

**NECESSARY AND SUFFICIENT CONDITION
FOR OSCILLATIONS
OF NEUTRAL DIFFERENTIAL EQUATIONS**

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Abstract

Consider the neutral delay differential equation

$$\frac{d}{dt} [y(t) + py(t - \tau)] + \sum_{i=1}^k q_i y(t - \sigma_i) = 0 \quad (1)$$

where $p \in \mathbf{R}$, $\tau \geq 0$, $q_i > 0$, and $\sigma_i \geq 0$ for $i = 1, 2, \dots, k$. We prove the following result.

THEOREM. *A necessary and sufficient condition for the oscillation of all solutions of Eq. (1) is that the characteristic equation*

$$\lambda + p\lambda e^{-\lambda\tau} + \sum_{i=1}^k q_i e^{-\lambda\sigma_i} = 0$$

has no real roots.

1. Introduction

Neutral delay differential equations (NDDE for short) are differential equations in which the highest order derivative of the unknown function is evaluated both at the present time t and at some past time $t - \tau$.

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In this paper we prove the following result:

THEOREM. *Consider the NDDE*

$$\frac{d}{dt} [y(t) + py(t - \tau)] + \sum_{i=1}^k q_i y(t - \sigma_i) = 0, \quad (1)$$

where $p \in \mathbf{R}$, $\tau \geq 0$, $q_i > 0$, and $\sigma_i \geq 0$, for $i = 1, 2, \dots, k$. Then a necessary and sufficient condition for the oscillation of all solutions of Eq. (1) is that the characteristic equation

$$\lambda + p\lambda e^{-\lambda\tau} + \sum_{i=1}^k q_i e^{-\lambda\sigma_i} = 0 \quad (2)$$

has no real roots.

The special case of this results when $k = 1$ was proved by Sficas and Stavroulakis [7]. Also for $k = 1$ and τ , σ , p , and $q \in \mathbf{R}$ the above theorem was proved by Grove, Ladas, and Meimaridou [3]. When $p = 0$ or $\tau = 0$, the above result is due to Trammov [10]. For a simple proof see Ladas, Sficas and Stavroulakis [6].

There are drastic differences in the behavior of solutions of NDDEs compared to nonneutral equations. For example Snow [9] has shown, see also [8], that even though the characteristic roots of (2) may lie in the negative half-plane, it is still possible for (1) to have unbounded solutions. Our theorem establishes the fact that unlike stability, the oscillatory nature of the solutions of (1) is determined by the roots of the characteristic equation (2).

Several results concerning the oscillation and asymptotic behavior of the solutions of (1), when $k = 1$, were recently obtained by Ladas and Sficas [5] and Grammatikopoulos, Grove and Ladas [2].

Neutral equations have applications in electric networks containing lossless transmission lines. Such networks arise in high speed computers where the lossless transmission lines are used to interconnect switching circuits. See [1] and [4].

Let $\rho = \max\{\tau, \sigma_1, \sigma_2, \dots, \sigma_k\}$ and $t_0 \in \mathbf{R}$. By a solution of (1), in the interval $[t_0, \infty)$, we mean a function $y \in C([t_0 - \rho, \infty), \mathbf{R})$ such that $y(t) + py(t - \tau)$ is continuously differentiable for $t \geq t_0$ and $y(t)$ satisfies (1) for all $t \geq t_0$. Using the method of steps, it follows that for every $\phi \in C([t_0 - \rho, t_0], \mathbf{R})$ there exists a unique solution of (1) in the interval $[t_0, \infty)$ such that $y(t) = \phi(t)$ for $t_0 - \tau \leq t \leq t_0$.

As usual, a solution of (1) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

In the sequel, for convenience, when we write a functional inequality, we shall mean that it holds for all sufficiently large values of the argument t .

2. Proof of the theorem

The proof of the necessity part of the theorem is quite simple. If it is false, the characteristic equation (2) would have a real root λ_0 and therefore (1) would have the nonoscillatory solution

$$y(t) = e^{\lambda_0 t}.$$

But this contradicts the hypothesis that every solution of (1) oscillates.

