in $W(\Psi)$. The action of $W(\Psi)_x$ on Ψ_x identifies $W(\Psi)_x$ with $W(\Psi_x)$.*
(6) *If $\mathcal{C}$ is a chamber whose closure contains $\mathcal{F}$, then $\Psi(\mathcal{C})^0 \cap \Psi_{\mathcal{F}}$ is a basis for $\Psi_{\mathcal{F}}$.*
(7) *The set of chambers $\mathcal{C}$ whose closure contains $\mathcal{F}$ is in bijection with the set of Weyl chambers in $\Psi_{\mathcal{F}}$, the bijection being given by $\mathcal{C} \mapsto \Psi_{\mathcal{F}}(\mathcal{C})^+ = \Psi_{\mathcal{F}} \cap \Psi(\mathcal{C})^+$.*
(8) *More generally, the set of facets whose closure contains $\mathcal{F}$ is in bijection with the set of parabolic subsets of $\Psi_{\mathcal{F}}$, the bijection being given by $\mathcal{F}' \mapsto \Psi_{\mathcal{F}}(\mathcal{F}')^+ = \{\psi \in \Psi_{\mathcal{F}} | \psi(\mathcal{F}') \geqslant 0\}$. If $\mathcal{F}_1' \prec \mathcal{F}_2'$ then $\Psi_{\mathcal{F}}(\mathcal{F}_2')^+ \subset \Psi_{\mathcal{F}}(\mathcal{F}_1')$.*

It is obvious that if Ψ is reduced then so is Ψ_x.

Proof (2) If $\psi, \eta \in \Psi$ have equal derivative, then there exists a $c \in \mathbf{R}$ such that $\eta = \psi + c$. Hence, unless $c = 0$, that is $\eta = \psi$, both of these affine roots cannot vanish at x. This shows that ∇ restricted to Ψ_x is injective. As $\Phi = \nabla\Psi$ is finite by Proposition 1.3.3, we conclude that Ψ_x is also finite.

(1) The axioms in the definition of finite root systems for Ψ_x follow from the axioms in Definition 1.3.1 for Ψ and the finiteness established in (2).

(3) If $\psi \in \Psi$ vanishes on x, it vanishes on $\mathcal{F}$. Thus Ψ_x only depends on $\mathcal{F}$ and is contained in $A^*_{\mathcal{F}}$.

(4) is immediate.

(5) follows from Lemma 1.3.17(5).

(6) We will use [Bou02, Chapter VI, §1, no. 7, Corollary 3]. Let $\Psi_x(\mathcal{C})^\pm = \Psi_x \cap \Psi(\mathcal{C})^\pm$ and $\Psi_x(\mathcal{C})^0 = \Psi_x \cap \Psi(\mathcal{C})^0$. All elements of $\Psi_x(\mathcal{C})^0$ are indivisible by construction. By Proposition 1.3.22(4) every element of $\Psi_x(\mathcal{C})^+$ can be written as a non-negative integral linear combination $\sum n_\psi \psi$ with $\psi \in \Psi(\mathcal{C})^0$. Each such ψ evaluates non-negatively at x, while the linear combination vanishes at x. This shows $n_\psi = 0$ if $\psi(x) \neq 0$. To see that $\Psi_x(\mathcal{C})^0$ is linearly independent we use Lemma 1.3.23. Writing $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ and $x = (x_1, \dots, x_n)$, a linear relation among $\Psi_x(\mathcal{C})$ would imply an affine relation among $\Psi_{i,x_i}(\mathcal{C}_i)$, but since all members of the latter vanish at x_i, that relation is in fact linear, and by Proposition 1.3.22(5) it must be trivial.

(7) Any other basis of $\Psi_{\mathcal{F}}$ is obtained by applying an element of $W(\Psi_{\mathcal{F}}) = W(\Psi)_{\mathcal{F}}$ to $\Psi_x(\mathcal{C})^0$. Applying this element to $\mathcal{C}$ produces another chamber containing $\mathcal{F}$ in its closure. Conversely, any other chamber containing $\mathcal{F}$ in its

closure is obtained by applying an element $w \in W(\Psi)$ to $\mathcal{C}$. Then $\mathcal{F}, w^{-1}\mathcal{F}$ are both contained in the closure of $\mathcal{C}$, so Lemma 1.3.17 implies $w \in W(\Psi)_{\mathcal{F}}$.

(8) Let now $\mathcal{F}'$ be a facet whose closure contains $\mathcal{F}$. It is immediate that $\Psi_{\mathcal{F}}(\mathcal{F}')^+$ is a parabolic subset. It is also immediate that if $\mathcal{F}'' \prec \mathcal{F}'$ then $\Psi_{\mathcal{F}}(\mathcal{F}')^+ \subset \Psi_{\mathcal{F}}(\mathcal{F}'')^+$. Conversely let $P \subset \Psi_{\mathcal{F}}$ be a parabolic subset. According to [Bou02, Chapter VI, §1, no. 7, Proposition 20] and the previous point, there exists a chamber $\mathcal{C}$ and a subset $S \subset \Psi_{\mathcal{F}}(\mathcal{C})^0$ such that P is the union of $\Psi_{\mathcal{F}}(\mathcal{C})^+$ and the subset of $\Psi_{\mathcal{F}}$ consisting of linear combinations of elements of S with non-positive integer coefficients. Then

$$\mathcal{F}' := \bigcap_{\psi \in S} H_\psi \cap \overline{\mathcal{C}}$$

is a facet contained in the closure of $\mathcal{C}$. We have

$$\Psi_{\mathcal{F}}(\mathcal{F}')^+ = \Psi_{\mathcal{F}}(\mathcal{C})^+ \cup (\Psi_{\mathcal{F}}(\mathcal{C})^- \cap \Psi_{\mathcal{F}}(\mathcal{F}')^+).$$

An element of $\Psi_{\mathcal{F}}(\mathcal{C})^- \cap \Psi_{\mathcal{F}}(\mathcal{F}')^+$ is a non-negative linear combination of elements of $\Psi_{\mathcal{F}}(\mathcal{C})^0$ that vanishes on $\mathcal{F}'$, thus a non-negative linear combination of elements of S. We conclude $P = \Psi_{\mathcal{F}}(\mathcal{F}')^+$. □

The subset $\Phi_x := \nabla\Psi_x$ of $\Phi = \nabla\Psi$ is a subsystem and the isomorphism $\nabla : A_x^* \to V^*$ identifies Ψ_x with Φ_x. The relation $\nabla(\psi^\vee) = (\nabla\psi)^\vee$ implies that $\{a^\vee \in \Phi^\vee \mid a \in \Phi_x\}$ is the dual of Φ_x.

Corollary 1.3.36 *Let $x \in A$. The subset*

$$\Phi'_x := \Phi^{\mathrm{nd}} \cap (\Phi_x \cup \tfrac{1}{2}\Phi_x)$$

is a reduced root system in V^* with the same Weyl group as Ψ_x, where $(-)^{\mathrm{nd}}$ denotes the set of non-divisible roots.

Proof This is immediate from the fact that Ψ_x is a root system. □

Remark 1.3.37 When Φ is reduced then $\Phi'_x = \Phi_x$. In general neither of Φ_x and Φ'_x is contained in the other.

Both Φ_x and Φ'_x will be relevant for Bruhat–Tits theory. We will see (cf. Theorem 8.4.10) that $\Psi_x \simeq \Phi_x$ is the root system of the maximal reductive quotient of the special fiber of the parahoric group scheme associated to x, while the root system Φ'_x is the set of $a \in \Phi^{\mathrm{nd}}$ for which the filtration of the root subgroup $U_a(k)$ associated to the point x has a break at 0.

Example 1.3.38 Let $\Phi = \{-2a, -a, a, 2a\}$ be the root system of type BC_1 and let

$$\Psi = \{-a, a\} \times \mathbf{Z} \cup \{-2a, 2a\} \times (2\mathbf{Z} + 1)$$

as in Construction 1.3.27. Let $x, y \in A = V = \mathbf{R}$ be determined by $a(x) = 0$ and $a(y) = 1/2$. Then $\Phi_x = \Phi'_x = \Phi'_y = \{-a, a\}$, while $\Phi_y = \{-2a, 2a\}$.

Let Φ be the root system of type BC_2 and let

$$\Psi = \{\pm a, \pm b, \pm(a+b), \pm(a+2b)\} \times \mathbf{Z} \cup \{\pm 2b, \pm(2a+2b)\} \times (2\mathbf{Z}+1)$$

be as in Construction 1.3.27. Let $\{a, b\}$ be a set of simple roots so that the corresponding positive roots are $\{a, b, a+b, 2b, a+2b, 2a+2b\}$. Let $x, y \in A = V = \mathbf{R}^2$ be the elements specified by $a(x) = b(x) = 0$, $a(y) = 1/2$, $b(y) = 0$. Then $\Phi_x^+ = \Phi_x'^+ = \{a, b, a+b, a+2b\}$, $\Phi_y^+ = \{b, 2a+2b\}$, $\Phi_y'^+ = \{b, a+b\}$.

These examples show that Φ_x need not be a closed subsystem of Φ. For further examples we point the reader to Figure 1.3.1, from which one sees visually that, if x is a vertex, the following cases occur: when Ψ is of type A_2, then Ψ_x is always of type A_2; when Ψ is of type C_2, then Ψ_x is either of type C_2 or $A_1 \times A_1$; when Ψ is of type G_2, then Ψ_x is of type G_2 or A_2 or $A_1 \times A_1$; if Ψ is of type BC_2, then Ψ_x is of type $C_2 \subset BC_2$, or $B_2 \subset BC_2$, or $A_1 \times A_1$.

We now recall the concept of special points. There are in fact two different definitions of this notion.

Definition 1.3.39 A point $x \in A$ is called

(1) *special*, if for each $\psi \in \Psi$ there exists $\psi' \in \Psi_x$ such that H_ψ and $H_{\psi'}$ are parallel, and
(2) *extra special*, if there exist $\psi_1, \dots, \psi_\ell \in \Psi_x$ such that $\{\dot\psi_1, \dots, \dot\psi_\ell\}$ is a basis of $\Phi = \nabla\Psi$.

Remark 1.3.40 There is some discrepancy in the literature regarding the concept of a "special" point. We have decided to follow the convention used in [Bou02], [BT72], and [Tit74]. On the other hand, [Mac72] calls "special" what we have called here "extra special."

Remark 1.3.41 The notions of "special" and "extra special" do not change if we replace Ψ by Ψ^{nd}, cf. Remark 1.3.5. Therefore we can always reduce considerations to the case that Ψ is reduced.

If $\Psi = \Psi_1 \oplus \Psi_2$, then a point $x = (x_1, x_2)$ is (extra) special if and only if x_i is such for the affine root system Ψ_i. This allows one to reduce considerations to the case that Ψ is irreducible.

Lemma 1.3.42 *Let $x \in A$. The map $\nabla\colon W(\Psi) \to W(\Phi)$ restricts to an injection $W(\Psi)_x \to W(\Phi)$, which is surjective if and only if x is special, in which case it realizes $W(\Psi)$ as the semi-direct product of $W(\Phi)$ with the subgroup of $W(\Psi)$ consisting of translations.*

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Proof This is [Bou02, Chapter V, §3, no. 10, Proposition 9]. □

Proposition 1.3.43 (1) *Extra special points exist.*

(2) *An extra special point is special.*

(3) *A (extra) special point is a vertex.*

(4) *If $\Phi(= \nabla\Psi)$ is reduced, then a point is special if and only if it is extra special.*

(5) *If $\mathcal{C}$ is a chamber, then there is a vertex of $\mathcal{C}$ which is extra special.*

(6) *If x is extra special, then ∇ identifies $(\Psi_x)^{\mathrm{nd}}$ with Φ^{nd}.*

(7) *If Ψ is irreducible, a given basis can be enumerated $\{\psi_0, \dots, \psi_\ell\}$ so that $\{\dot\psi_1, \dots, \dot\psi_\ell\}$ is a basis of Φ.*

Proof (1) Let $a_1, \dots, a_\ell$ be a basis of Φ. Choose $\psi_i \in \Psi$ with $\dot\psi_i = a_i$. The linear independence of $\dot\psi_i$s implies that the hyperplanes H_{ψ_i} intersect in a single point of A. This point is extra special.

(2) Let $\psi_1, \dots, \psi_\ell \in \Psi_x$ be such that $\{\dot\psi_1, \dots, \dot\psi_\ell\}$ is a basis of Φ. The map $\nabla : W(\Psi) \to W(\Phi)$, which is surjective by Proposition 1.3.3, must then remain surjective when restricted to $W(\Psi_x) \subset W(\Psi)$. From Proposition 1.3.35(5) we have $W(\Psi_x) = W(\Psi)_x$. The claim follows from Lemma 1.3.42.

(3) If the point $x \in \mathcal{A}$ is special or extra special there exist $\psi_1, \dots, \psi_\ell \in \Psi_x$ whose derivatives are linearly independent, so x is a vertex.

(4) Assume that Φ is reduced and that x is special. Let $\Delta \subset \Phi$ be a basis. For each $a \in \Delta$ there exists $\psi \in \Psi$ with $\dot\psi = a$. Let $\psi' \in \Psi_x$ be such that H_ψ and $H_{\psi'}$ are parallel. Since Φ is reduced, $\dot\psi' = \pm\dot\psi$. Since $-\psi' \in \Psi_x$ we are done.

(5) From (1) and (3) we know that there exists a chamber $\mathcal{C}'$ that has an extra special vertex. Since the property of being an extra special vertex is preserved under the action of $W(\Psi)$, the claim follows from Lemma 1.3.17.

(6) We have already noted that ∇ is an isomorphism $\Psi_x \to \Phi_x$. By assumption Φ_x contains a basis of Φ. Since a basis consists of indivisible roots, it lies in Φ_x^{nd} and hence comes from $(\Psi_x)^{\mathrm{nd}}$.

(7) A basis corresponds to a chamber, and this chamber has an extra special vertex x according to (5). Enumerate the basis as $\{\psi_0, \dots, \psi_\ell\}$, with ψ_0 corresponding to x. By Proposition 1.3.35(6) $\{\psi_1, \dots, \psi_\ell\}$ is a basis of Ψ_x and the claim follows from (6). □

The following example shows the existence of a point that is special but not extra special.

Example 1.3.44 Let $A = V = \mathbf{R}$. Let $a \in A^*$ be the identity function and $\Phi = \{-2a, -a, a, 2a\}$ be the root system of type BC_1. Let $\Psi = \Psi_\Phi$. Thus $\Psi = \{-2a + 2\mathbf{Z} + 1, -a + \mathbf{Z}, a + \mathbf{Z}, 2a + 2\mathbf{Z} + 1\}$. The point $0 \in A$ is extra special. Since $\Psi_{1/2} = \{-2a, 2a\}$, the point $1/2 \in A$ is special, but not extra special.

In type BC_2, the difference between special and extra special vertices can be seen visually in Figure 1.3.1: the extra special vertices are those where 4 solid lines meet, while the special but not extra special vertices are those where 2 solid and 2 dotted lines meet.

Lemma 1.3.45 *Let Ψ be an affine root system and let $\mathcal{C}$ be a chamber. For all $\psi, \eta \in \Psi(\mathcal{C})^0$, $\psi \neq \eta$, we have $\dot{\psi}(\dot{\eta}^\vee) \leqslant 0$.*

Proof This is [Bou02, Chapter V, §3, no 4, Proposition 3]. □

Lemma 1.3.46 *Let Φ be a finite root system. If $a \in \Phi$ is divisible in $\mathbf{Z}[\Phi]$, it is divisible in Φ.*

Proof Assume that a is indivisible in Φ. Then there exists a basis Δ of Φ containing a. Since $\mathbf{Z}[\Phi]$ is freely generated by Δ, a cannot be divisible in $\mathbf{Z}[\Phi]$. □

Proposition 1.3.47 *Assume that Ψ is irreducible and let Δ be a basis corresponding to a chamber $\mathcal{C}$.*

(1) *There exists a collection of non-negative integers $(n_\psi)_{\psi \in \Delta}$ without common denominator such that*
$$c(\Psi) := \sum_{\psi \in \Delta} n_\psi \psi \tag{1.3.1}$$
is a constant functional.
(2) *This collection is uniquely determined, all integers are positive, and the constant $c(\Psi)$ is positive.*
(3) *A vertex x of $\mathcal{C}$ is extra special if and only if it is special and the integer n_ψ, with $\psi \in \Delta$ corresponding to the vertex x (cf. Proposition 1.3.22(6)), is equal to 1.*
(4) *If a finite integral linear combination of elements of Ψ is a constant, then this constant is an integral multiple of $c(\Psi)$.*
(5) *$c(\Psi)$ is independent of $\mathcal{C}$.*

Proof (1) Let x be an extra special vertex of $\mathcal{C}$. Enumerate the simple affine roots $\Psi(\mathcal{C})^0 = \{\psi_0, \dots, \psi_\ell\}$ so that $\psi_0(x) > 0$, cf. Proposition 1.3.22(6). Then, according to Lemma 1.3.45, $-\dot{\psi}_0$ is a positive root in Φ. So there exists a collection of non-negative integers $n_0, \dots, n_\ell$ without a common denominator such that $-\dot{\psi}_0 = \sum_{i=1}^{\ell} n_i \dot{\psi}_i$. Thus $c(\Psi) = \psi_0 + \sum_{i=1}^{\ell} n_i \psi_i$ has zero derivative, and is therefore constant.

(2) The value of the linear functional $c(\Psi)$ at x equals $\psi_0(x) > 0$, which shows that $c(\Psi)$ is positive. Evaluating $c(\Psi)$ at every other vertex of $\mathcal{C}$ shows that each n_i is positive. Since $\Psi(\mathcal{C})^0$ is a basis of A^* by Proposition 1.3.22(5)

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and since the constant linear functionals form a 1-dimensional subspace of A^* we see that the collection (n_i) is uniquely determined by the properties that it consist of positive integers without common denominator.