On the other hand, the proof of the sufficiency part is quite involved and will be divided into several cases. Here the assumption is that (2) has no real roots. Set

$$F(\lambda) \equiv \lambda + p\lambda e^{-\lambda\tau} + \sum_{i=1}^k q_i e^{-\lambda\sigma_i}.$$

Then $F(0) = \sum_{i=1}^k q_i > 0$ and so $F(\lambda) > 0$ for every $\lambda \in \mathbf{R}$. Without loss of generality we will assume that $0 \leq \sigma_1 < \sigma_2 < \dots < \sigma_k$. Clearly,

$$\text{if } p > 0 \text{ then } \tau < \sigma_k. \quad (3)$$

Otherwise, $F(-\infty) = -\infty$ which is impossible. Also $F(+\infty) = F(-\infty) = +\infty$ and so there exists a positive constant m such that for every $\lambda \in \mathbf{R}$,

$$\lambda + p\lambda e^{-\lambda\tau} + \sum_{i=1}^k q_i e^{-\lambda\sigma_i} \geq m. \quad (4)$$

Assume, for the sake of contradiction, that (1) has an eventually positive solution $y(t)$. Set

$$z(t) = y(t) + py(t - \tau).$$

Using arguments similar to those in [2] and [5] the following lemma is easily established.

LEMMA 1.

(a) Let $p \geq -1$. Then $z(t)$ decreases monotonically to zero and

$$\lim_{t \rightarrow \infty} y(t) = 0.$$

(b) Let $p < -1$. Then $z(t)$ decreases monotonically to $-\infty$ and

$$\lim_{t \rightarrow \infty} y(t) = \infty.$$

(c) $z(t)$ is a differentiable solution of (1), that is,

$$\dot{z}(t) + pz(t - \tau) + \sum_{i=1}^k q_i z(t - \sigma_i) = 0. \tag{5}$$

(d) Set

$$w(t) = z(t) + pz(t - \tau).$$

Then $w(t)$ is a twice-differentiable solution of (5). Furthermore,

$$w(t) > 0 \text{ and } \ddot{w}(t) > 0.$$

In particular, it follows from Lemma 1(d) that

$$p \neq -1.$$

Otherwise,

$$\dot{w}(t) - \dot{w}(t - \tau) + \sum_{i=1}^k q_i w(t - \sigma_i) = 0,$$

and so

$$\dot{w}(t) - \dot{w}(t - \tau) < 0$$

which contradicts the fact that $\ddot{w}(t) > 0$.

As we remarked in the introduction, when $p\tau = 0$ the theorem reduces to a known result about nonneutral equations (see [6]). Thus we will assume that $p\tau \neq 0$.

In view of the above observations, in the sequel, we will assume that $\tau \neq 0$ and $p \in (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$.

Case 1. $p > 0$.

Set

$$v(t) = [y(t) + py(t - \tau)] + \sum_{i=1}^{k-1} q_i \int_{t-\sigma_k}^{t-\sigma_i} y(s) ds,$$

with the convention that for $k = 1$ the sum in the above notation is zero. Then

$$\dot{v}(t) = -\left(\sum_{i=1}^k q_i\right)y(t - \sigma_k) < 0$$

and so $v(t)$ is a positive and strictly decreasing function. Next, we claim that $v(t)$ is a differentiable solution of the equation

$$\dot{v}(t) + pv(t - \tau) + \sum_{i=1}^k q_i v(t - \sigma_i) = 0.$$

The proof of this claim is a consequence of Lemma 1(c) and the following interesting result.

LEMMA 2. Let $y(t)$ be a solution of (1) for $t \geq t_0$ and let α and β be any constants. Then

$$x(t) = \int_{t-\alpha}^{t-\beta} y(s) ds$$

is a solution of (1) for $t \geq t_0 + \max\{\alpha, \beta\}$.

PROOF. We have

$$\begin{aligned} \frac{d}{dt} [x(t) + px(t - \tau)] &= y(t - \beta) - y(t - \alpha) + p[y(t - \tau - \beta) - y(t - \tau - \alpha)] \\ &= [y(t - \beta) + py(t - \beta - \tau)] - [y(t - \alpha) + py(t - \alpha - \tau)] \\ &= \int_{t-\alpha}^{t-\beta} \frac{d}{ds} [y(s) + py(s - \tau)] ds = \int_{t-\alpha}^{t-\beta} - \left[\sum_{i=1}^k q_i y(s - \sigma_i) \right] ds \\ &= - \sum_{i=1}^k q_i \int_{t-\sigma_i-\alpha}^{t-\sigma_i-\beta} y(s) ds = - \sum_{i=1}^k q_i x(t - \sigma_i) \end{aligned}$$

and the proof is complete.

Set

$$v_n(t) = \begin{cases} v(t), & n = 0 \\ [v_{n-1}(t) + pv_{n-1}(t - \tau)] + \sum_{i=1}^{k-1} q_i \int_{t-\sigma_k}^{t-\sigma_i} v_{n-1}(s) ds, & n = 1, 2, \dots \end{cases} \tag{6}$$

Then, for $n = 1, 2, \dots$ we have

$$v_n(t) > 0, \quad \dot{v}_n(t) < 0, \quad \ddot{v}_n(t) > 0, \tag{7}$$

$$\dot{v}_n(t) = - \left(\sum_{i=1}^k q_i \right) v_{n-1}(t - \sigma_k), \tag{8}$$

and

$$\dot{v}_n(t) + p\dot{v}_n(t - \tau) + \sum_{i=1}^k q_i v_n(t - \sigma_i) = 0. \tag{9}$$

Set

$$\Lambda_n = \{ \lambda \geq 0 : \dot{v}_n(t) + \lambda v_n(t) \leq 0 \}.$$

The proof will be accomplished by showing that Λ_n has the following contradictory properties.

(P₁) Λ_n is a nonempty and bounded interval of nonnegative numbers. In particular, there exist numbers λ_1 and λ_2 , independent of n , such that

$$\lambda_1 \in \Lambda_n \text{ and } \lambda_2 \notin \Lambda_n \text{ for } n = 1, 2, \dots$$

(P₂) There is a positive number μ , independent of n , such that

$$\lambda \in \Lambda_n \text{ with } \lambda \geq \lambda_1 \Rightarrow (\lambda + \mu) \in \Lambda_{n+1} \text{ for } n = 1, 2, \dots$$

We will need the following lemma from [6].

LEMMA 3. *Let A and α be positive constants. Assume that $u(t) > 0$ satisfies the delay inequality*

$$\dot{u}(t) + Au(t - \alpha) \leq 0.$$

Then

$$u(t - \alpha) < Bu(t)$$

where $B = 4/(\alpha A)^2$.

From (7) and (9) we have,

$$\dot{v}_n(t - \tau) + p\dot{v}_n(t - \tau) + q_k v_n(t - \sigma_k) < 0$$

or

$$\dot{v}_n(t) + \frac{q_k}{1+p} v_n(t - (\sigma_k - \tau)) < 0. \tag{10}$$

Thus,

$$\dot{v}_n(t) + \frac{q_k}{1+p} v_n(t) < 0$$

which proves that

$$\lambda_1 \equiv \frac{q_k}{1+p} \in \Lambda_n.$$

Applying Lemma 3 to (10) we find

$$v_n(t - \alpha) < Bv_n(t) \tag{11}$$

where $\alpha = \sigma_k - \tau$ and $B = 4(1+p)^2/(\alpha^2 q_k^2)$. Integrating (8) from $t - \alpha$ to t we get

$$v_n(t) - v_n(t - \alpha) + \left(\sum_{i=1}^k q_i \right) \alpha v_{n-1}(t - \sigma_k) < 0$$

and so

$$\left(\sum_{i=1}^k q_i \right) v_{n-1}(t - \sigma_k) < \frac{1}{\alpha} v_n(t - \alpha). \tag{12}$$

Hence, from (8), using (11) and (12), we obtain

$$0 = \dot{v}_n(t) + \left(\sum_{i=1}^k q_i \right) v_{n-1}(t - \sigma_k) < \dot{v}_n(t) + \frac{B}{\alpha} v_n(t),$$

which proves that Λ_n is bounded above by

$$\lambda_2 \equiv B/\alpha = 4(1 + p)^2 / [(\sigma_k - \tau)^3 q_k^2].$$

The proof of (P_1) is complete. We now turn to proving (P_2) .