(3) The construction in (1) had the byproduct that $n_{\psi_0} = 1$ for $\psi_0 \in \Delta$ corresponding to the extra special vertex x that was used to produce the collection (n_ψ). The uniqueness statement of (2) shows that the collection (n_ψ) is independent of the choice of x and we conclude that $n_\psi = 1$ for ψ corresponding to any extra special vertex of $\mathcal{C}$.

Conversely, let x be a special vertex of $\mathcal{C}$ with $n_\psi = 1$, where $\psi \in \Psi(\mathcal{C})^0$ is the affine root associated to x. We numerate $\Psi(\mathcal{C})^0 = \{\psi_0, \ldots, \psi_\ell\}$ so that $\psi_0 = \psi$. Then $\{\psi_1, \ldots, \psi_\ell\}$ is a basis of Ψ_x by Proposition 1.3.35(6). We will show that $\{\dot\psi_1, \ldots, \dot\psi_\ell\}$ is a basis of Φ. According to [Bou02, Chapter VI, §1, no. 7, Corollary 3], it is enough to show that every element of $a \in \Phi$ can be expressed as an integral linear combination of $\{\dot\psi_1, \ldots, \dot\psi_\ell\}$ with the same sign. Since x is special, there exists $b \in \Phi_x$ that is proportional to a, i.e. $a = rb$ with $r \in \{\pm 1, \pm 2, \pm 1/2\}$. If b is divisible in Φ_x, after replacing it with $b/2$, we assume that b is not divisible in Φ_x. It is enough to show that $r \neq \pm 1/2$, i.e. that b is indivisible in Φ. For this, the fact that a is an integral linear combination of $\{\dot\psi_0, \ldots, \dot\psi_\ell\}$ and that $n_{\psi_0} = 1$ shows that a is an integral linear combination of $\{\dot\psi_1, \ldots, \dot\psi_\ell\}$. If $r = \pm 1/2$, then b would be an integral linear combination of $\{2\dot\psi_1, \ldots, 2\dot\psi_\ell\}$ and would therefore be a divisible element of the root lattice of Ψ_x. This is a contradiction to our choice by Lemma 1.3.46.

(4) Assume that c' is a constant that is a finite integral linear combination of elements of Ψ. Using Proposition 1.3.22(5) we see that c' is a unique integral linear combination of elements of $\Psi(\mathcal{C})^0$. Thus $c' = \sum n'_i \psi_i$ with $n'_i \in \mathbf{Z}$, and we find that $n'_i/n_i = c'/c(\Psi)$ for all i. Taking i so that the vertex x_i is extra special, we see from (3) that $c'/c(\Psi) = n'_i \in \mathbf{Z}$.

(5) By (4), $c(\Psi)$ is the smallest positive constant that is an integral linear combination of elements of Ψ, and hence it does not depend on $\mathcal{C}$. □

Remark 1.3.48 Let Φ be an irreducible reduced finite root system and $\Psi = \Psi_\Phi$. If $a_1, \ldots, a_\ell$ is a basis for Φ, a_0 is the highest root, and we set $\psi_1 = a_1, \ldots, \psi_\ell = a_\ell$, and $\psi_0 = 1 - a_0$, then $\psi_0, \ldots, \psi_\ell$ is a basis for Ψ. The integers $n_0, \ldots, n_\ell$ of (1.3.1) are specified by $n_0 = 1$ and $a_0 = \sum_{i=1}^{\ell} n_i a_i$.

Consider now the dual affine root system $\Psi^\vee$. By Remark 1.3.33 we know that $\psi_0^\vee, \ldots, \psi_\ell^\vee$ is a basis for $\Psi^\vee$. Thus $\sum_{i=0}^{\ell} \frac{1}{2} n_i \langle \psi_i, \psi_i \rangle \psi_i^\vee$ is constant. So the integers (1.3.1) for $\Psi^\vee$ are obtained by taking the sequence

$$n_0 \langle \psi_0, \psi_0 \rangle, \ldots, n_\ell \langle \psi_\ell, \psi_\ell \rangle$$

and dividing each term in it by the greatest common divisor of the sequence.

This procedure produces the integers (1.3.1) for all irreducible reduced affine root systems except BC_n, in which case one can compute them by hand. The results are compiled in Table 1.3.5.

The following proposition will be used in the study of isomorphisms (cf. Definition 1.3.50) as well as the classification of affine root systems.

Proposition 1.3.49 *Let Ψ be an affine root system. For each $\psi \in \Psi$ let $u_\psi \in \mathbf{R}$ be the smallest positive number such that $\psi' = \psi + u_\psi \in \Psi$. Let $t_\psi = r_{\psi'} r_\psi$. Then*

(1) *For $x \in A$ and $\eta \in A^*$ we have $t_\psi(x) = x - u_\psi \dot\psi^\vee$ and $t_\psi(\eta) = \eta + u_\psi \dot\eta(\dot\psi^\vee)$.*
(2) *For any $\psi \in \Psi$ and $r \in \mathbf{R}$, $\psi + r \in \Psi$ if and only if r is an integral multiple of u_ψ.*
(3) *u_ψ depends only on the Weyl orbit of the derivative of ψ; write u_a with $a = \dot\psi$.*
(4) *u_ψ is a positive integral multiple of $c(\Psi)$.*

Proof (1) is an immediate computation.

(2) Let $m \in \mathbf{Z}$ and $r \in \mathbf{R}$. Then (1) implies $t_\psi^m(\psi + r) = \psi + r + 2u_\psi m$. Taking $r = 0$ and $r = u_\psi$ we see $\psi + mu_\psi \in \Psi$ for all $m \in \mathbf{Z}$. Conversely, if $\psi + r \in \Psi$, then $\psi + r + 2u_\psi m = t_\psi^m(\psi + r) \in \Psi$ for all $m \in \mathbf{Z}$. Choosing m appropriately we obtain an affine root $\psi + r'$ with $-u_\psi < r' \leqslant u_\psi$. It is enough to show that $r' = 0$ or $r' = u_\psi$. If that were not the case, then either $r' \in (-u_\psi, 0)$ or $(0, u_\psi)$. In the first case we apply the translation $r_\psi r_{\psi+r'}$ to $\psi + r'$ to obtain the affine root $\psi - r'$, which reduces to the second case, namely $r' \in (0, u_\psi)$. That is however a contradiction to the minimality of u_ψ.

(3) Since the difference of any two affine roots with equal derivative is a constant, it follows from (2) that u_ψ depends only on the derivative of ψ. The surjectivity of $W(\Psi) \to W(\nabla\Psi)$ due to Proposition 1.3.3 reduces to showing $u_{w\psi} = u_\psi$ with $w \in W(\Psi)$. However $w(\psi + u_a) = w\psi + u_a$ and (2) implies $u_{w\psi} | u_\psi$. Replacing ψ by $w\psi$ and w by w^{-1} we see the opposite divisibility relation, hence $u_{w\psi} = u_\psi$.

(4) This follows from Proposition 1.3.47(4). □

We will now introduce and study the concept of isomorphisms of affine root systems, as a preparation for the classification of affine root systems.

Definition 1.3.50 Let A_i be an affine space under the $\mathbf{R}$-vector space V_i, $\Psi_i \subset A_i^*$ an affine root system, for $i = 1, 2$.

(1) A *strong isomorphism* $\Psi_1 \to \Psi_2$ is an isomorphism $f\colon A_2 \to A_1$ of

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affine spaces such that $f^* : A_1^* \to A_2^*$ induces a bijection $\Psi_1 \to \Psi_2$ such that for $\eta := (f^*)^{-1}(\psi)$, $(\nabla f)(\dot\psi^\vee) = \dot\eta^\vee$ for all $\psi \in \Psi_2$.

(2) An *isomorphism* $\Psi_1 \to \Psi_2$ is a set theoretic bijective map $\varphi\colon \Psi_1 \to \Psi_2$ with the following property. If $\psi, \eta \in \Psi_1$ and r, s are integers such that $r\psi + s\eta \in \Psi_1$, then $\varphi(r\psi + s\eta) = r\varphi(\psi) + s\varphi(\eta)$. The analogous property is required of φ^{-1}.

Remark 1.3.51 A strong isomorphism $\Psi_1 \to \Psi_2$ is uniquely determined by the bijective map $\Psi_1 \to \Psi_2$ that it induces, because Ψ_1 generates A_1^* and an affine map $A_2 \to A_1$ is uniquely determined by its dual $A_1^* \to A_2^*$. Therefore, being a strong isomorphism is a property of a bijection $\Psi_1 \to \Psi_2$. It is clear that this property is stronger than the property of being an isomorphism as in (2) of the above definition.

A strong isomorphism is thus an example of an isomorphism. Another example of an isomorphism is the natural bijection $\Psi \to s\Psi$ for any $s \in \mathbf{R}^\times$, cf. Example 1.3.55 below. We will show in Proposition 1.3.54 below that a general isomorphism $\Psi_1 \to \Psi_2$ arises as the composition of those two examples applied to each individual irreducible factor of Ψ_1, equivalently Ψ_2.

Proposition 1.3.52 *Assume that Ψ_1, Ψ_2 are irreducible affine root systems. Any isomorphism $\varphi\colon \Psi_1 \to \Psi_2$ extends uniquely to a vector space isomorphism $f^*\colon A_1^* \to A_2^*$ that sends the line of constants $\mathbf{R} \subset A_1^*$ to the line of constants $\mathbf{R} \subset A_2^*$ and hence descends to vector space isomorphism $V_1^* \to V_2^*$.*

Proof Since Ψ_1 generates A_1^*, a linear extension $A_1^* \to A_2^*$ of φ is necessarily unique. To show that it exists, fix a basis Δ of Ψ_1. According to Proposition 1.3.22(5) Δ is also a basis of the vector space A_1^*. Let $f^*\colon A_1^* \to A_2^*$ be the linear map extending $\varphi|_\Delta$. We will now show that f^* extends φ. For this purpose, let Θ be the subset of Ψ_1 consisting of $\psi \in \Psi_1$ such that $f^*(\psi) = \varphi(\psi)$. Obviously, $\Delta \subset \Theta$. We claim that for all $\psi, \eta \in \Theta$, $r_\psi(\eta) \in \Theta$. We have the following:

$$\begin{aligned} f^*(r_\psi(\eta)) &= f^*(\eta - \dot\eta(\dot\psi^\vee)\psi) = f^*(\eta) - \dot\eta(\dot\psi^\vee) f^*(\psi) = \varphi(\eta) - \dot\eta(\dot\psi^\vee)\varphi(\psi) \\ &= \varphi(r_\psi(\eta)). \end{aligned}$$

This proves that $r_\psi(\eta) \in \Theta$.

Now since $\Delta \subset \Theta$, and the r_α, for $\alpha \in \Delta$, generate the affine Weyl group $W(\Psi_1)$, we see that Θ is stable under the action of $W(\Psi_1)$. As $W(\Psi_1) \cdot \Delta = \Psi_1^{\mathrm{nd}}$ (Proposition 1.3.22(3)), and for a multipliable root $\psi \in \Psi_1$, $\varphi(2\psi) = 2\varphi(\psi)$, we conclude that $\Theta = \Psi_1$, that is, f^* indeed extends φ.

To see that f^* is an isomorphism, we apply the same argument to φ^{-1} and obtain a linear extension $g^*\colon A_2^* \to A_1^*$. The compositions $f^* \circ g^*$ and $g^* \circ f^*$

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are equal to the identity when restricted to a basis of Ψ_2 resp. Ψ_1, and hence are equal to the identity on A_2^* resp. A_1^*.

It remains to prove that $f^*(\mathbf{1}_{A_1}) \in \mathbf{R} \cdot \mathbf{1}_{A_2}$. Choose any affine root $\psi_1 \in \Psi_1$ and let $\psi_2 = f^*(\psi_1)$. Consider the sequence $\psi_1 + nu_{\psi_1} \in \Psi_1$ for $n = 1, 2, 3, \ldots$, cf. Proposition 1.3.49. The image sequence $f^*(\psi_1 + nu_{\psi_1}\mathbf{1}_{A_1})$ consists of elements of Ψ_2. Since according to Proposition 1.3.3, the set of derivatives of elements of Ψ_2 is finite, there exist $n_1 < n_2$ such that $f^*(\psi_1 + n_1 u_{\psi_1}\mathbf{1}_{A_1})$ and $f^*(\psi_1 + n_2 u_{\psi_1}\mathbf{1}_{A_1})$ have equal derivative. Hence,

$$f^*(\psi_1 + n_2 u_{\psi_1}\mathbf{1}_{A_1}) - f^*(\psi_1 + n_1 u_{\psi_1}\mathbf{1}_{A_1}) = c\mathbf{1}_{A_2},$$

where $c \in \mathbf{R}$. On the other hand,

$$c\mathbf{1}_{A_2} = f^*((\psi_1 + n_2 u_{\psi_1}\mathbf{1}_{A_1}) - (\psi_1 + n_1 u_{\psi_1}\mathbf{1}_{A_1})) = (n_2 - n_1)u_{\psi_1} f^*(\mathbf{1}_{A_1}),$$

so $f^*(\mathbf{1}_{A_1}) = (c/((n_2 - n_1)u_{\psi_1}))\mathbf{1}_{A_2}$. □

Lemma 1.3.53 *Let $\psi, \eta \in \Psi$. Assume that their derivatives $\dot{\psi}$ and $\dot{\eta}$ are linearly independent. Let r and s be the largest non-negative integers such that $\psi - r\eta, \psi + s\eta \in \Psi$. Then $\dot{\psi}(\dot{\eta}^\vee) = r - s$.*

Proof Since $\dot{\psi}$ and $\dot{\eta}$ have been assumed to be linearly independent, there is a $x \in A$ where both ψ and η vanish. So $\psi - r\eta, \psi + s\eta \in \Psi_x$, which is a finite root system according to the Proposition 1.3.35(1). The claim now follows from [Bou02, Chapter VI, §1, no. 3, Proposition 9]. □

Proposition 1.3.54 *Let Ψ_1 and Ψ_2 be irreducible affine root systems and $\varphi\colon \Psi_1 \to \Psi_2$ be an isomorphism . Let $f^*\colon A_1^* \to A_2^*$ be the extension of φ as in Proposition 1.3.52.*

(1) *Let $\alpha_1 \in A_1^*$ and $\eta_1 \in \Psi_1$. We denote $f^*(\alpha_1)$ by α_2 and $\varphi(\eta_1)$ by η_2. Then $\dot{\alpha}_1(\dot{\eta}_1^\vee) = \dot{\alpha}_2(\dot{\eta}_2^\vee)$.*
(2) *There exists an $\varepsilon \in \{\pm 1\}$ such that $f^*(c(\Psi_1)\mathbf{1}_{A_1}) = \varepsilon c(\Psi_2)\mathbf{1}_{A_2}$.*
(3) *If $c(\Psi_1) = c(\Psi_2)$, and $\varepsilon = +1$, then φ is a strong isomorphism.*

Proof (1) Since Ψ_1 spans A_1^* it is enough to assume $\alpha_1 \in \Psi_1$. By Proposition 1.3.52, f^* descends to an isomorphism $V_1^* \to V_2^*$. Therefore, if $\dot{\alpha}_1$ and $\dot{\eta}_1$ are linearly independent, then so are $\dot{\alpha}_2$ and $\dot{\eta}_2$, and the claim then follows from Lemma 1.3.53. If $\dot{\alpha}_1$ and $\dot{\eta}_1$ are linearly dependent, then $\dot{\alpha}_1 = c\dot{\eta}_1$ for some $c \in \mathbf{R}$ (in fact, $c \in \{\pm 1, \pm 2, \pm\frac{1}{2}\}$), and then $\dot{\alpha}_2 = c\dot{\eta}_2$, implying $\dot{\alpha}_1(\dot{\eta}_1^\vee) = 2c = \dot{\alpha}_2(\dot{\eta}_2^\vee)$.

(2) To see that $f^*(c(\Psi_1)\mathbf{1}_{A_1}) = \varepsilon c(\Psi_2)\mathbf{1}_{A_2}$, let $(n_\psi)_{\psi \in \Delta}$ be positive integers such that $\sum_{\psi \in \Delta} n_\psi \psi = c(\Psi_1)\mathbf{1}_{A_1}$. Then according to Proposition 1.3.52,

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$f^*(c(\Psi_1)\mathbf{1}_{A_1})$ is a constant functional on A_2. But

$$f^*(c(\Psi_1)\mathbf{1}_{A_1}) = \sum_{\psi\in\Delta} n_\psi\,\varphi(\psi),$$

hence according to Proposition 1.3.47(4), there exists an integer n_2 such that $f^*(c(\Psi_1)\mathbf{1}_{A_1}) = n_2 c(\Psi_2)\mathbf{1}_{A_2}$.

Applying the same reasoning to φ^{-1}, we conclude that there exits an integer n_1 such that $(f^*)^{-1}(c(\Psi_2)\mathbf{1}_{A_2}) = n_1 c(\Psi_1)\mathbf{1}_{A_1}$. Hence,

$$c(\Psi_1)\mathbf{1}_{A_1} = (f^*)^{-1} f^*(c(\Psi_1)\mathbf{1}_{A_1}) = (f^*)^{-1}(n_2 c(\Psi_2)\mathbf{1}_{A_2}) = n_1 n_2 c(\Psi_1)\mathbf{1}_{A_1}.$$

So, $n_1 n_2 = 1$, implying that $n_1 = n_2 = \pm 1$.

(3) If $c(\Psi_1) = c(\Psi_2)$, and $\varepsilon = +1$, then (2) implies $f^*(\mathbf{1}_{A_1}) = \mathbf{1}_{A_2}$. Now according to Proposition 1.2.10, f^* is the dual of an affine isomorphism $f\colon A_2 \to A_1$. □

Example 1.3.55 Let $\Psi \subset A^*$ be an irreducible affine root system. Let $s \in \mathbf{R}^\times$, $s\Psi$ be the affine root system as in 1.3.29, and Δ be a basis of Ψ. Then $|s|\Delta := \{|s|\psi \mid \psi \in \Delta\}$ is a basis of $s\Psi$. So $c(s\Psi) = |s|c(\Psi)$. Now let $\varphi\colon \Psi \to s\Psi$ be the natural isomorphism $\psi \mapsto s\psi$, for $\psi \in \Psi$. Then the extended isomorphism f^* (as in Proposition 1.3.52) is clearly the automorphism of A^* defined by $x \mapsto sx$ for $x \in A^*$. Hence,

$$f^*(c(\Psi)\mathbf{1}_{A^*}) = sc(\Psi)\mathbf{1}_{A^*} = \varepsilon c(s\Psi)\mathbf{1}_{A^*},$$

where, $\varepsilon = s/|s|$. Thus $\varepsilon = +1$ if and only if s is positive.