Let $\lambda \in \Lambda_n$ and set

$$\phi_n(t) = e^{\lambda t} v_n(t).$$

Then

$$\dot{\phi}_n(t) = e^{\lambda t} [\dot{v}_n(t) + \lambda v_n(t)] \leq 0$$

and so $\phi_n(t)$ is a decreasing function. Now for $\lambda \in \Lambda_n$ with $\lambda \geq \lambda_1$ and with

$$\mu = m / [1 + pe^{\lambda_2 \tau} + (e^{\lambda_2 \sigma_k} / \lambda_1) \sum_{i=1}^{k-1} q_i]$$

we have, using (3) and (4),

$$\begin{aligned} & \dot{v}_{n+1}(t) + (\lambda + \mu)v_{n+1}(t) \\ &= - \left(\sum_{i=1}^k q_i \right) v_n(t - \sigma_k) + (\lambda + \mu)v_n(t) + (\lambda + \mu)pv_n(t - \tau) \\ & \quad + (\lambda + \mu) \sum_{i=1}^{k-1} q_i \int_{t-\sigma_k}^{t-\sigma_i} v_n(s) ds \\ & \leq e^{-\lambda t} \left[- \left(\sum_{i=1}^k q_i \right) e^{\lambda \sigma_k} \phi_n(t - \sigma_k) + (\lambda + \mu)\phi_n(t) \right. \\ & \quad \left. + (\lambda + \mu)pe^{\lambda \tau} \phi_n(t - \tau) + (\lambda + \mu) \sum_{i=1}^{k-1} q_i e^{\lambda t} \int_{t-\sigma_k}^{t-\sigma_i} \phi_n(s) e^{-\lambda s} ds \right] \\ & \leq e^{-\lambda t} \phi_n(t - \sigma_k) \left[- \left(\sum_{i=1}^k q_i \right) e^{\lambda \sigma_k} + (\lambda + \mu) \right. \\ & \quad \left. + (\lambda + \mu)pe^{\lambda \tau} + (\lambda + \mu) \sum_{i=1}^{k-1} q_i (e^{\lambda \sigma_k} - e^{\lambda \sigma_i}) / \lambda \right] \\ & = e^{-\lambda t} \phi_n(t - \sigma_k) \left[- \sum_{i=1}^k q_i e^{\lambda \sigma_i} + \lambda + \lambda pe^{\lambda \tau} \right. \\ & \quad \left. + \mu \left(1 + pe^{\lambda \tau} + \frac{1}{\lambda} e^{\lambda \sigma_k} \sum_{i=1}^{k-1} q_i \right) - \frac{\mu}{\lambda} \sum_{i=1}^{k-1} q_i e^{\lambda \sigma_i} \right] \\ & \leq e^{-\lambda t} \phi_n(t - \sigma_k) \left[-m + \mu \left(1 + pe^{\lambda_2 \tau} + \frac{1}{\lambda_1} e^{\lambda_2 \sigma_k} \sum_{i=1}^{k-1} q_i \right) \right] = 0 \end{aligned}$$

which proves (P_2) . The proof of the theorem in Case 1 is complete.

Case 2. $-1 < p < 0$.

Set

$$z_n(t) = \begin{cases} z(t) = y(t) + py(t - \tau) & \text{if } n = 0, \\ z_{n-1}(t) + pz_{n-1}(t - \tau) & \text{if } n = 1, 2, \dots \end{cases}$$

Then, in view of Lemma 1(a), (c), and (d), for each $n = 1, 2, \dots$ we have

$$z_n(t) > 0, \quad \dot{z}_n(t) < 0, \quad \ddot{z}_n(t) > 0, \tag{13}$$

$$\dot{z}_n(t) = - \sum_{i=1}^k q_i z_{n-1}(t - \sigma_i), \tag{14}$$

and

$$\dot{z}_n(t) + p\dot{z}_n(t - \tau) + \sum_{i=1}^k q_i z_n(t - \sigma_i) = 0. \tag{15}$$

Set

$$\Lambda_n = \{ \lambda \geq 0 : \dot{z}_n(t) + \lambda z_n(t) \leq 0 \}.$$

As in Case 1, it suffices to establish that Λ_n has the contradictory properties (P_1) and (P_2) .