Proposition 1.3.56 *Let $\varphi\colon \Psi \to \Psi'$ be an isomorphism of affine root systems. Then there exists an affine root system $\Psi'' \subset A^*$ such that*

(1) *If $\Psi = \Psi_1 \oplus \cdots \oplus \Psi_n$ is the decomposition of Ψ into irreducible pieces and $A = A_1 \times \cdots \times A_n$ is the corresponding decomposition of the affine space A, then $\Psi'' = r_1\Psi_1 \oplus \cdots \oplus r_n\Psi_n$ for some $r_1, \ldots, r_n \in \mathbf{R}^\times$.*
(2) *The composition $\varphi'\colon \Psi'' \to \Psi'$ of φ with the obvious bijection $\Psi'' \to \Psi$ is a strong isomorphism.*

Proof Let $\Psi'_i = \varphi(\Psi_i) \subset \Psi'$. As $\Psi = \bigcup_i \Psi_i$, we see that $\Psi' = \bigcup_i \Psi'_i$. Hence,

$$\Psi' = \Psi'_1 \oplus \cdots \oplus \Psi'_n.$$

For each i, $\varphi|_{\Psi_i}\colon \Psi_i \to \Psi'_i$ is an isomorphism. Let $f_i^*\colon A_i^* \to A_i'^*$ be its extension and let $r_i \in \mathbf{R}$ such that $f_i^*(c(\Psi_i)\mathbf{1}_{A_i}) = r_i c(\Psi'_i)\mathbf{1}_{A'_i}$, and let $\Psi''_i = r_i\Psi_i$. Then Proposition 1.3.52 shows that the composition of $\varphi|_{\Psi_i}$ with the natural bijection $\Psi''_i \to \Psi_i$ is a strong isomorphism. □

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Corollary 1.3.57 *Let $\varphi\colon \Psi \to \Psi'$ be an isomorphism. There exists an isomorphism $f\colon A' \to A$ that identifies the hyperplane arrangement of Ψ' with that of Ψ, and in particular is equivariant for the action of the extended affine Weyl groups.*

Proof Using Proposition 1.3.21 decompose $\Psi = \Psi_1 \oplus \cdots \oplus \Psi_n$. Let $\Psi'' = r_1\Psi_1 \oplus \cdots \oplus r_n\Psi_n$ as in Proposition 1.3.56(1). The hyperplane arrangement of Ψ is the same as that of Ψ'', and that in turn is the same as that of Ψ' due to Proposition 1.3.56(2). □

Remark 1.3.58 Our notion of a strict isomorphism is the analog of the notion of isomorphism introduced in [Mac72, §2]. On the other hand, in [Mac72, §3] the notion of "similarity" is introduced, according to which Ψ, Ψ' are called similar, if there exist irreducible affine root systems $\Psi_1, \ldots, \Psi_n$ and non-zero real numbers $r_1, \ldots, r_n$ such that

$$\Psi \simeq \Psi_1 \oplus \cdots \oplus \Psi_n \text{ and } \Psi' \simeq r_1\Psi_1 \oplus \cdots \oplus r_n\Psi_n.$$

Corollary 1.3.57 shows that our notion of isomorphism of Definition 1.3.50(2) recovers Macdonald's notion of similarity.

We now turn to the problem of classifying affine root systems. We will see in Proposition 1.3.67 below that every irreducible reduced affine root system is isomorphic to either Ψ_Φ or $\Psi_\Phi^\vee$ of Construction 1.3.27 for some irreducible (possibly non-reduced) finite root system Φ. The non-reduced affine root systems can be easily enumerated separately.

As before, we denote the finite root system $\nabla\Psi$ by Φ in what follows.

Lemma 1.3.59 *Let Ψ be a reduced affine root system. If $\psi \in \Psi$ and $r \in \mathbf{R}$ are such that $2\psi + r \in \Psi$, then $r = mu_\psi$ for an odd integer m. In particular, $u_{2a} = 2u_a$ for $a = \dot{\psi}$.*

Proof Let $\mu \in \mathbf{R}$ be the smallest non-negative number such that $2\psi + \mu \in \Psi$. We claim that $\mu = u_a$. Since Ψ is reduced we know $\mu > 0$. We have $r_{2\psi+\mu}(\psi) = -(\psi + \mu)$, thus $\psi + \mu \in \Psi$, so Proposition 1.3.49(2) implies $\mu = ru_a$ with some positive integer r. If $r = 2$ then both $\psi + u_a$ and $2\psi + 2u_a$ are affine roots, which contradicts the assumption that Ψ is reduced. If $r \geqslant 3$ there exists an integer $r/4 \leqslant r' < r/2$. Then

$$-r_{\psi+r'u_a}(2\psi + ru_a) = 2\psi + (4r' - r)u_a \in \Psi.$$

But then $0 < (4r' - r)u_a < ru_a = \mu$ contradicts the minimality of μ. Therefore $r = 1$ is the only possibility, confirming $\mu = u_a$.

From the definition of μ it follows that there is no affine root between 2ψ

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and $2\psi + \mu$. But $-r_\psi(2\psi + \mu) = 2\psi - \mu \in \Psi$ and we see that there is no affine root between $2\psi - \mu$ and 2ψ. Thus there is no affine root between $2\psi - \mu$ and $2\psi + \mu$ and Proposition 1.3.49(2) shows that $u_{2a} = 2\mu = 2u_a$. □

Proposition 1.3.60 *Let* $\overline{\Psi} = \{u_a^{-1} a \mid a \in \Phi\}$. *Then* $\overline{\Psi}$ *is a root system in* V^* *with the same Weyl group as* Φ *and its dual is* $\{u_a a^\vee \mid a \in \Phi\} \subset V$. *If* Ψ *is reduced, then* $\overline{\Psi} = \{u_a^{-1} a \mid a \in \Phi^{\mathrm{nd}}\}$, *so* $\overline{\Psi}$ *is also reduced.*

Proof We verify [Bou02, Chapter VI, §1, no. 1, Definition 1]. By construction $\overline{\Psi}$ is finite, does not contain 0, and spans V^*, hence ($\mathrm{RS_I}$). It is stable under $W(\Phi)$ according to Proposition 1.3.49(3). The reflection associated to the vector $u_a^{-1} a$ and the covector $u_a a^\vee$ is the same as the reflection associated to a and $a^\vee$, which is r_a, hence ($\mathrm{RS_{II}}$). Finally $u_a^{-1} a(u_b b^\vee) = u_a^{-1} u_b a(b^\vee)$. Choose $\alpha, \beta \in \Psi$ with gradients a, b respectively. By Proposition 1.3.49(1) we have

$$\Psi \ni t_\beta(\alpha) = \alpha + u_b \dot{\alpha}(\dot{\beta}^\vee) = \alpha + u_b a(b^\vee)$$

and Proposition 1.3.49(2) implies $u_b a(b^\vee) \in u_a \mathbf{Z}$, hence ($\mathrm{RS_{III}}$).

Assume now that Ψ is reduced. If Φ is also reduced the statement is immediate. Otherwise Lemma 1.3.59 shows that $u_{2a} = 2u_a$, hence $2u_{2a}^{-1} a = u_a^{-1} a$. □

Proposition 1.3.61 *Assume that* Φ *is reduced. Then the translation subgroup in* $W(\Psi)$ *is given by the lattice in* V *spanned by* $\overline{\Psi}^\vee$.

Proof Let $T \subset V$ be the translation subgroup of $W(\Psi)$ and let T' be the lattice spanned by $\overline{\Psi}^\vee$.

Let $\psi \in \Psi$ and let $a = \dot{\psi}$. Then $t_\psi = -u_a a^\vee$ by Proposition 1.3.49(1), hence $T' \subset T$. Conversely, fix a special vertex x and apply Lemma 1.3.42 to write $W(\Psi) = T \rtimes W(\Phi)$. Since T' is stable under $W(\Phi)$ we can form the subgroup $W(\Psi)' := T' \rtimes W(\Phi)$ of $W(\Psi)$. It is enough to show $W(\Psi)' = W(\Psi)$. In turn, it is enough to show $r_\psi \in W(\Psi)'$ for all $\psi \in \Psi$.

By Proposition 1.3.43 we have the isomorphism $\nabla\colon \Psi_x \to \Phi$, so there exists $\psi' \in \Psi_x$ with $\dot{\psi}' = a$. By Proposition 1.3.49 we have $\psi = \psi' + nu_a$ with $n \in \mathbf{Z}$. Then $r_\psi = t_{-nu_a a^\vee} r_{\psi'}$, where $t_{-nu_a a^\vee}$ is the translation by the vector $-nu_a a^\vee$. Therefore $r_\psi \in W(\Psi)'$ as claimed. □

Lemma 1.3.62 $\overline{\Psi}^\vee = \overline{\Psi}$.

Proof Let $\psi \in \Psi$ and $n \in \mathbf{Z}$. Write $a = \dot{\psi}$. Then

$$(\psi + nu_a)^\vee = \psi^\vee + 2nu_a \langle a, a\rangle^{-1}; \quad \text{hence,}\ u_{a^\vee} = 2u_a \langle a, a\rangle^{-1}.$$

Therefore,

$$\overline{\Psi}^\vee = \{u_{a^\vee}^{-1} a^\vee \mid a^\vee \in \Phi^\vee\} = \{u_a^{-1} a \mid a \in \Phi\} = \overline{\Psi}.$$

□

Theorem 1.3.63 *Let Ψ be an irreducible reduced affine root system. Then it is isomorphic to a system obtained from Construction 1.3.27, or to its dual.*

Proof We distinguish the following cases.

(1) The function $a \mapsto u_a$ is constant on Φ^{nd}: Up to rescaling Ψ we can assume $u_a = 1$ for all $a \in \Phi^{\mathrm{nd}}$. Let $x \in A$ be an extra special vertex. Then Proposition 1.3.49 and Lemma 1.3.59 show that every $\psi \in \Psi$ is of the form $\psi_x + n$ for some $\psi_x \in \Psi_x = \Phi^{\mathrm{nd}}$ and $n \in \mathbf{Z}$, or to $2\psi_x + 2n + 1$. Thus Ψ is obtained by applying Construction 1.3.27 to the finite root system Φ.

(2) Φ is reduced and there are at least two different values of u_a for $a \in \Phi$: By Proposition 1.3.49(3) there must exist exactly two root lengths in Φ. Let us again rescale Ψ to acheive $u_a = 1$ when $a \in \Phi$ is short. When $a \in \Phi$ is long, then $u_a \neq 1$, so $u_a^{-1}a$ has a different length from a. In order for $\overline{\Psi} = \{u_a^{-1}a \mid a \in \Phi\}$ to be a root system, a short $a \in \Phi$ must become a long $a \in \overline{\Psi}$ and a long $a \in \Phi$ must become a short $u_a^{-1}a \in \overline{\Psi}$. This implies $u_a = \langle a, a\rangle$ when $a \in \Phi$ is long, where $\langle -, -\rangle$ is rescaled so that $\langle a, a\rangle = 1$ when $a \in \Phi$ is short. A simple computation shows that $u_{a^\vee} = 2u_a/\langle a, a\rangle$ for all $a \in \Phi$. Therefore the dual system $\Psi^\vee$ falls under case (1) and we see that it is obtained by applying Construction 1.3.27 to the finite root system $\Phi^\vee$.

(3) Φ is non-reduced and there are at least two different values of u_a for $a \in \Phi^{\mathrm{nd}}$: In this case, Φ must be of type BC_n with $n \geqslant 1$. As in (2) we rescale Ψ so that $u_a = 1$ when $a \in \Phi^{\mathrm{nd}}$ is short and conclude that $u_a = 2$ when $a \in \Phi^{\mathrm{nd}}$ is long. Choose an extra special vertex $x \in A$. Then $\nabla : \Psi_x^{\mathrm{nd}} \to \Phi^{\mathrm{nd}}$ is an isomorphism by Proposition 1.3.43(6). Let $a, b \in \Phi^{\mathrm{nd}}$ be orthogonal short roots so that $a \pm b, 2a, 2b \in \Phi$. Let $\alpha, \beta \in \Psi_x^{\mathrm{nd}}$ have gradients a, b respectively. Then $\alpha \pm \beta \in \Psi_x$. By Lemma 1.3.59 also $2\beta + 1 \in \Psi$. Then $r_{2\beta+1}(\alpha + \beta) = \alpha - \beta + 1$, showing $u_{a-b} \leqslant 1$. But $u_{a-b} = 2$ as $a - b \in \Phi^{\mathrm{nd}}$ is long. Thus case (3) cannot exist. □

Construction 1.3.64 Let Ψ be an irreducible affine root system. One can associate to it an affine Dynkin diagram $\widetilde{\mathscr{D}}$ as follows. Let $\mathcal{C}$ be a chamber and let $\Psi(\mathcal{C})^0$ be the corresponding basis as in Definition 1.3.19. The nodes of the affine Dynkin diagram are the elements of $\Psi(\mathcal{C})^0$, and the bonds and arrows are inserted according to the same rules as for finite root systems: two $\psi, \eta \in \Psi(\mathcal{C})^0$, such that $\dot\psi$ is not divisible in Φ, are joined by a bond with multiplicity $f(\psi, \eta) := \dot\psi(\dot\eta^\vee) \cdot \dot\eta(\dot\psi^\vee)$. Hence the only possible values for $f(\psi, \eta)$ are $0, 1, 2, 3, 4$ according to [Bou02, Chapter VI, §1, no. 3]. This is just like for finite root systems, except for the possibility of the value 4, which according to loc. cit. occurs if either $\dot\psi = \pm\dot\eta$ or $\dot\psi = \pm 2, \dot\eta$. Both of these possibilities do

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occur, the first for Ψ_Φ with Φ of type A_1, and the second with Φ of type BC_1. The value 0 indicates the absence of a bond. An arrow is placed on the bond if and only if $\dot\eta(\dot\psi^\vee) \neq \dot\psi(\dot\eta^\vee)$. In that case one of these two numbers has absolute value 1, say without loss of generality $\dot\eta(\dot\psi^\vee)$, and the arrow points towards η.

One can interpret most of this recipe also in terms of a Weyl group invariant scalar product $(-,-)$ on V^*. If the bond between η and ψ has an arrow on it, then the arrow points towards the shorter root (i.e. the root whose norm in terms of the scalar product is smaller), and the multiplicity of the bond equals $(\psi,\psi)/(\eta,\eta)$, assuming η is the shorter root. This however does not apply to bonds of multiplicity 4 without an arrow, because then $(\psi,\psi)/(\eta,\eta)$ is equal to 1, rather than 4.

Note that if Ψ is not reduced, the affine Dynkin diagram is the same as for the reduced subsystem $\Psi^{\mathrm{nd}} = \{\psi \in \Psi \mid \psi/2 \notin \Psi\}$, because $\Psi(\mathcal{C})^0 \subset \Psi^{\mathrm{nd}}$.

Fact 1.3.65 *The affine Dynkin diagram of the dual system $\Psi^\vee$ is obtained from the affine Dynkin diagram of Ψ by inverting all arrows.*

Fact 1.3.66 *Let Ψ be an irreducible reduced affine root system, $\mathcal{C}$ a chamber, and $\mathcal{F}$ a facet contained in its closure. The Dynkin diagram of $\Psi_{\mathcal{F}}$ is obtained from the Dynkin diagram of Ψ by removing all vertices of the facet $\mathcal{F}$ and all edges emanating from them.*

Proposition 1.3.67 *Let Ψ be an irreducible reduced affine root system.*

(1) *The Dynkin diagram of Ψ is among those given in Table 1.3.4, and each diagram in this table is the diagram of some Ψ.*

(2) *The isomorphism class of Ψ is determined by its affine Dynkin diagram. The label of the diagram is called the* type *of Ψ.*

Proof (1) Theorem 1.3.63 shows that either Ψ or $\Psi^\vee$ is produced by Construction 1.3.27. Computing the resulting Dynkin diagram is a simple exercise left to the reader.

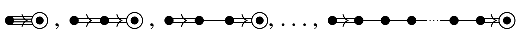
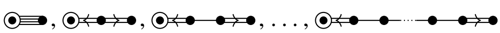
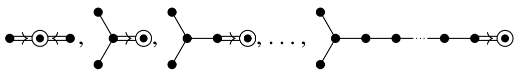
(2) Removing a node from the Dynkin diagram of Ψ produces the Dynkin diagram of the finite root system Ψ_x, where x is the vertex of the chamber $\mathcal{C}$ corresponding to the removed node, cf. Fact 1.3.66. Since Ψ_x is determined by its Dynkin diagram up to isomorphism, the same is true for Ψ. □

Remark 1.3.68 Another way to classify the possible affine Dynkin diagrams is as follows. Let S be the set of simple reflections corresponding to the basis $\Psi(\mathcal{C})^0$ of Ψ and let $W = W(\Psi)$. Since (W,S) is a Coxeter system by Proposition 1.3.20, one can appeal to the classification of Coxeter graphs in [Bou02, Chapter VI, §4, no. 3, Theorem 4]. One has to only replace a bond with label 4 with a double edge with orientation, and a bond with label 6 with a triple edge with orientation.

Theorem 1.3.69 *Let Ψ be an irreducible non-reduced affine root system. Consider the reduced subsystems Ψ^{nd} and Ψ^{nm}. The pair consisting of their types determines the isomorphism class of Ψ. The possibilities are given in Table 1.3.3.*

Proof The finite root system $\Phi = \nabla\Psi$ is irreducible and non-reduced, hence of type BC_n. Thus Φ^{nd} and Φ^{nd} are the subsystems of type B_n and C_n, respectively. Moreover, we have $\Phi^{\mathrm{nd}} \subset \nabla(\Psi^{\mathrm{nd}}) \subset \Phi$ and $\Phi^{\mathrm{nm}} \subset \nabla(\Psi^{\mathrm{nm}}) \subset \Phi$. Therefore $\nabla(\Psi^{\mathrm{nd}})$ is of type B_n or BC_n, showing that Ψ^{nd} is of type B_n, $C_n^\vee$, or BC_n. In the same way, $\nabla(\Psi^{\mathrm{nm}})$ is of type C_n or BC_n, showing that Ψ^{nm} is of type $B_n^\vee$, C_n, or BC_n. But since the Weyl groups of Ψ^{nd} and Ψ^{nm} agree, the only possible options are those listed in Table 1.3.3. □

Table 1.3.3 *The affine Dynkin diagrams of the non-reduced irreducible affine root systems*

Label		Diagrams
(BC_n, C_n)	$(n \geqslant 1)$	
$(C_n^\vee, BC_n)$	$(n \geqslant 1)$	
$(B_n, B_n^\vee)$	$(n \geqslant 2)$	
$(C_n^\vee, C_n)$	$(n \geqslant 1)$	

Remark 1.3.70 In Table 1.3.3 we have listed the types of the non-reduced irreducible affine root systems as discussed in Theorem 1.3.69. Thus the type is a pair (X, Y), with X the Dynkin type of Ψ^{nd} and Y the Dynkin type of Ψ^{nm}. Since the affine Dynkin diagram is the same as that for Ψ^{nd}, and the special and extra special vertices are also the same, instead of recording those, we have recorded in Table 1.3.3 the information about which simple root is multipliable: the non-multipliable simple roots are labeled by a solid node ($\bullet$), while the multipliable simple roots are labeled by a solid node with a circle around it.

In Table 1.3.4 we have recorded the special and extra special vertices as follows. According to Proposition 1.3.22(6) the vertices of a chamber are in

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bijection with the corresponding set of simple roots. We have labeled by an empty node (∘) those simple roots that correspond to extra special vertices. We have labeled by a crossed node (⊗) those simple roots that correspond to vertices that are special, but not extra special. By Proposition 1.3.43 such vertices exist only if the derivative root system Φ is non-reduced. The simple roots that correspond to non-special vertices are labeled by a solid node (•).

Table 1.3.4 *The affine Dynkin diagrams of the reduced irreducible affine root systems*

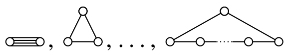
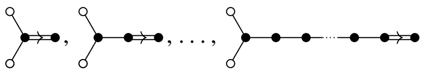
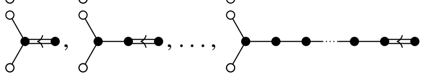
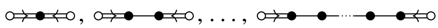
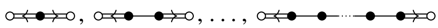
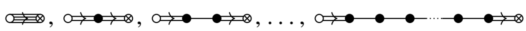
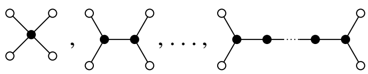
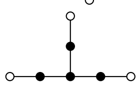
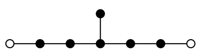
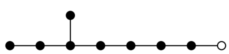
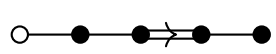
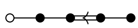
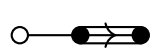
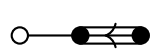
Label		Diagrams
A_n	$(n \geqslant 1)$	
B_n	$(n \geqslant 3)$	
$B_n^\vee$	$(n \geqslant 3)$	
C_n	$(n \geqslant 2)$	
$C_n^\vee$	$(n \geqslant 2)$	
BC_n	$(n \geqslant 1)$	
D_n	$(n \geqslant 4)$	
E_6		
E_7		
E_8		
F_4		
$F_4^\vee$		
G_2		
$G_2^\vee$		

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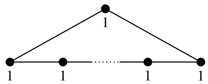
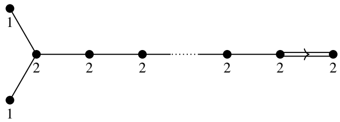
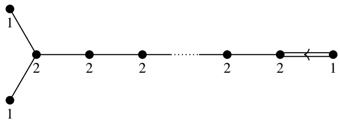
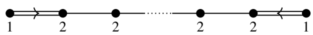
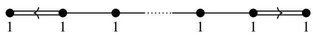
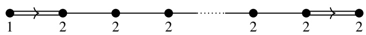
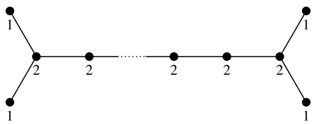
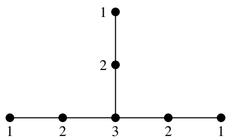
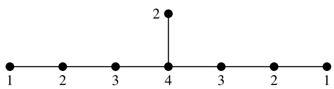
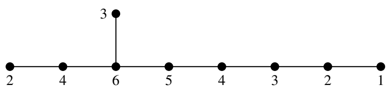
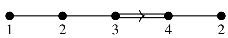
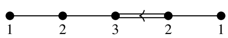
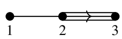
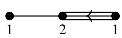
Table 1.3.5 *The integers* (1.3.1) *for the reduced irreducible affine root systems*

Label		Diagram
A_n	$(n \geqslant 1)$	
B_n	$(n \geqslant 3)$	
$B_n^\vee$	$(n \geqslant 3)$	
C_n	$(n \geqslant 2)$	
$C_n^\vee$	$(n \geqslant 2)$	
BC_n	$(n \geqslant 1)$	
D_n	$(n \geqslant 4)$	
E_6		
E_7		
E_8		
F_4		
$F_4^\vee$		
G_2		
$G_2^\vee$		

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Definition 1.3.71 The *extended affine Weyl group* of Ψ is the group $W(\Psi)^{\mathrm{ext}}$ consisting of those automorphisms of the affine space A that preserve Ψ and whose derivative is an element of $W(\Phi)$.

1.3.72 If $\mathcal{C} \subset A$ is any chamber and $\Xi_{\mathcal{C}}$ is the stabilizer of $\mathcal{C}$ in $W(\Psi)^{\mathrm{ext}}$, then Lemma 1.3.17 implies $W(\Psi)^{\mathrm{ext}} = W(\Psi) \rtimes \Xi_{\mathcal{C}}$.

Remark 1.3.73 Consider an irreducible affine root system Ψ. It is clear from Table 1.3.5 that the set of vertices, the information whether or not two vertices are linked, and the sequence of integers (1.3.1), completely determines the affine Dynkin diagram of the reduced subsystem Ψ^{nd}. Table 1.3.3 shows that, if one adds to this the information about which affine simple root is multipliable, then this completely determines both Ψ^{nd} and Ψ^{nm}, hence also Ψ.

Lemma 1.3.74 *Assume that Ψ is reduced.*

(1) *The translation subgroup $W(\Psi)^{\mathrm{ext}} \cap V$ is the lattice*

$$\{v \in V \mid \langle v, a\rangle \in u_a\mathbf{Z} \text{ for all } a \in \Phi^{\mathrm{nd}}\}.$$

(2) *The translation subgroup $W(\Psi)^{\mathrm{aff}} \cap V$ is the lattice generated by the $(u_a/\delta_a)a^\vee$ for $a \in \Phi^{\mathrm{nd}}$, where $\delta_a = 1$ if a is non-multipliable, and $\delta_a = 2$ if a is multipliable.*

Proof (1) follows from Proposition 1.3.49 and Lemma 1.3.59.

(2) follows from Proposition 1.3.61 when Φ is reduced. Since the statement respects direct sums, the only remaining case to check is that of type BC_n, which can be checked by hand. □

Lemma 1.3.75 *Let Ψ be an irreducible affine root system. Assume that it is either reduced, or as in Example 1.3.28. Let $\mathcal{C}$ be a chamber. Then* $\mathrm{Aut}(\mathcal{C}) \subset W(\Psi)^{\mathrm{ext}}$ *acts simply transitively on the set of extra special vertices of $\mathcal{C}$.*

Proof When Φ is reduced this is [Bou02, Chapter VI, §2, no. 3, Proposition 6], where 0 in loc. cit. is by construction a special, hence also extra special, vertex. When Φ is not reduced one checks directly that there is a unique extra special vertex. At the same time $W(\Psi)^{\mathrm{aff}} = W(\Psi)^{\mathrm{ext}}$, so $\mathrm{Aut}(\mathcal{C}) = \{1\}$. □

Remark 1.3.76 Consider an irreducible affine root system $\Psi \subset A^*$ and a chamber $\mathcal{C} \subset A$. Recall the stabilizer $\Xi = \Xi_{\mathcal{C}}$ of $\mathcal{C}$ in $W(\Psi)^{\mathrm{ext}}$. The action of Ξ on $\mathcal{C}$ is faithful, because the action of $W(\Psi)^{\mathrm{ext}}$ on A is faithful by definition of $W(\Psi)^{\mathrm{ext}}$ as a subgroup of affine transformations of A. Since there is a bijection between the set of vertices of $\mathcal{C}$ and the set of simple affine roots (cf. Proposition 1.3.22(6)), hence the set of vertices of the affine Dynkin diagram, one obtains

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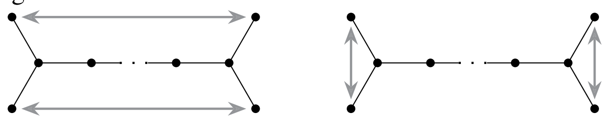
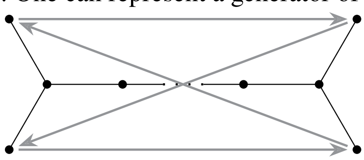
a faithful action of Ξ on the affine Dynkin diagram. With respect to this action, Ξ is realized as a normal subgroup of the full symmetry group of the affine Dynkin diagram, which we shall call $\mathrm{Aut}(\widetilde{\mathscr{D}})$. Let us describe the two groups $\Xi \subset \mathrm{Aut}(\widetilde{\mathscr{D}})$. It is enough to assume that Ψ is reduced, since in the non-reduced cases we have $\Xi = \mathrm{Aut}(\widetilde{\mathscr{D}})$. More generally, if the irreducible root system Φ contains roots of two different lengths or if it is of type A_1, E_7 or E_8, then it does not admit a non-trivial automorphism that preserves a basis, and in these cases we have $\Xi = \mathrm{Aut}(\widetilde{\mathscr{D}})$ (see (1.3.2)).

(1) If Ψ is of type A_1, then $\Xi = \mathrm{Aut}(\widetilde{\mathscr{D}}) = \mathbf{Z}/2\mathbf{Z}$.
(2) If Ψ is of type A_n, $n > 1$, then $\mathrm{Aut}(\widetilde{\mathscr{D}}) = \mathbf{Z}/(n+1)\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$ is the dihedral group of order $2(n+1)$, and Ξ is the subgroup of index 2 consisting of all rotations.
(3) If Ψ is of type B_n, $B_n^\vee$, C_n, $C_n^\vee$, then $\Xi = \mathrm{Aut}(\widetilde{\mathscr{D}}) = \mathbf{Z}/2\mathbf{Z}$.
(4) If Ψ is of type BC_n then $\Xi = \mathrm{Aut}(\widetilde{\mathscr{D}}) = \{1\}$.
(5) If Ψ is of type D_4, then $\mathrm{Aut}(\widetilde{\mathscr{D}}) = S_4$, and Ξ is the unique Sylow-2 subgroup of A_4, hence isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$. One can represent two of its generators as follows:

(6) If Ψ is of type D_{2n} with $n > 2$, then $\mathrm{Aut}(\widetilde{\mathscr{D}}) = (\mathbf{Z}/2\mathbf{Z})^2 \rtimes \mathbf{Z}/2\mathbf{Z}$ is generated by the following three automorphisms of order 2: switch the two left nodes and fix all others, switch the two right nodes and fix all others, switch the left and right branches. In fact, there are two distinct automorphisms of order 2 that switch the left and right branches; they commute and their product is the unique central element in the symmetry group of the affine Dynkin diagram. These two automorphisms generate the subgroup Ξ, which is hence isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$. One can represent two of its generators as follows:



(7) If Ψ is of type D_{2n-1}, $n > 2$, then again $\mathrm{Aut}(\widetilde{\mathscr{D}}) = (\mathbf{Z}/2\mathbf{Z})^2 \rtimes \mathbf{Z}/2\mathbf{Z}$ with the same description as for D_{2n}. The subgroup Ξ is the unique subgroup isomorphic to $\mathbf{Z}/4\mathbf{Z}$. One can represent a generator of it as follows:



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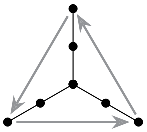
(8) If Ψ is of type E_6, then $\mathrm{Aut}(\widetilde{\mathscr{D}}) = S_3 = (\mathbf{Z}/3\mathbf{Z}) \rtimes \mathbf{Z}/2\mathbf{Z}$ is the dihedral group of order 6 and Ξ is the unique subgroup $\mathbf{Z}/3\mathbf{Z}$. One can represent a generator of it as follows:



(9) If Ψ is of type E_7, then $\Xi = \mathrm{Aut}(\widetilde{\mathscr{D}}) = \mathbf{Z}/2\mathbf{Z}$.

(10) If Ψ is of type E_8, F_4, $F_4^\vee$, G_2, $G_2^\vee$, then $\Xi = \mathrm{Aut}(\widetilde{\mathscr{D}}) = \{1\}$.

More information about Ξ can be found in [Ree10, §3.6] and [Bou02, Chapter VI, §2, no. 3]. Note in particular that, when $\Psi = \Psi_\Phi$ for an irreducible reduced finite root system Φ, and x is a special node in the affine Dynkin diagram, then the map sending $\omega \in \Xi$ to ωx is a bijection between the group Ξ and the set of special nodes. Removing from $\widetilde{\mathscr{D}}$ the node x and all edges emanating from x we obtain the Dynkin diagram $\mathscr{D}$ of Φ. If $\mathrm{Aut}(\widetilde{\mathscr{D}})_x$ is the stabilizer of x in $\mathrm{Aut}(\widetilde{\mathscr{D}})$, then $\mathrm{Aut}(\mathscr{D}) = \mathrm{Aut}(\widetilde{\mathscr{D}})_x$ and $\mathrm{Aut}(\widetilde{\mathscr{D}}) = \Xi \rtimes \mathrm{Aut}(\widetilde{\mathscr{D}})_x$. The extension

$$1 \to \Xi \to \mathrm{Aut}(\widetilde{\mathscr{D}}) \to \mathrm{Aut}(\mathscr{D}) \to 1 \tag{1.3.2}$$

obtained in this way does not depend on the choice of x, and the choice of x gives a splitting of this extension. Another description of the map $\mathrm{Aut}(\widetilde{\mathscr{D}}) \to \mathrm{Aut}(\mathscr{D})$, which does not involve the choice of x, can be given as follows. The group $\mathrm{Aut}(\widetilde{\mathscr{D}})$ is the quotient of the automorphism group of Ψ by the affine Weyl group $W(\Psi)$; the group $\mathrm{Aut}(\mathscr{D})$ is the quotient of the automorphism group of $\Phi = \nabla\Psi$ by the finite Weyl group $W(\Phi)$; the derivative map $\mathrm{Aut}(\Psi) \to \mathrm{Aut}(\Phi)$ induces the homomorphism $\mathrm{Aut}(\widetilde{\mathscr{D}}) \to \mathrm{Aut}(\mathscr{D})$.

Recall from [Bou02, Chapter VI, §1, no. 7, Definition 4] the notions of a closed, parabolic, and symmetric subset X of a root system Φ: it is *symmetric* if $-X = X$; *closed* if $a, b \in X$ and $a + b \in \Phi$ implies $a + b \in X$; and *parabolic* if it is closed and $X \cup -X = \Phi$. A closed symmetric subset is the same as a closed subroot system; cf. [Bou02, Chapter VI, §1, no. 7, Proposition 23].

Lemma 1.3.77 *Let A be an affine space over V, $\Psi \subset A^*$ an affine root system, and $\Phi' \subset \Phi$ a closed symmetric subset. Let $W \subset V$ be the subspace annihilated by Φ'. Then $\Psi' = \{\psi \in \Psi \mid \dot\psi \in \Phi'\}$ is an affine root system in $(A/W)^*$. It is reduced if Ψ is.*

Proof Since $\Psi' \subset \Psi$ we know that Ψ' does not contain 0. The subspace of V^* generated by Φ' is $W^\perp = (V/W)^*$. For each $a \in \Phi$ the set $\{\psi \in \Psi \mid \dot\psi = a\}$ is a free abelian group of rank 1 (cf. [Mac72, Proposition 6.9]), therefore Ψ' generates $(A/W)^*$, hence satisfies Axiom AR 1. For $\psi \in \Psi'$ the reflection $r_{\dot\psi}$

preserves Φ', therefore Axiom AR 2 for Ψ' follows from Axiom AR 2 for Ψ. The remaining axioms for Ψ' follow immediately from those for Ψ. □

1.4 Tits Systems

In this section we review the notion of a Tits systems and some of its properties. Tits systems are very closely related to Tits buildings, a notion reviewed in the next section.

Definition 1.4.1 A *Tits system* is a tuple (G, B, N, S) consisting of a group G, two subgroups B and N of G, and a subset S of $N/(B \cap N)$, subject to the following axioms.

TS 1 The set $B \cup N$ generates G and $B \cap N$ is a normal subgroup of N.

TS 2 The set S generates the group $W = N/(B \cap N)$ and consists of elements of order 2.

For $w \in W$ and n any lift of w in N, we define $wB := nB$ and $Bw := Bn$. These are well-defined cosets of B in G.

TS 3 Given $s \in S$ and $w \in W$ one has $sBw \subset BwB \cup BswB$.

TS 4 Given $s \in S$ one has $sBs \neq B$.

The system is called *saturated*, if in addition the following axiom holds.

TS 5 $\bigcap_{n \in N} nBn^{-1} = B \cap N$.