Clearly, $\lambda_1 \equiv 0 \in \Lambda_n$ for $n = 1, 2, \dots$. Next, we will prove that Λ_n is bounded above. From (13) and (15) we find,

$$\dot{z}_n(t) + q_k z_n(t - \sigma_k) < 0$$

which, in view of Lemma 2, implies that

$$z_n(t - \sigma_k) < \tilde{B} z_n(t), \tag{16}$$

where

$$\tilde{B} = 4 / (\sigma_k q_k)^2.$$

Note that if $\sigma_k = 0$ then Eq. (1) has $k = 1$ and this case has been covered in [3]. So, without loss of generality, we will assume that $\sigma_k > 0$. Now, integrating (14) from $t - \alpha$ to t , $\alpha > 0$ we get

$$z_n(t) - z_n(t - \alpha) + \sum_{i=1}^k q_i \int_{t-\alpha}^t z_{n-1}(s - \sigma_i) ds = 0,$$

or

$$\sum_{i=1}^k q_i z_{n-1}(t - \sigma_i) < \frac{1}{\alpha} z_n(t - \alpha). \tag{17}$$

Hence, from (14), using (16), (17) and $\alpha = \sigma_k$ we obtain

$$0 = \dot{z}_n(t) + \sum_{i=1}^k q_i z_{n-1}(t - \sigma_i) < \dot{z}_n(t) + \frac{\tilde{B}}{\sigma_k} z_n(t),$$

which proves that

$$\lambda_2 \equiv \tilde{B}/\sigma_k = 4/(\sigma_k^3 q_k^2)$$

is an upper bound of Λ_n . The proof of (P_1) is complete. Next, we will prove (P_2) with $\mu = m$. Let $\lambda \in \Lambda_n$ and set

$$\psi_n(t) = e^{\lambda t} z_n(t).$$

Then

$$\dot{\psi}_n(t) = e^{\lambda t} [\dot{z}_n(t) + \lambda z_n(t)] \leq 0$$

and so $\psi_n(t)$ is a decreasing function. Finally, for $\lambda \in \Lambda_n$ and using (4), we find

$$\begin{aligned} & \dot{z}_{n+1}(t) + (\lambda + m)z_{n+1}(t) \\ &= - \sum_{i=1}^k q_i z_n(t - \sigma_i) + (\lambda + m)z_n(t) + p(\lambda + m)z_n(t - \tau) \\ &= e^{-\lambda t} \left[- \sum_{i=1}^k q_i e^{\lambda \sigma_i} \psi_n(t - \sigma_i) + (\lambda + m)\psi_n(t) + p(\lambda + m)e^{\lambda \tau} \psi_n(t - \tau) \right] \\ &\leq e^{-\lambda \tau} \left[- \sum_{i=1}^k q_i e^{\lambda \sigma_i} + \lambda + p\lambda e^{\lambda \tau} + m \right] \psi_n(t) \leq e^{\lambda t} (-m + m) \psi_n(t) \\ &= 0, \end{aligned}$$

which proves (P_2) and completes the proof of the theorem in Case 2.

Case 3. $p < -1$.

This case will be divided into the following two subcases:

Subcase 3(i). $\sigma_k \leq \tau$. Set

$$u_n(t) = \begin{cases} -z(t) = -[y(t) + py(t - \tau)] & \text{if } n = 0 \\ -[u_{n-1}(t) + pu_{n-1}(t - \tau)] & \text{if } n = 1, 2, \dots \end{cases}$$

Then, in view of Lemma 1(b), (c), and (d), u_n satisfies the following properties for each $n = 1, 2, \dots$:

$$u_n(t) > 0, \quad \dot{u}_n(t) > 0, \quad \ddot{u}_n(t) > 0, \tag{18}$$

$$\dot{u}_n(t) = \sum_{i=1}^k q_i u_{n-1}(t - \sigma_i), \tag{19}$$

and

$$\dot{u}_n(t) + p\dot{u}_n(t - \tau) + \sum_{i=1}^k q_i u_n(t - \sigma_i) = 0. \tag{20}$$