Remark 1.4.2 Set $T = B \cap N$. The group $W = N/T$ is called the *Weyl group* of the Tits system. According to [Bou02, Chapter IV,§2,no. 5, Corollary] the set S consists precisely of those non-trivial elements of W for which the set $B \cup BwB$ is a subgroup of G. Since G will usually be fixed, we may also refer to the pair (B, N) as a Tits system. Sometimes this pair is called a *BN-pair*, but this can cause confusion when its members are not called B and N.

Example 1.4.3 If G is a connected reductive group over a field k, P is a minimal parabolic k-subgroup, and N the normalizer of a maximal k-split torus S contained in P, then $(P(k), N(k))$ is a Tits system in $G(k)$ with finite Weyl group; see [Bor91, Theorem 21.15]. The role of T is then played by $M(k)$, where M is the centralizer of S in P, equivalently in G; it is a Levi k-subgroup of P. This is usually called the standard Tits system of $G(k)$. We may call it the *spherical Tits system*; see Example 1.5.11. Remarkably, when k is infinite, any Tits system in $G(k)$ with finite Weyl group that satisfies a mild natural condition is the spherical Tits system for some choice of P and N; see [Pra14, Theorem B].

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In this book we will be primarily concerned with another fundamental example of a Tits system. For this the abstract group G will be a certain subgroup $G(k)^0$ of the group $G(k)$ of k-points of a connected reductive group G over a discretely valued Henselian field k, and the role of B will be played by a certain bounded subgroup of $G(k)^0$, called an Iwahori subgroup. The Weyl group of this Tits system will be infinite.

Definition 1.4.4 Let (G, B, N, S) be a Tits system.

(1) For any subset $X \subset S$ let $W_X \subset W$ be the subgroup generated by X and let $G_X = BW_XB$. The group G_X is called a *standard parabolic subgroup*.
(2) Any subgroup of G containing a conjugate of B is called a *parabolic* subgroup.

We have the following properties of Tits systems; see [Bou02, Chapter IV, §2] and the summary in [Tit74, §3.2].

Proposition 1.4.5
(1) *(Bruhat Decomposition) The map $w \mapsto BwB$ is a bijection from the Weyl group W to the set of B-double cosets in G. In particular, $G = BWB$.*
(2) *Any parabolic subgroup is conjugate to a unique standard parabolic subgroup.*
(3) *Each parabolic subgroup is equal to its normalizer.*
(4) *Let Q be a subgroup of G that contains two parabolic subgroups Q_1 and Q_2 of G. Then any $g \in G$ such that $gQ_1g^{-1} = Q_2$ belongs to Q.*

Definition 1.4.6 Let $P \subset G$ be a parabolic subgroup. The subset $X \subset S$ such that P is conjugate to $G_X = BW_XB$ is called the *type* of P.

According to [Bou02, Chapter IV, §2, no. 4, Theorem 2], the tuple (W, S) is a Coxeter system in the sense of [Bou02, Chapter IV, §1, no. 3, Definition 3]. Recall from the end of [Bou02, Chapter IV, §1] that a Coxeter system (W, S) is called *irreducible* if one cannot write S as a disjoint union $S = S_1 \cup S_2$ of two non-empty subsets such that each element of S_1 commutes with each element of S_2; equivalently the Coxeter graph of (W, S) is connected and non-empty. We call (G, B, N, S) irreducible if its Coxeter system is irreducible. More generally we will be interested in Tits systems for which (W, S) may not be irreducible, but S is finite. There exists a unique smallest disjoint union decomposition $S = S_1 \cup \cdots \cup S_n$ such that for $i \neq j$ each element of S_i commutes with each element of S_j. Then $W = \prod_i W_i$, where W_i is the subgroup of W generated by S_i; see [Bou02, Chapter IV, §1, no. 9, Proposition 8]. Each (W_i, S_i) is an irreducible Coxeter system, and the graphs of (W_i, S_i) are the irreducible components of the graph of (W, S). The groups G, B, and N, need not have an analogous direct

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product decomposition. However, the set of parabolic subgroups of (G, B, N, S) does have such a decomposition, as we will now discuss.

Let (G, B, N, S) be a Tits system and let $S = S_1 \cup S_2$ be a disjoint union such that each element of S_1 commutes with each element of S_2. Write N_i for the preimage of W_i in N. Then N_1 and N_2 normalize each other, their intersection is T, and their product is N.

Lemma 1.4.7 $G_{S_i} \cap N = N_i$.

Proof By construction $G_{S_i} = BN_iB$, so $N_i \subset G_{S_i} \cap N$. At the same time, $G_{S_1} \cap N = G_{S_1} \cap (N_1 \cdot N_2) = N_1 \cdot (G_{S_1} \cap N_2) \subset N_1 \cdot (G_{S_1} \cap G_{S_2} \cap N) = N_1 \cdot (B \cap N) = N_1$ and analogously $G_{S_2} \cap N \subset N_2$. □

Lemma 1.4.8 *(G, G_{S_1}, N, S_2) is a Tits system with Weyl group W_2.*

Proof Since $B \cup N$ generates G, so does $G_{S_1} \cup N$. The group $G_{S_1} \cap N$ equals N_1 by Lemma 1.4.7 and is thus normal in N. The quotient N/N_1 is isomorphic to $N_2/(N_1 \cap N_2) = N_2/T = W_2$. To verify Axiom TS 3 we first claim that for any $w \in W$ we have $W_1 Bw \subset BW_1 wB$. Since S_1 generates W_1 this is equivalent to $s_1^{(1)} \cdots s_1^{(n)} Bw \subset W_1 wB$ for $s_1^{(1)}, \ldots, s_1^{(n)} \in S_1$. We work by induction on n starting with the trivial case $n = 0$. For the induction step we compute using Axiom TS 3 for the system (G, B, N, S) and the induction hypothesis that

$$\begin{aligned} s_1^{(1)} \cdots s_1^{(n)} Bw &\subset (s_1^{(1)} \cdots s_1^{(n-1)} BwB) \cup (s_1^{(1)} \cdots s_1^{(n-1)} Bs_1^{(n)} wB) \\ &\subset BW_1 wB \cup BW_1 s_1^{(n)} wB \\ &= BW_1 wB. \end{aligned}$$

The claim is proved. We now check Axiom TS 3 for (G, G_{S_1}, N, S_2) by taking $s_2 \in S_2$, $w_2 \in W_2$, and computing

$$\begin{aligned} s_2 G_{S_1} w_2 &= s_2 BW_1 Bw_2 \subset s_2 BW_1 w_2 B \\ &= s_2 Bw_2 W_1 B \subset Bw_2 W_1 B \cup Bs_2 w_2 W_1 B \\ &\subset G_{S_1} w_2 G_{S_1} \cup G_{S_1} s_2 w_2 G_{S_1}. \end{aligned}$$

Finally let $s \in S$ and let $n \in N$ be a lift. If $nG_{S_1} n^{-1} = G_{S_1}$ then Proposition 1.4.5 and Lemma 1.4.7 imply $n \in N_1$. Thus if $s \in S_2$ then $sG_{S_1} s \neq G_{S_1}$, hence Axiom TS 4 holds for (G, G_{S_1}, N, S_2). □

Let $S_i^c = S - S_i$. Of course we have $S_1^c = S_2$ and $S_2^c = S_1$. Let $\mathcal{P}$ be the set of parabolic subgroups of the Tits system (G, B, N, S) and let $\mathcal{P}_i$ be the set of parabolic subgroups of the Tits system $(G, G_{S_i^c}, N, S_i)$. There is a tautological order-preserving G-equivariant inclusion $\iota_i \colon \mathcal{P}_i \to \mathcal{P}$, defined by $\iota_i(P_i) = P_i$. We have $\operatorname{type}_{\mathcal{P}}(\iota_i(P_i)) = \operatorname{type}_{\mathcal{P}_i}(P_i) \cup S_i^c$,

Lemma 1.4.9 *Let $P \in \mathcal{P}$. Then $\bigcap_{P \subset P_i \in \mathcal{P}_i} P_i$ is an element of $\mathcal{P}_i$, which we shall call $\pi_i(P)$. If $P = G_X$ with $X \subset S$ then $\pi_i(P) = G_{X \cup S_i^c}$. In particular,* $\mathrm{type}_{\mathcal{P}_i}(\pi_i(P)) = \mathrm{type}_{\mathcal{P}}(P) \cap S_i$. *The map π_i is an order-preserving G-equivariant map $\mathcal{P} \to \mathcal{P}_i$ that is a section of the inclusion $\iota_i : \mathcal{P}_i \to \mathcal{P}$.*

Proof To each $P \in \mathcal{P}$ we can assign the set of all elements of $\mathcal{P}_i$ that contain P. The assignment of this set to P is equivariant for the action of G by conjugation on $\mathcal{P}$ and $\mathcal{P}_i$, respectively. Therefore we may assume that P is standard, say $P = G_X$ for some $X \subset S$. We claim that any $P_i \in \mathcal{P}_i$ containing G_X must also contain $G_{X \cup S_i^c}$. Indeed, $P_i \in \mathcal{P}_i$ is equivalent to the existence of $g \in G$ such that $gG_{S_i^c}g^{-1} \subset P_i$. Therefore $gBg^{-1} \subset P_i$. At the same time $B \subset G_X \subset P_i$. Thus B is contained in both P_i and $g^{-1}P_i g$. Proposition 1.4.5 implies $g \in P_i$. Therefore $P_{S_i^c} \subset P_i$ and $G_X \subset P_i$, from which the claim follows. But the claim immediately implies that $G_{X \cup S_i^c}$ is the intersection of all elements of $\mathcal{P}_i$ containing G_X. □

Proposition 1.4.10 *The map $\pi : \mathcal{P} \to \mathcal{P}_1 \times \mathcal{P}_2$ sending P to $(\pi_1(P), \pi_2(P))$ is an order-preserving G-equivariant bijection. It satisfies* $\mathrm{type}_{\mathcal{P}_1}(\pi_1(P)) \cup \mathrm{type}_{\mathcal{P}_2}(\pi_2(P)) = \mathrm{type}_{\mathcal{P}}(P)$. *Its inverse is given by $(P_1, P_2) \mapsto P_1 \cap P_2$.*

Proof The G-equivariance is clear. Using it, injectivity is reduced to the claim $G_X = G_{X \cup S_1^c} \cap G_{X \cup S_2^c}$. But for any two subsets $X_1, X_2 \subset X$ we have $G_{X_1} \cap G_{X_2} = G_{X_1 \cap X_2}$, and the claim is immediate.

To prove surjectivity and the claim about the inverse, consider a pair (P_1, P_2). We claim that $P_1 \cap P_2 \in \mathcal{P}$. Again by G-equivariance we are free to conjugate both P_1 and P_2 by the same element of G. Since both P_1, P_2 are parabolic subgroups of (G, B, N, S) we may assume, after conjugating both by an element of G, that at least P_1 contains B, therefore $P_1 = G_{X_1}$ for some $S_1^c \subset X_1 \subset S$. Let $g \in G$ be such that gP_2g^{-1} contains B and thus equals G_{X_2} for some $S_2^c \subset X_2 \subset S$. Using Proposition 1.4.5 write $g = b_1 n b_2$ with $b_1, b_2 \in B$ and $n \in N$. Write $n = n_1 n_2$ with $n_i \in N_i$. Then $P_2 = g^{-1}G_{X_2}g = b_2^{-1}n_2^{-1}G_{X_2}n_2 b_2$ contains $b_2^{-1}n_2^{-1}Bn_2b_2$. Since $n_2, b_2 \in G_{X_1}$, the latter also contains $b_2^{-1}n_2^{-1}Bn_2b_2$. We conclude that $P_1 \cap P_2$ contains a conjugate of B and therefore lies in $\mathcal{P}$, and the claim is proved.

Let now $P = P_1 \cap P_2$. We want to show $P_i = \pi_i(P)$. Again by G-conjugation we may assume that P, hence also P_1 and P_2, contain B. Thus $P_i = G_{X_i}$ with $S_i^c \subset X_i \subset S$ and then $P = G_{X_1 \cap X_2}$, while $\pi_i(P) = G_{(X_1 \cap X_2) \cup S_i^c}$. We want to show $(X_1 \cap X_2) \cup S_i^c = X_i$. The case $i = 2$ is shown by $(X_1 \cap X_2) \cup S_1 \subset X_2 = S_1 \cup (S_2 \cap X_2) \subset S_1 \cup (X_1 \cap X_2)$ and the case $i = 1$ is entirely analogous. □

Example 1.4.11 Let (G_i, B_i, N_i, S_i) for $i = 1, 2$ be two Tits systems. Let $\mathcal{P}_i'$ be the corresponding sets of parabolic subgroups. Set $G = G_1 \times G_2$, $B = B_1 \times B_2$,

$N = N_1 \times N_2$, $S = S_1 \cup S_2$. Then (G, B, N, S) is also a Tits system. The set $\mathcal{P}$ of parabolic subgroups of that Tits system is $\{P_1 \times P_2 | P_i \in \mathcal{P}'_i\} \simeq \mathcal{P}'_1 \times \mathcal{P}'_2$.

The Tits system (G, G_{S_1}, N, S_2) of Lemma 1.4.8 is given by

$$(G_1 \times G_2, G_1 \times B_2, N_1 \times N_2, S_2).$$

As above let $\mathcal{P}_2$ be the set of parabolic subgroups of that Tits system. The map $P_2 \mapsto G_1 \times P_2$ is an order-preserving bijection $\psi_2 : \mathcal{P}'_2 \to \mathcal{P}_2$. The map $\pi_2 \colon \mathcal{P} \to \mathcal{P}_2$ of Lemma 1.4.9 sends $P_1 \times P_2$ to $G_1 \times P_2$. Thus $\psi_2 \circ \pi_2 \colon \mathcal{P} = \mathcal{P}'_1 \times \mathcal{P}'_2 \to \mathcal{P}'_2$ is the natural projection.

The bijection $\pi \colon \mathcal{P} \to \mathcal{P}_1 \times \mathcal{P}_2$ of Proposition 1.4.10 sends $P = P_1 \times P_2$ to $(P_1 \times G_2, G_1 \times P_2)$. Therefore the composition $(\psi_1, \psi_2) \circ \pi \colon \mathcal{P} \to \mathcal{P}'_1 \times \mathcal{P}'_2$ is the identity map.

Next we will discuss ways to modify a Tits system while preserving the set of parabolic subgroups. Our guiding example is that of the spherical Tits system for a connected reductive group G and the maps $G \to G_{\mathrm{ad}}$ and $G_{\mathrm{sc}} \to G$, where G_{ad} is the adjoint group of G and G_{sc} is the simply connected cover of the derived subgroup of G.

Lemma 1.4.12 *Let (G, B, N, S) be a Tits system.*

(1) *Let $Z \subset T$ be a subgroup that is normal in G. Set $G' = G/Z$, $B' = B/Z$, $N' = N/Z$. Then (G', B', N', S) is a Tits system with the same Weyl group as (G, B, N, S). It is saturated if (G, B, N, S) is.*

(2) *Let $G \to G'$ be an inclusion with normal image, $T' \subset G'$ a subgroup normalizing B and N and normalized by N such that $G' = GT'$ and $T' \cap G = T$. Set $B' = BT'$ and $N' = NT'$. Then (G', B', N', S) is a Tits system with the same Weyl group as (G, B, N, S). It is saturated if (G, B, N, S) is.*

Proof Consider (1). It is immediate that B/Z and N/Z generate G/Z and that $B/Z \cap N/Z = T/Z$ is normal in N/Z. We have $(N/Z)/(B/Z \cap N/Z) = (N/Z)/(T/Z) = N/T$. The inclusion

$$s(B/Z)w \subset (B/Z)w(B/Z) \cup (B/Z)sw(B/Z)$$

is also immediate. If we assume $s(B/Z)s = B/Z$, then taking preimage in G we obtain $sBs = B$, a contradiction. Assume (G, B, N, S) is saturated. Since $\bigcap_n n(B/Z)n^{-1}$ is the image of $\bigcap_n nBn^{-1}$ under $G \to G/Z$, it equals $(B \cap N)/Z = T/Z = B/Z \cap N/Z$.

Consider (2). It is immediate that B' and N' generate G'. Using $G \cap T' = T$ we see that for any collection $(A_i)_i$ of subgroups $T \subset A_i \subset G$ we have $\bigcap_i (A_i T') = (\bigcap_i A_i)T'$. Indeed, an element of the left-hand side is given by a collection $a_i \in A_i$ and $t'_i \in T'$ such that $a_i t'_i = a_j t'_j$ for all i, j. Fix one index i.

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For each index j we have $a_i t_i' = a_j t_j'$, thus $a_j^{-1} a_i = t_j' t_i'^{-1} \in G \cap T' = T$, so we can write $t_j' = t_j t_i'$ with $t_j \in T$, and hence $a_j t_j' = (a_j t_j) t_i'$. Replacing each a_j by $a_j t_j$ we obtain $a_i = a_j$ for all j, thus an element of the right-hand side.

In particular $B' \cap N' = (B \cap N)T' = TT' = T'$, normal in N'. We have $N'/(B' \cap N') = NT'/T' = N/(N \cap T') = N/T = W$. We have $sB'w = sBT'w = sBwT' \subset BwBT' \cup BswBT' \subset B'wB' \cup B'swB'$. If $sB's = B'$, then $sBsT' = BT'$. Intersecting with G and using $G \cap T' = G$ we obtain $sBs = B$, a contradiction. Assume (G, B, N, S) is saturated. We have $\bigcap nB'n^{-1} = (\bigcap_n nBn)T' = TT' = T'$. ◻

Lemma 1.4.13 *Let (G, B, N, S) and (G', B', N', S') be Tits systems. Assume that (G, B, N, S) is saturated. Let $f\colon G \to G'$ be a group homomorphism mapping B to B' and N to N'. Assume that*

(1) $\ker(f)$ *is contained in T,*
(2) *$f(G)$ is normal in G' and $G'/f(G)$ is abelian,*
(3) *T' normalizes $f(B)$,*
(4) *$B' = f(B) \cdot T'$, $N' = f(N) \cdot T'$.*

Then

(1) *$G' = T' f(G) = f(G)T'$,*
(2) *$f^{-1}(T') = T$,*
(3) *T' normalizes $f(BwB)$ for any $w \in W$,*
(4) *the map $W \to W'$ induced by f is an isomorphism that carries S bijectively onto S'.*

Proof Since $B' = f(B)T'$ and $N' = f(N)T'$ generate G', so do T' and $f(G)$. Moreover, since $f(G)$ is normal in G', this reduces to $G' = T' f(G) = f(G)T'$.

We claim that $f^{-1}(T') \subset B$. Let $g \in G$ be such that $f(g) \in T'$. Since T' normalizes $f(B)$ we have $f(B) = f(gBg^{-1})$. Since the kernel of f is contained in B and is normal in G, it is contained in both B and gBg^{-1}, hence we have $B = gBg^{-1}$, which implies $g \in B$ and the claim is proved. Together with $B' = f(B)T'$ this implies $f^{-1}(B') = B$. We can apply this argument to the Tits system $(G, nBn^{-1}, N, nSn^{-1})$ for any $n \in N$ and conclude $f^{-1}(T') \subset \bigcap_{n \in N} nBn^{-1}$. In particular, if (G, B, N, S) is saturated, then $f^{-1}(T') = T$.

The surjectivity of the map $W \to W'$ induced by f is immediate from the assumption $N' = f(N) \cdot T'$, while its injectivity is immediate from $f^{-1}(T') = T$.

Next we claim that for any $w \in W$ the subset $f(BwB)$ of G' is normalized by T'. Indeed, letting $n' \in N'$ represent $f(w)$ and taking $t' \in T'$ we see $t' f(B) n' f(B) t'^{-1} = f(B)(t' n' t'^{-1} n'^{-1}) n' f(B)$, using that t' normalizes $f(B)$. The commutator $t' n' t'^{-1} n'^{-1}$ vanishes in $G'/f(G)$ since that quotient is assumed abelian, and hence lies in $T' \cap f(G) = f(T)$, and the claim is proved.

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Next we show that the isomorphism $W \to W'$ maps S bijectively to S'. Given $s \in S$ we know that $B \cup BsB$ is a subgroup of G. Let $s' = f(s)$. Then $f(B \cup BsB) = f(B) \cup f(BsB)$ is a subgroup of G'. Since T' normalizes both $f(B)$ and $f(BsB)$, we have that

$$T'(f(B) \cup f(BsB)) = T'f(B) \cup T'f(B)s'f(B) = B' \cup B's'B'$$

is a subgroup of G', hence $s' \in S'$.

Conversely, let $w \in W$ be such that its image in W' lies in S'. Thus $B' \cup B'w'B'$ is a subgroup of G' and by the above equation equals $T'(f(B) \cup f(BwB))$. The following two elementary facts, valid for any homomorphism $f : G \to G'$ of groups, imply that $B \cup BwB$ is a subgroup of G, hence $w \in S$.

Fact 1: If a subset $X \subset G$ is stable under left multiplication by $\ker(f)$ and $f(X)$ is a subgroup of G', then X is a subgroup of G.

Fact 2: If a subset $X' \subset f(G)$ is normalized by a subgroup $T' \subset G'$ and stable under left multiplication by $T' \cap f(G)$, and if $T' \cdot X'$ is a subgroup of G', then X' is a subgroup of G'.

The proofs of these facts are immediate and left to the reader. □

Lemma 1.4.14 *Let (G, B, N, S) and (G', B', N', S') be Tits systems. Let $f : G \to G'$ be a group homomorphism satisfying the assumptions of Lemma 1.4.13. If $P \subset G$ is a parabolic subgroup, then so is $P' = N_{G'}(f(P)) = f(P) \cdot T'$. The maps*

$$P \mapsto f(P) \cdot T', \qquad P' \mapsto f^{-1}(P')$$

are type-preserving, order-preserving, f-equivariant, mutually inverse bijections between the sets of parabolic subgroups.

Proof First we prove that the two maps $P \mapsto T'f(P)$ and $P' \mapsto f^{-1}(P')$ induce mutually inverse type-preserving bijections between the sets of standard parabolic subgroups. In fact, the bijection $S \to S'$ induced by f as in Lemma 1.4.13 already establishes such a bijection, under which the standard parabolic subgroups BW_XB of G and $B'W'_{X'}B'$ of G' correspond, when $X \subset S$ and $X' \subset S'$ correspond to each other. So we just need to check that the above maps recover the two directions of this bijection, which is immediate from $T'f(BW_XB) = T'f(B)W'_{X'}f(B) = B'W'_{X'}B'$ and $f^{-1}(T'f(P)) = f^{-1}(T')P = TP = P$, the latter by Lemma 1.4.13. That these bijections are order reversing is then also clear.

Let us now check that $T'f(P) = N_{G'}(f(P))$. Lemma 1.4.13 states that $G' = T'f(G)$ and that T' normalizes $f(P)$, which in turn implies that $N_{G'}(f(P)) = N_{T'f(G)}(f(P)) = T'N_{f(G)}(f(P))$. Now $f(P)$ is a parabolic subgroup for the Tits system $(f(G), f(B), f(N), S)$, hence $N_{f(G)}(f(P)) = f(P)$, and we conclude $N_{G'}(f(P)) = T'f(P)$ as desired.

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Finally, since B' and N' generate G', the assumptions $B' = f(B)T'$ and $N' = f(N)T'$ imply that T' and $f(G)$ generate G', which by the normality of $f(G)$ in G' implies $G' = f(G)T'$. Therefore any G'-conjugate of B' is also an $f(G)$-conjugate of B'. Since the two maps $P \mapsto N_{G'}(f(P))$ and $P' \mapsto f^{-1}(P')$ are equivariant under conjugation by G and $f(G)$, respectively, the proof is complete. □

1.5 Abstract Buildings

In this section we review the notion of a Tits building, which will be essential for our construction of the Bruhat–Tits building of a reductive group defined over a discretely valued Henselian field. Tits buildings are very closely related to Tits systems. This relationship is explored in [Tit74, §3.2], [BT72, §2], and the exercises to Chapter IV in [Bou02]. We give here just a brief summary.

We alert the reader that for a given Tits system one can define two buildings. They are closely related, but often distinct. The usual building, introduced by Tits, is reviewed here in Proposition 1.5.6. It is always a simplicial complex, even when the Tits system is not irreducible. A slight variant of it, which we call the "restricted building," is given in Proposition 1.5.18. The two buildings coincide when the Tits system is irreducible, but not otherwise. The restricted building of a Tits system that is not irreducible is a polysimplicial complex. The Bruhat–Tits building of a reductive group will be the restricted building associated to a particular Tits system.

Definition 1.5.1 (1) A *simplicial complex* is a pair $(V, \mathcal{B})$ consisting of a non-empty set V and a non-empty set $\mathcal{B}$ of non-empty finite subsets of V. We call V the set of *vertices*. We require $\{x\} \in \mathcal{B}$ for all $x \in V$, and further that $\varnothing \neq A \subset B \in \mathcal{B}$ implies $A \in \mathcal{B}$. Abusing notation, we will refer to $\mathcal{B}$ as the simplicial complex, and to V as the underlying set of vertices; see Remark 1.5.4.

(2) A *polysimplicial complex* $\mathcal{B}$ is a tuple $(\mathcal{B}_1, \dots, \mathcal{B}_n)$ of simplicial complexes. We set $V = V_1 \times \cdots \times V_n$ and $\mathcal{B} = \mathcal{B}_1 \times \cdots \times \mathcal{B}_n$. Abusing notation, we refer to $\mathcal{B}$ as the polysimplicial complex.

Let $\mathcal{B}$ be a simplicial or polysimplical complex.

(3) An element of $\mathcal{B}$ is called a *facet*. If $\mathcal{B}$ is a simplicial complex, it is also called a *simplex*.
(4) If $A, B \in \mathcal{B}$ and $A \subset B$, then A is called a *face* of B.
(5) If A is a face of B, we define $\operatorname{codim}(A, B)$ to be the largest n for which there exists a chain $A = A_0 \subsetneq A_1 \subset \cdots \subsetneq A_n = B$ and $A_i \in \mathcal{B}$.
(6) For $B \in \mathcal{B}$ we define $\dim(B)$ to be $\sup\{\operatorname{codim}(A, B) | A \in \mathcal{B}, A \subset B\}$.

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(7) A *subcomplex* of $\mathcal{B}$ is a subset $\mathcal{B}' \subset \mathcal{B}$ such that if $A \in \mathcal{B}'$, $B \subset A$, and $B \in \mathcal{B}$, then $B \in \mathcal{B}'$.
(8) The (open) *star* of $A \in \mathcal{B}$ is the complex consisting of all facets of $\mathcal{B}$ that contain A. It need not be a subcomplex.
(9) A *chamber complex* is a polysimplicial complex in which every element is contained in a maximal element, and given two maximal elements C, C' there exists a sequence $C = C_1, \dots, C_n = C'$ such that $C_i \cap C_{i+1} \in \mathcal{B}$ and $\mathrm{codim}(C_i \cap C_{i+1}, C_i) = \mathrm{codim}(C_i \cap C_{i+1}, C_{i+1}) = 1$ for all $i = 1, \dots, n-1$.
(10) The maximal elements in a chamber complex are called *chambers*, and a sequence $C = C_1, \dots, C_n = C'$ as above is called a *gallery joining C and C′*.
(11) A chamber complex is called *thick*, if each facet of codimension 1 is the face of at least three chambers. It is called *thin*, if each facet of codimension 1 is the face of exactly two chambers.
(12) An *isomorphism* $(V_1, \mathcal{B}_1) \to (V_2, \mathcal{B}_2)$ *of simplicial complexes* is a bijection $f : V_1 \to V_2$ such that $f(A_1) \in \mathcal{B}_2$ for all $A_1 \in \mathcal{B}_1$ and $f^{-1}(A_2) \in \mathcal{B}_1$ for all $A_2 \in \mathcal{B}_2$.
(13) An *isomorphism* $(\mathcal{B}_1, \dots, \mathcal{B}_n) \to (\mathcal{B}'_1, \dots, \mathcal{B}'_n)$ *of polysimplicial complexes* is a tuple $(\sigma, f_1, \dots, f_n)$, where σ is a permutation of $\{1, \dots, n\}$ and $f_i : \mathcal{B}_i \to \mathcal{B}'_{\sigma(i)}$ is an isomorphism of simplicial complexes.
(14) An *isomorphism of chamber complexes* is an isomorphism of (poly) simplicial complexes that maps chambers to chambers.

Remark 1.5.2 If $\mathcal{B}$ is a simplicial complex then $\dim(A) = \#A - 1$ and $\mathrm{codim}(A, B) = \#B - \#A$.

Remark 1.5.3 We have specifically required that the empty subset of V not be an element of a simplicial complex. This is not always done in the literature. This choice is more convenient for our purposes.

Remark 1.5.4 Let $(V, \mathcal{B})$ be a (poly)simplicial complex. Then the inclusion of subsets of V endows the set $\mathcal{B}$ with an order. One can recover V from the set $\mathcal{B}$ and that order relation: if we identify $v \in V$ with the singleton set $\{v\}$ then V is the subset of minimal elements in $\mathcal{B}$. This gives another way to think of a (poly)simplicial complex, as an ordered set subject to certain axioms, namely those translated from the axioms imposed on the pair $(V, \mathcal{B})$ above.

Definition 1.5.5 A *building* is a chamber complex $\mathcal{B}$ equipped with a collection of subcomplexes, called *apartments*, satisfying the following axioms.

BL 1 $\mathcal{B}$ is a thick chamber complex.
BL 2 Each apartment is a thin chamber complex.
BL 3 Any two chambers belong to an apartment.

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BL 4 Given two apartments $\mathcal{A}_1, \mathcal{A}_2$ and two facets $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{A}_1 \cap \mathcal{A}_2$, there exists an isomorphism $\mathcal{A}_1 \to \mathcal{A}_2$ that leaves invariant $\mathcal{F}_1$, $\mathcal{F}_2$, and all of their faces.

Note that in Axiom BL 4 it is not assumed that the isomorphism $\mathcal{A}_1 \to \mathcal{A}_2$ is the restriction of an automorphism of $\mathcal{B}$. This will however be the case for the buildings coming from Tits systems, cf. Proposition 1.5.13.

The relationship between Tits systems and buildings is expressed in the following two propositions (1.5.6 and 1.5.28) due to Tits, see [Tit74, Theorem 3.2.6, Proposition 3.11].

Proposition 1.5.6 *Let (G, B, N, S) be a Tits system with S finite. Let V be the set of all maximal proper parabolic subgroups of the Tits system. Let $\mathcal{B}$ the set of those finite sets $\{P_0, \dots, P_n\}$ of maximal proper parabolic subgroups such that $\bigcap_{i=0}^n P_i$ is itself a parabolic subgroup. Endow $\mathcal{B}$ with the action of G defined by $g\{P_0, \dots, P_n\} = \{gP_0g^{-1}, \dots, gP_ng^{-1}\}$. Let $\mathcal{C} \subset \mathcal{B}$ consist of all subsets of the set of standard maximal proper parabolic subgroups. Let $\mathcal{A} \subset \mathcal{B}$ be the union of all N-conjugates of $\mathcal{C}$.*

(1) *The pair $(V, \mathcal{B})$ is a simplicial complex.*
(2) *Given a facet $\mathcal{F} = \{P_0, \dots, P_n\} \in \mathcal{B}$, let $P_{\mathcal{F}} = \bigcap_{i=0}^n P_i$. The map $\mathcal{F} \mapsto P_{\mathcal{F}}$ is a G-equivariant bijection from the set of facets of $\mathcal{B}$ to the set of proper parabolic subgroups, which translates the face relation between facets to the opposite of the inclusion relation between parabolic subgroups. Thus the maximal facets (called chambers) correspond to the minimal parabolic subgroups of G. These minimal parabolic subgroups are gBg^{-1}, for $g \in G$.*
(3) *The subset $\mathcal{C}$ is a chamber, called the* standard chamber.
(4) *If $g \in G$ stabilizes a facet, then it fixes each of its vertices.*
(5) *Given two faces $\mathcal{F}$ and $\mathcal{F}'$ of the standard chamber $\mathcal{C}$ and elements $n \in N$ and $g \in G$, if $g\mathcal{A}$ contains $\mathcal{F}$ and $n\mathcal{F}'$, then there is an element of G that carries $\mathcal{A}$ to $g\mathcal{A}$ and fixes every vertex of $\mathcal{F}$ and $n\mathcal{F}'$ (note that $\mathcal{F}$ and $n\mathcal{F}'$ are contained in $\mathcal{A}$).*
(6) *The complex $\mathcal{B}$ is a building whose set of apartments is $\{g\mathcal{A} \mid g \in G\}$. It is called the* Tits building *of the Tits system. The subset $\mathcal{A}$ is called the* standard apartment.
(7) *The group G acts transitively on the set of pairs consisting of an apartment and a chamber contained in it.*

Note that the building $\mathcal{B}$ depends only on the G-conjugacy class of the pair (B, N). The standard chamber $\mathcal{C}$ depends on B, and the standard apartment

$\mathcal{A}$ depends on $T = B \cap N$. It is often, but not always, true that the standard apartment consists of those proper parabolic subgroups that contain T.

Proof (1) The fact that any subgroup of G that contains a parabolic subgroup is itself a parabolic subgroup implies (1).

(2) Since $P_{\mathcal{F}}$ is a parabolic subgroup, it contains a conjugate, say gBg^{-1} of B. So $g^{-1}P_ig$ contains B, hence it is the standard parabolic G_{X_i}, for a maximal proper subset X_i of S. Let $X = \bigcap_{i=0}^{n} X_i$. Then $\bigcap G_{X_i} = G_X$. Therefore, $P_{\mathcal{F}} = gG_Xg^{-1}$. Conversely, given a $g \in G$, and a proper subset X of S, let X_i for $0 \leqslant i \leqslant n$ be the proper maximal subsets of S that contain X and let $P_i = gG_{X_i}ng^{-1}$. Then $\mathcal{F} = \{P_0, \ldots, P_n\}$ is the unique facet of $\mathcal{B}$ corresponding to the parabolic subgroup gG_Xg^{-1}. It is obvious that the maximal facets of $\mathcal{B}$ correspond to the minimal parabolic subgroups gBg^{-1}, for $g \in G$, since $G_{\varnothing} = B$. The maximal facets are called *chambers* of $\mathcal{B}$. As every facet of $\mathcal{B}$ is clearly a face of a chamber, $\mathcal{B}$ is a chamber complex.

(3) Let $P_0, \ldots, P_{r-1}$, $r = \#S$, be the standard maximal proper parabolic subgroups. Then $\bigcap_{i=0}^{r-1} P_i = B$, hence $\mathcal{C} := \{P_0, \ldots, P_{r-1}\}$ is a maximal facet.

(4) Suppose $\mathcal{F}$ is a facet that is stable under the action of $g \in G$. Then the corresponding parabolic subgroup P is normalized by g. As the normalizer of P is itself, g lies in P, and therefore it normalizes all the subgroups of G containing P. This implies that every face of $\mathcal{F}$, so in particular every vertex of $\mathcal{F}$, is fixed by g.

(5) We will now establish the assertion (5). Let $P = G_X$ and $P' = G_{X'}$, with $X, X' \subset S$, be the standard parabolic subgroups corresponding to the facets $\mathcal{F}$ and $\mathcal{F}'$ respectively. Let Y and Y' be the subgroups of W generated by X and X' respectively. Then $P = G_X = BYB$ and $P' = G_{X'} = BY'B$. Since $\mathcal{F} \subset g\mathcal{A}$, using Proposition 1.4.5(2) we see that $P = gn_0Pn_0^{-1}g^{-1}$ for some $n_0 \in N$. We replace g with gn_0 to assume that $P = gPg^{-1}$. As P is equal to its own normalizer in G, $g \in P$. Again using Proposition 1.4.5(2), we see that the condition $n\mathcal{F}' \subset g\mathcal{A}$ implies that $nP'n^{-1} = gn'P'n'^{-1}g^{-1}$ for some $n' \in N$. So $n'^{-1}g^{-1}n$ normalizes P' and hence it belongs to P'. As $g \in P$, we infer that $n \in Pn'P'$. Let w, w' be the images of n, n' in W. From the fact that $n \in Pn'P'$, using Axiom TS 3 of Definition 1.4.1 we see that $w \in Yw'Y'$. Let $w_1 \in Y$ be such that $w \in w_1w'Y'$, and let n_1 be a representative of w_1^{-1} in N. Then n_1 is in P and $n_1n \in n'P'$. Therefore, $gn_1 \in P$, $gn_1nP' = gn'P' = nP'$ and $gn_1\mathcal{A} = g\mathcal{A}$, so the left multiplication by gn_1 is the desired isomorphism. Thus we have shown that (5) holds.

(6) Now we will show that $\mathcal{B}$ is a building by verifying the axioms listed in Definition 1.5.5. Axiom BL 4 is just (5).

To prove that the apartments are thin, it is enough to show that $\mathcal{A}$ is thin. For

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this, it suffices to show that given a face $\mathcal{P}$ of $\mathcal{C}$ of codimension 1 (such faces are called *panels*), there is a unique chamber $\mathcal{C}' \neq \mathcal{C}$ in $\mathcal{A}$ that shares $\mathcal{P}$. The faces of codimension 1 of $\mathcal{C}$ correspond to the parabolic subgroups $G_{\{s\}}$ for $s \in S$. Hence, chambers in $\mathcal{A}$ that share the panel corresponding to $G_{\{s\}} = B \cup BsB$ are the chambers corresponding to nBn^{-1}, for $n \in N$, such that $nBn^{-1} \subset G_{\{s\}}$. But the minimal parabolic subgroups B and $nBn^{-1} (\neq B)$ that are contained in $G_{\{s\}}$ are conjugate to each other under $G_{\{s\}}$. Thus, if $nBn^{-1} \neq B$, there is a $b \in B$ such that $nBn^{-1} = bsBs^{-1}b^{-1}$. Then $n \in BsB$ and hence, by the Bruhat decomposition, $n \in s(B \cap N)$. This implies that the only chambers in $\mathcal{A}$ that share the panel corresponding to the parabolic subgroup $G_{\{s\}}$ are the chambers $\mathcal{C}$ (which corresponds to B) and the chamber $s\mathcal{C}$ (which corresponds to the parabolic subgroup $sBs^{-1} = sBs$). Thus $\mathcal{A}$ is thin.

We will now show that $\mathcal{B}$ is thick by determining all the chambers in it that share the panel corresponding to $G_{\{s\}}$. The set of such chambers is in natural bijective correspondence with the set of conjugates of B in $G_{\{s\}} = B \cup BsB$. As we saw in the preceding paragraph, besides B itself, its other conjugates in $G_{\{s\}}$ are $bsBs^{-1}b^{-1}$, with $b \in B$. Moreover, $bsBs^{-1}b^{-1} = b'sBs^{-1}b'^{-1}$ for $b, b' \in B$, if and only if $b^{-1}b' \in sBs^{-1} = sBs$. Therefore, for any $b \in B - sBs$, the conjugates sBs and $bsBs^{-1}b^{-1}$ are different. This shows that $\mathcal{B}$ is thick.

We will now show that given two chambers $\mathcal{C}_1$ and $\mathcal{C}_2$, there is an apartment that contains both of them. Let $g_i\mathcal{C} = \mathcal{C}_i$, for $g_i \in G$. By the Bruhat decomposition, $g_1^{-1}g_2 = b_1nb_2$, with $b_i \in B$ and $n \in N$. Then

$$(\mathcal{C}_1, \mathcal{C}_2) = g_1(\mathcal{C}, g_1^{-1}g_2\mathcal{C}) = g_1(\mathcal{C}, b_1nb_2\mathcal{C}) = g_1b_1(\mathcal{C}, n\mathcal{C}).$$

Thus $\mathcal{C}_1$ and $\mathcal{C}_2$ are contained in the apartment $g_1b_1\mathcal{A}$.

We will now show that given any two chambers, there is a gallery in $\mathcal{B}$ joining them. In view of the result in the preceding paragraph, it is enough to show that for any $n \in N$, the chambers $\mathcal{C}$ and $n\mathcal{C}$ can be joined by a gallery. We denote by w, the image of n in W, and let $w = s_1s_2 \cdots s_n$, with $s_i \in S$ for $i \leqslant n$. Let $w_0 = 1$, and for $j > 0$, let $w_j = s_1 \cdots s_j$ and $\mathcal{C}_j = w_j\mathcal{C}$. Then $\mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n = w\mathcal{C}$ is a gallery in $\mathcal{A}$ joining $\mathcal{C}$ to $w\mathcal{C}$. Thus we have verified all the axioms in the definition of buildings for $\mathcal{B}$, and the apartments $g\mathcal{A}$, $g \in G$. Hence $\mathcal{B}$ is a building.

(7) Observe that, given a pair $(\mathcal{A}', \mathcal{C}')$ consisting of an apartment $\mathcal{A}' = g\mathcal{A}$ and a chamber $\mathcal{C}'$ in it, there is an element $h \in G$ such that $h(\mathcal{A}, \mathcal{C}) = (\mathcal{A}', \mathcal{C}')$. As $g^{-1}\mathcal{C}'$ is a chamber of $\mathcal{A}$, we see that $g^{-1}\mathcal{C}' = n\mathcal{C}$ for an $n \in N$. Hence, $\mathcal{C}' = gn\mathcal{C}$. So the given pair $(\mathcal{A}', \mathcal{C}')$ is $(g\mathcal{A}, gn\mathcal{C}) = gn(\mathcal{A}, \mathcal{C})$. Setting $h = gn$, we see that the pair $(\mathcal{A}', \mathcal{C}') = h(\mathcal{A}, \mathcal{C})$ with $h \in G$. This implies (7). $\square$

Remark 1.5.7 Using the bijection $\mathcal{F} \to P_{\mathcal{F}}$ one can identify the simplices of $\mathcal{B}$ with the proper parabolic subgroups of G. Under this identification, the face relation becomes the opposite of the inclusion relation between proper parabolic subgroups.

In fact, following Remark 1.5.4, we can also interpret $\mathcal{B}$ as the set of all parabolic subgroups of the Tits system together with the opposite of the inclusion order.

Definition 1.5.8 A Tits system (G, B, N, S) is called

(1) *spherical*, if each apartment is the triangulation of a sphere;
(2) *affine*, if each apartment is the triangulation of a Euclidean space.

Remark 1.5.9 It can be shown that a Tits system is spherical if and only if its Weyl group $W = N/(B \cap N)$ is finite.

Example 1.5.10 The simplest example of an affine building is a tree (cf. [Ser03]), provided each vertex has at least three edges emanating from it. An apartment is an infinite path through the tree, thus a simplicial subcomplex whose geometric realization is a line. The chambers are the edges of the tree. Figure 1.5.1 illustrates the case of a 3-regular tree. We will see in §3.1 that this is the (affine) Bruhat–Tits building associated to the group $\mathrm{SL}_2/\mathbf{Q}_2$. Non-regular trees also occur as (affine) Bruhat–Tits buildings of reductive groups over local fields. This is the case with the group $\mathrm{SU}_3/\mathbf{Q}_p$ associated to an unramified quadratic extension of $\mathbf{Q}_p$, cf. §3.2.

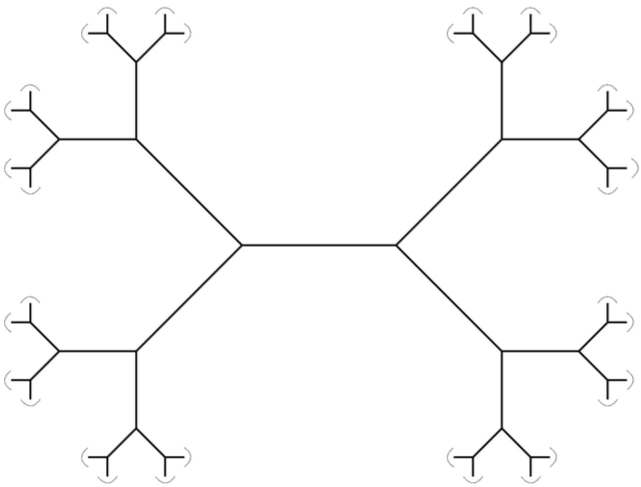

Figure 1.5.1 The 3-regular tree.

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Example 1.5.11 Consider the Tits system of Example 1.4.3. The building associated to it by Proposition 1.5.6 is called the *spherical building* of G. Its facets are in 1–1 correspondence with the proper parabolic k-subgroups of G; see [Tit74, Theorem 5.2].

Example 1.5.12 Let k be any field. The simplest examples of a spherical building is the set $\mathbf{P}^1(k)$, seen as a 0-dimensional simplicial complex. The vertices, which are also the chambers, are the elements of that set. Every pair of points is an apartment. This building arises as the spherical building of the spherical Tits system of the reductive group SL_2/k.

Proposition 1.5.13 *Let (G, B, N, S) be a Tits system and $\mathcal{B}$ the building of Proposition 1.5.6.*

(1) *Given two apartments $\mathcal{A}_1, \mathcal{A}_2$, both containing a facet $\mathcal{F}$ of $\mathcal{B}$, there exists a $g \in G$ that transports $\mathcal{A}_1$ to $\mathcal{A}_2$ and fixes all the vertices of $\mathcal{F}$.*
(2) *Given a facet $\mathcal{F} \in \mathcal{B}$, its stabilizer $\{g \in G \mid g\mathcal{F} = \mathcal{F}\}$ is equal to the parabolic subgroup $P_{\mathcal{F}}$.*
(3) *The fixed point set of $P_{\mathcal{F}}$ in $\mathcal{B}$ is the set of faces of $\mathcal{F}$ (including $\mathcal{F}$ itself).*
(4) *If $\mathcal{F}_1, \mathcal{F}_2$ are two facets of the same chamber and $g \in G$ satisfies $g\mathcal{F}_1 = \mathcal{F}_2$, then $\mathcal{F}_1 = \mathcal{F}_2$ and $g \in P_{\mathcal{F}_1}$.*
(5) *Two facets $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{A}$ are conjugate under G if and only if they are conjugate under N.*

Proof (1) For $i = 1, 2$, let $\mathcal{C}_i$ be a chamber in $\mathcal{A}_i$ such that $\mathcal{F}$ is a face of $\mathcal{C}_i$. We fix an apartment $\mathcal{A}$ that contains both $\mathcal{C}_1$ and $\mathcal{C}_2$. Then $\mathcal{C}_i$ is contained in both $\mathcal{A}$ and $\mathcal{A}_i$. Now Proposition 1.5.6 (7),(4) imply, for $i = 1, 2$, that there is a $g_i \in G$ that transports $\mathcal{A}$ to $\mathcal{A}_i$ and fixes every vertex of the chamber $\mathcal{C}_i$. Then $g := g_2 g_1^{-1} (\in G)$ transports $\mathcal{A}_1$ to $\mathcal{A}_2$ and fixes every vertex of $\mathcal{F}$.

(2) Since proper parabolic subgroups of G and facets of $\mathcal{B}$ are the same, the second assertion is equivalent to the statement that each proper parabolic subgroup is equal to its own normalizer, which is part of Proposition 1.4.5.

(3) The third assertion follows from the order-reversing bijective correspondence between the facets of $\mathcal{B}$ and the parabolic subgroups of G and the fact that each parabolic subgroup equals its own normalizer in G.

(4) To prove the fourth assertion, we consider the parabolic subgroups $P_1 := P_{\mathcal{F}_1}$ and $P_2 := P_{\mathcal{F}_2}$. Both are standard with respect to the chamber of which $\mathcal{F}_1$ and $\mathcal{F}_2$ are assumed to be faces. Then $gP_1g^{-1} = P_2$ implies via Proposition 1.4.5 that $P_1 = P_2$ and $g \in P_1$.

(5) To prove the fifth assertion, choose a chamber $\mathcal{C}$ in $\mathcal{A}$ of which $\mathcal{F}_1$ is a face and $n \in N$ so that $n^{-1}\mathcal{F}_2$ is also a face of $\mathcal{C}$. Let $g \in G$ be such that $g\mathcal{F}_1 = \mathcal{F}_2$. Then (4) implies that $\mathcal{F}_1 = n^{-1}\mathcal{F}_2$ and $n^{-1}g \in P_{\mathcal{F}_1}$. Thus $n\mathcal{F}_1 = g\mathcal{F}_1 = \mathcal{F}_2$. □

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Proposition 1.5.14 *Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{B}$ be two facets.*

(1) *There exists an apartment $\mathcal{A}$ containing both $\mathcal{F}_1$ and $\mathcal{F}_2$.*
(2) *Assuming $\mathcal{A}$ is the standard apartment, the mixed Bruhat decomposition $G = \mathcal{P}_{\mathcal{F}_1} \cdot W \cdot \mathcal{P}_{\mathcal{F}_2}$ holds.*

Proof Without loss of generality we may replace $\mathcal{F}_i$ with a chamber $\mathcal{C}_i$ containing $\mathcal{F}_i$ in its closure and we may further assume that $\mathcal{C}_1$ is the standard chamber. Let $g \in G$ be such that $\mathcal{C}_2 = g\mathcal{C}_1$. Write $g = bnb' \in BWB$ according to the Bruhat decomposition, Proposition 1.4.5. Thus both $\mathcal{C}_2 = bn\mathcal{C}_1$ and $\mathcal{C}_1 = b\mathcal{C}_1$ are contained in the apartment $b\mathcal{A}$.

Assuming now that $b\mathcal{A} = \mathcal{A}$, we have $P_{\mathcal{C}_1} = B$ and $P_{\mathcal{C}_2} = nBn^{-1}$ and the Bruhat decomposition implies $G = BNB = P_{\mathcal{C}_1} N P_{\mathcal{C}_2} n$, hence $G = P_{\mathcal{C}_1} N P_{\mathcal{C}_2}$. □

Remark 1.5.15 With $\mathcal{B}$ as in Proposition 1.5.13, using the bijection $\mathcal{F} \to P_{\mathcal{F}}$ between facets and parabolic subgroups, we can associate via Definition 1.4.6 to each facet of $\mathcal{B}$ a subset of S, called its *type*. Thus the type of $\mathcal{F}$ is the subset $X \subset S$ such that $P_{\mathcal{F}}$ is conjugate to $G_X = BW_XB$.

Remark 1.5.16 We have made a very minor change in the definition of the building of a Tits system as compared to [Tit74] and other sources, by considering only proper parabolic subgroups, that is, excluding G from consideration. This corresponds to excluding the empty subset of V from the definition of a simplicial complex.

We will find it useful to make a further change. Namely, consider the basic example where G is (the set of k-points of) a connected linear algebraic group defined over an algebraically closed field k, B is a Borel subgroup, and N is the normalizer of a maximal torus. As defined so far, the building is the set of all proper parabolic subgroups. It is a simplicial complex. Consider now the situation where $G = G_1 \times G_2$ for two connected algebraic groups G_1 and G_2. If $P_1 \subset G_1$ is a proper parabolic subgroup, then $P = P_1 \times G_2$ is a proper parabolic subgroup of G. For our future purposes we would like to have a variant of the building that excludes such parabolic subgroups from consideration, and instead only contains parabolic subgroups of the form $P_1 \times P_2$, where $P_i \subset G_i$ is a proper parabolic subgroup.

Definition 1.5.17 Let (G, B, N, S) be a Tits system with S finite and non-empty. Write $S = S_1 \cup \cdots \cup S_n$ with S_i pairwise commuting and irreducible.

(1) A subset $X \subset S$ is called *admissible* if $S_i \cap X$ is a proper subset of S_i for all i.

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(2) A *facet* (respectively a parabolic subgroup) is called *admissible* if its type is admissible.

Proposition 1.5.18 *Let (G, B, N, S) be a Tits system with S finite and non-empty. Let $\mathcal{B}$ be its Tits building of Proposition 1.5.6.*

(1) *The subset $\mathcal{B}'$ of $\mathcal{B}$ consisting of all admissible facets forms itself a building, where the apartments are given by $\mathcal{A}' = \mathcal{B}' \cap \mathcal{A}$ for apartments $\mathcal{A} \subset \mathcal{B}$.*
(2) *Every chamber in $\mathcal{B}$ is admissible and hence lies in $\mathcal{B}'$.*
(3) *$\mathcal{B}'$ is a polysimplicial complex, invariant under the action of G.*
(4) *$\mathcal{B}' = \mathcal{B}$ if and only if S is irreducible.*

Proof Using Remark 1.5.7 we interpret $\mathcal{B}$ as the set of parabolic subgroups of (G, B, N, S) with the opposite inclusion order. Write $S = S_1 \cup \cdots \cup S_n$ as a disjoint union of mutually commuting subsets, with each S_i irreducible. Let $S_i^c = S - S_i$. Then Proposition 1.4.10 gives a bijection $\mathcal{B}' \to \prod_i \mathcal{B}_i$, where $\mathcal{B}_i$ is the building of the Tits system $(G, P_{S_i^c}, N, S_i)$. This is a bijection of sets that preserves the order relation, hence endows the ordered set $\mathcal{B}$ with the structure of a polysimplicial complex. The remaining claims are immediate. □

Remark 1.5.19 The vertices in $\mathcal{B}'$ are the facets of $\mathcal{B}$ of type $X_1 \cup \cdots \cup X_n$, where $X_i \subset S_i$ is a maximal proper subset. On the other hand, the vertices in $\mathcal{B}$ are the facets of $\mathcal{B}$ of type $X_1 \cup \cdots \cup X_n$, where for some $i_0 \leqslant n$, X_{i_0} is a maximal proper subset of S_{i_0} and for all $i \neq i_0$, $X_i = S_i$. Thus, unless S is irreducible, the vertices in $\mathcal{B}$ do not lie in $\mathcal{B}'$, and the vertices of $\mathcal{B}'$ are facets of $\mathcal{B}$ whose dimension is greater than 1.

Definition 1.5.20 We will call the building $\mathcal{B}'$ of Proposition 1.5.18 the *restricted building*.

Remark 1.5.21 Except for not necessarily being a simplicial complex, all properties of $\mathcal{B}$ stated in Proposition 1.5.6 and Proposition 1.5.13 immediately carry over to $\mathcal{B}'$.

Definition 1.5.22 A *panel* is face of codimension 1 of a chamber of $\mathcal{B}$. Two chambers are said to be *adjacent* if they share a common panel. Given two facets $\mathcal{F}$ and $\mathcal{F}'$ in $\mathcal{B}$, a *gallery* of *length n* joining them is a sequence $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n$ of chambers of $\mathcal{B}$ such that for $i < n$, $\mathcal{C}_i$ is adjacent to $\mathcal{C}_{i+1}$ and $\mathcal{F}$ is a face of $\mathcal{C}_0$ and $\mathcal{F}'$ is a face of $\mathcal{C}_n$. A gallery of length n joining $\mathcal{F}$ and $\mathcal{F}'$ is said to be *minimal* if there is no gallery of length smaller than n joining $\mathcal{F}$ and $\mathcal{F}'$.

Lemma 1.5.23 *Let $\mathcal{A}$ be an apartment of $\mathcal{B}$ and $\mathcal{C}$ a chamber in $\mathcal{A}$. If f is an automorphism of $\mathcal{A}$ that keeps every vertex of $\mathcal{C}$ fixed, then f is trivial.*

Proof Let $\mathcal{C}'$ be a chamber in $\mathcal{A}$ and let $\mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n = \mathcal{C}'$ be a gallery in $\mathcal{A}$ joining $\mathcal{C}$ to $\mathcal{C}'$ such that for all $i < n$, $\mathcal{C}_i \neq \mathcal{C}_{i+1}$. We will prove by induction that the vertices of $\mathcal{C}_i$, for $i \leqslant n$, are fixed under f. This will prove the lemma. Let us assume that for some $j < n$, every vertex of $\mathcal{C}_j$ is fixed under f and let $\mathcal{P}$ be the panel shared by $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$. Then as $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$ are the only chambers in $\mathcal{A}$ that have $\mathcal{P}$ as a face, and as f fixes $\mathcal{C}_j$, we infer that it fixes $\mathcal{C}_{j+1}$ also. Since the vertices of $\mathcal{P}$ are fixed under f, the remaining vertex of $\mathcal{C}_{j+1}$ is also fixed under f. Now by induction, we see that for all $i \leqslant n$, every vertex of $\mathcal{C}_i$ is fixed under f. □

Proposition 1.5.24 *Let $\mathcal{A} \subset \mathcal{B}$ be an apartment and $\mathcal{C} \subset \mathcal{A}$ a chamber. There exists a unique polysimplicial map $\rho = \rho_{\mathcal{A},\mathcal{C}} : \mathcal{B} \to \mathcal{A}$ with the following properties.*

(1) *$\rho|_{\mathcal{A}}$ is the identity.*
(2) *For every apartment $\mathcal{A}'$ that contains $\mathcal{C}$, the map $\rho|_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}$ is a polysimplicial isomorphism.*
(3) *For any vertex x of $\mathcal{C}$, $\rho^{-1}(x) = \{x\}$.*

Proof Consider an arbitrary facet $\mathcal{F}$ of $\mathcal{B}$. Choose an apartment $\mathcal{A}'$ containing $\mathcal{F}$ and $\mathcal{C}$; such an apartment exists by Definition 1.5.5. By BL 4 of Definition 1.5.5 and Lemma 1.5.23 there exists a unique isomorphism $\sigma_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}$ that fixes every vertex of $\mathcal{C}$. Define $\rho(\mathcal{F}) = \sigma_{\mathcal{A}'}(\mathcal{F})$. We claim that $\rho(\mathcal{F})$ does not depend on the choice of $\mathcal{A}'$. Let $\mathcal{A}''$ be another apartment containing $\mathcal{F}$ and $\mathcal{C}$. By BL 4 of Definition 1.5.5, there exists an isomorphism $\tau : \mathcal{A}'' \to \mathcal{A}'$ that fixes every vertex of $\mathcal{F}$ and $\mathcal{C}$. It is obvious that $\sigma_{\mathcal{A}''} : \mathcal{A}'' \to \mathcal{A}$ is just $\sigma_{\mathcal{A}'} \circ \tau$. This implies that $\rho(\mathcal{F})$ is independent of the choice of $\mathcal{A}'$.

Consider now a vertex y of $\mathcal{B}$ such that $x := \rho(y)$ is a vertex of $\mathcal{C}$, and let $\mathcal{A}'$ be an apartment containing y and $\mathcal{C}$. Then since $\rho|_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}$ is an isomorphism, that maps x, y to x, we conclude that $y = x$. □

Definition 1.5.25 The polysimplicial map $\rho_{\mathcal{A},\mathcal{C}} : \mathcal{B} \to \mathcal{A}$ constructed in the preceding proposition is called the *retraction* of $\mathcal{B}$ onto $\mathcal{A}$ with center $\mathcal{C}$.

Example 1.5.26 In Example 1.5.10, where $\mathcal{B}$ is a 3-regular tree, the retraction to an apartment centered at a fixed edge is depicted in Figure 1.5.2. An intuitive way to describe it may be to imagine holding the tree with your hand at the fixed edge and shaking it until all branches collapse onto a single line (the apartment onto which the building is being retracted).

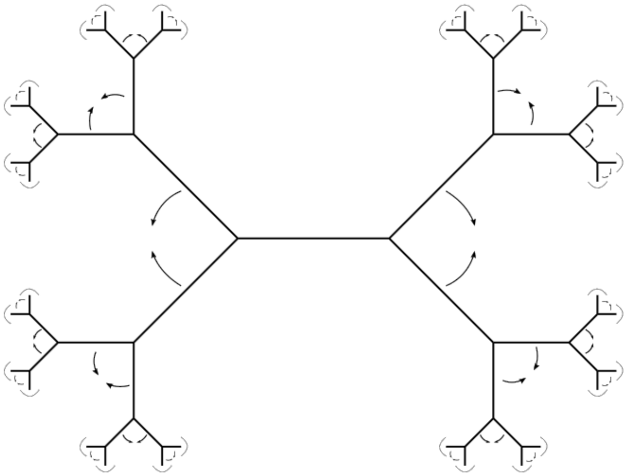

Figure 1.5.2 The retraction of a 3-regular tree.

Corollary 1.5.27 *Let $\mathcal{A}$ be an apartment of $\mathcal{B}$, $\mathcal{C}'$ a chamber in $\mathcal{A}$ and $\mathcal{F}$ a facet in $\mathcal{A}$. Let $\mathcal{G} := \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n = \mathcal{C}'$ be a minimal gallery joining $\mathcal{F}$ to $\mathcal{C}'$. Then all the chambers $\mathcal{C}_i$ are contained in $\mathcal{A}$.*

Proof Assume, if possible, that not all chambers in $\mathcal{G}$ are contained in $\mathcal{A}$. Let $j < n$ be the largest integer such that the chamber $\mathcal{C}_j$ is not contained in $\mathcal{A}$. Let $\mathcal{P}$ be the common panel of $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$ and $\mathcal{C}$ be the unique chamber in $\mathcal{A}$ different from $\mathcal{C}_{j+1}$ that has $\mathcal{P}$ as a face. According to the last assertion of Proposition 1.5.24, the retraction $\rho := \rho_{\mathcal{A},\mathcal{C}}$ does not map $\mathcal{C}_j$ onto $\mathcal{C}$. Hence, $\rho(\mathcal{C}_j) = \mathcal{C}_{j+1}$. Therefore, the gallery $\rho(\mathcal{G}) := \rho(\mathcal{C}_0), \rho(\mathcal{C}_1), \cdots, \rho(\mathcal{C}_n) = \mathcal{C}'$ is a gallery joining $\mathcal{F}$ and $\mathcal{C}'$ contained in $\mathcal{A}$ and as $\rho(\mathcal{C}_j) = \mathcal{C}_{j+1} = \rho(\mathcal{C}_{j+1})$ since $\mathcal{C}_{j+1}$ is contained in $\mathcal{A}$, the gallery $\mathcal{G}$ is not minimal. A contradiction. □

The following proposition is a converse of Proposition 1.5.6.

Proposition 1.5.28 *Let G be a group that operates on $\mathcal{B}$ by simplicial automorphisms. We assume that this action has the following two properties:*

(1) *Given two pairs $(\mathcal{A}', \mathcal{C}')$ and $(\mathcal{A}'', \mathcal{C}'')$ consisting of an apartment of $\mathcal{B}$ and a chamber in it, there exists an element $g \in G$ that carries the first pair onto the second pair, and fixes every vertex common to $\mathcal{C}'$ and $\mathcal{C}''$.*
(2) *If an element of G fixes a chamber, then it fixes all its vertices.*

We choose a pair $(\mathcal{A}, \mathcal{C})$ and let B be the subgroup consisting of all elements of G that keep $\mathcal{C}$ stable, and N be the subgroup consisting of all elements of G

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which keep $\mathcal{A}$ stable. Then (B, N) is a saturated Tits system (Definition 1.4.1) in G. The Weyl group of this Tits system is $N/B \cap N$.

A simplicial automorphism of $\mathcal{B}$, or $\mathcal{A}$, which has the property that if it stabilizes a chamber, then it fixes all its vertices, will be called a *special* automorphism.

Proof From conditions (1) and (2) it is obvious that the subgroup N acts transitively on the set of chambers in $\mathcal{A}$ and it maps onto the group W of all special automorphisms of $\mathcal{A}$. Using Lemma 1.5.23 we see that the kernel of the homomorphism $N \to W$ is $T := B \cap N$, so $T(= B \cap N)$ is a normal subgroup of N and $N/T \simeq W$. On the other hand, in view of (2), an element of G acts trivially on $\mathcal{A}$ if and only if it keeps every chamber in $\mathcal{A}$ stable, thus the kernel of $N \to W$ is $\bigcap_{n\in N} nBn^{-1}$. Hence, $T = \bigcap_{n\in N} nBn^{-1}$.

Now let $g \in G$ and $\mathcal{C}' := g\mathcal{C}$. Let $\mathcal{A}'$ be an apartment that contains both $\mathcal{C}$ and $\mathcal{C}'$. Then there is an element $b \in B$ that carries the pair $(\mathcal{A}', \mathcal{C})$ to the pair $(\mathcal{A}, \mathcal{C})$. In particular, $bg\mathcal{C}$ is a chamber in $\mathcal{A}$. Hence there is $n \in N$ such that $bg\mathcal{C} = n\mathcal{C}$, which implies that $n^{-1}bg \in B$, and so $g \in BNB$. Thus we have shown that $G = BNB$ and condition TS 1 of 1.4.1 has been verified.

Given a panel of $\mathcal{C}$, let $\mathcal{C}'$ be the other chamber in $\mathcal{A}$ that shares this panel. Then there is an element in N that carries $\mathcal{C}$ to $\mathcal{C}'$. Let s be its image in W. Then it is clear that s carries $\mathcal{C}'$ back to $\mathcal{C}$; condition (2) and Lemma 1.5.23 imply that $s^2 = 1$, that is, s is a reflection. By considering all the panels of $\mathcal{C}$ we obtain a set S of reflections. We will now show that S generates W. For $w \in W$, we define its length $\ell(w)$ to be the length of a minimal gallery joining $\mathcal{C}$ to $w\mathcal{C}$. Let $\mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_m = w\mathcal{C}$ be a minimal gallery (so $\ell(w) = m$). Let $\mathcal{P}$ be the panel shared by $\mathcal{C} = \mathcal{C}_0$ and $\mathcal{C}_1$ and s be the reflection in $\mathcal{P}$. Then $s\mathcal{C}_1 = \mathcal{C}$, hence applying s to the gallery $\mathcal{C}_1, \ldots, \mathcal{C}_m(= w\mathcal{C})$, we obtain the gallery $\mathcal{C}, \ldots, s\mathcal{C}_n = sw\mathcal{C}$ of length $m - 1$ joining $\mathcal{C}$ to $sw\mathcal{C}$, therefore, $\ell(sw) \leqslant n - 1$. So by induction on length, we conclude that sw lies in the group generated by S, and hence so does w. This proves that S generates W and we have verified condition TS 2 of Definition 1.4.1.

We will now verify TS 3. Let $w \in W$, $s \in S$ and $b \in B$. We fix $n \in N$ that maps onto w. The chambers $\mathcal{C}$, $s\mathcal{C}$, and $bs\mathcal{C}$ share a panel $\mathcal{P}$. Hence, $n\mathcal{C}$, $ns\mathcal{C}$ and $nbs\mathcal{C}$ share the panel $n\mathcal{P}$. Let $\mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_m$ be a minimal gallery joining $\mathcal{C}$ to $n\mathcal{P}$, and $\mathcal{A}'$ be an apartment containing $\mathcal{C}$ and $nbs\mathcal{C}$. Let $b' \in B$ be an element that carries the pair $(\mathcal{A}', \mathcal{C})$ to the pair $(\mathcal{A}, \mathcal{C})$. Since $\mathcal{A}'$ contains $n\mathcal{P}$, according to Corollary 1.5.27 this apartment contains $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_m$. By induction on j we see that b' leaves invariant $\mathcal{C}_j$ for all $j \leqslant m$. It follows that $b'n\mathcal{P} = n\mathcal{P}$. Therefore , the chamber $b'nbs\mathcal{C}$ contains the panel $n\mathcal{P}$ and so it is either $n\mathcal{C}$ or

$ns\mathcal{C}$. Hence, $b'nbsB = nB$ or nsB, which implies that $nBsB \subset BnB \cup BnsB$. This completes the verification of TS 3.

To verify TS 4, let $s \in S$ and $\mathcal{P}$ be the panel shared by $\mathcal{C}$ and $s\mathcal{C}$. As the building $\mathcal{B}$ is thick, there is a chamber $\mathcal{C}'$ different from $\mathcal{C}$ and $s\mathcal{C}$, which shares the panel $\mathcal{P}$. Let $\mathcal{A}'$ be an apartment that contains $\mathcal{C}$ and $\mathcal{C}'$. Then an element of G that carries $(\mathcal{A}, \mathcal{C})$ to $(\mathcal{A}', \mathcal{C})$ belongs to B but not to sBs since it does not fix the chamber $s\mathcal{C}$ (in fact, $b's\mathcal{C} = \mathcal{C}'$). □

Lemma 1.5.29 *In the setting of Proposition 1.5.28, let $\mathcal{A}$ be an apartment of $\mathcal{B}$ and $\mathcal{C}$ a chamber of $\mathcal{A}$, and let $\rho\colon \mathcal{B} \to \mathcal{A}$ be the retraction with center $\mathcal{C}$. For each apartment $\mathcal{A}'$ containing $\mathcal{C}$ there exists $g \in G$ such that $\rho(x) = gx$ for all $x \in \mathcal{A}'$. In particular, for a facet $\mathcal{F} \subset \mathcal{A}$ and $g \in G_{\mathcal{C}}$ one has $\rho(g\mathcal{F}) = \mathcal{F}$.*

Proof By Proposition 1.5.28 there exists an element $g \in G$ that fixes $\mathcal{C}$ and maps $\mathcal{A}'$ to $\mathcal{A}$. The composition of the action of g^{-1} with ρ is a polysimplicial automorphism of $\mathcal{A}$ that fixes $\mathcal{C}$ pointwise. By Lemma 1.5.23, this automorphism is trivial. We conclude that $\rho|_{\mathcal{A}'}\colon \mathcal{A}' \to \mathcal{A}$ coincides with the action of g. To prove the second statement, take $\mathcal{A}' = g\mathcal{A}$. □

1.6 The Monoid $\widetilde{\mathbf{R}}$

This section does not belong to the discussion of affine root systems and abstract buildings; it introduces useful notation that will be applied throughout the book.

Consider a group X equipped with a descending filtration X_r indexed by real numbers $r \in \mathbf{R}$. It is oftentimes useful to consider the group $X_{r+} = \bigcup_{s>r} X_s$, which is contained in X_r and could be a proper subgroup. We can think of $r+$ as a number that is infinitesimally larger than r; it is larger than r, but smaller than any real number that is larger than r. Moreover, we can set $X_\infty = \bigcap_r X_r$. The filtration is called *separated* if $X_\infty = \{1\}$.

This leads us to introduce the set $\widetilde{\mathbf{R}} = (\mathbf{R} \times \{0,1\}) \cup \{\infty\}$. We will write r in place of $(r,0)$ and $r+$ in place of $(r,1)$, and we think of $r+$ as a number infinitesimally larger than r.

The set $\widetilde{\mathbf{R}}$ is made into a totally ordered commutative monoid that contains $\mathbf{R}$ as an ordered submonoid via the following rules.

(1) $r + (s+) = (r+) + s = (r+) + (s+) = (r + s)+$.

(2) $r + \infty = (r+) + \infty = \infty$.

(3) $\infty > r+ > r$ for any $r \in \mathbf{R}$.

(4) $r+ > s+$ if $r > s$.

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We define an operation $\widetilde{r} \mapsto \widetilde{r}+$ on $\widetilde{\mathbf{R}}$ by setting $(r+)+ = r+$ and $\infty+ = \infty$.

Another way to think about the monoid $\widetilde{\mathbf{R}}$ is as the monoid of intervals of $\mathbf{R}$ of the form $[r, \infty)$ or (r, ∞) for $-\infty < r \leqslant \infty$. Then $r \in \mathbf{R}$ corresponds to $[r, \infty)$, $r+$ corresponds to (r, ∞), and ∞ corresponds to the empty interval. Addition corresponds to pointwise addition of intervals. The operation $\widetilde{r} \to \widetilde{r}+$ corresponds to taking the interior. The order is the opposite of the inclusion order.

A descending filtration X_r of a group X indexed by $\mathbf{R}$ extends, as was just discussed, to a descending filtration indexed by $\widetilde{\mathbf{R}}$ in a natural way. A filtration indexed by $\mathbf{R}_{\geqslant 0}$ can be extended in the same way to $\widetilde{\mathbf{R}}_{\geqslant 0}$.

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