Set

$$\Lambda_n = \{ \lambda \geq 0: -\dot{u}_n(t) + \lambda u_n(t) \leq 0 \}.$$

As in Case 1, it suffices to establish that Λ_n has the properties (P₁) and (P₂). Clearly $\lambda_1 \equiv 0 \in \Lambda_n$ for $n = 1, 2, \dots$. We have,

$$u_{n-1}(t - \tau) = \frac{u_n(t) + u_{n-1}(t)}{-p}$$

and so (19) yields

$$\begin{aligned} \dot{u}_n(t) &= \frac{1}{-p} \sum_{i=1}^k q_i [u_n(t + \tau - \sigma_i) + u_{n-1}(t + \tau - \sigma_i)] \\ &> \frac{1}{-p} \sum_{i=1}^k q_i u_{n-1}(t + \tau - \sigma_i). \end{aligned}$$

Integrating from $t - \tau$ to t we obtain

$$u_n(t) - u_n(t - \tau) > \frac{1}{-p} \sum_{i=1}^k q_i \int_{t-\tau}^t u_{n-1}(s + \tau - \sigma_i) ds$$

and so

$$\sum_{i=1}^k q_i u_{n-1}(t - \sigma_i) < \frac{-p}{\tau} u_n(t). \tag{21}$$

From (21) and (19) we find

$$-\dot{u}_n(t) + \left(\frac{-p}{\tau}\right) u_n(t) > 0$$

which proves that Λ_n is bounded above by $\lambda_2 \equiv -p/\tau$.

Let $\lambda \in \Lambda_n$ and set $\phi_n(t) = e^{-\lambda t} u_n(t)$. Then

$$\dot{\phi}_n(t) = e^{-\lambda t} [\dot{u}_n(t) - \lambda u_n(t)] = -e^{-\lambda t} [-\dot{u}_n(t) + \lambda u_n(t)] \geq 0$$

and so $\phi_n(t)$ is an increasing function.

For $\lambda \in \Lambda_n$ and $\mu = m/(-p)$ we have

$$\begin{aligned} & -\dot{u}_{n+1}(t) + (\lambda + \mu)u_{n+1}(t) \\ &= -\sum_{i=1}^k q_i u_n(t - \sigma_i) - (\lambda + \mu)u_n(t) - p(\lambda + \mu)u_n(t - \tau) \\ &= e^{\lambda t} \left[-\sum_{i=1}^k q_i e^{-\lambda \sigma_i} \phi_n(t - \sigma_i) - (\lambda + \mu)\phi_n(t) - p(\lambda + \mu)e^{-\lambda \tau} \phi_n(t - \tau) \right] \\ &\leq e^{\lambda t} \phi_n(t - \sigma_k) \left[-\sum_{i=1}^k q_i e^{-\lambda \sigma_i} - \lambda - \mu - p\lambda e^{-\lambda \tau} - p\mu e^{-\lambda \tau} \right] \\ &\leq e^{\lambda t} \phi_n(t - \sigma_k)(-m + me^{-\lambda \tau}) \leq 0, \end{aligned}$$

which completes the proof of the theorem in this subcase.

Subcase 3(ii). $\tau < \sigma_k$. Let $j = \min\{i: \tau < \sigma_i\}$. That is, $j = 1$ if $\tau < \sigma_1$ and $j = \nu$ if $\sigma_{\nu-1} \leq \tau < \sigma_\nu$ for some $\nu = 2, 3, \dots, k$. Set

$$w_n(t) = \begin{cases} -[y(t) + py(t - \tau)] + \sum_{i=j}^k q_i \int_{t-\sigma_i}^{t-\tau} y(s) ds & \text{if } n = 0, \\ -[w_{n-1}(t) + pw_{n-1}(t - \tau)] + \sum_{i=j}^k q_i \int_{t-\sigma_i}^{t-\tau} w_{n-1}(s) ds & \text{if } n = 1, 2, \dots \end{cases}$$

Then, one can prove that for each $n = 1, 2, \dots$ the following are true:

$$w_n(t) > 0, \quad \dot{w}_n(t) > 0, \quad \ddot{w}_n(t) > 0, \tag{22}$$

$$\dot{w}_n(t) = \sum_{i=1}^{j-1} q_i w_{n-1}(t - \sigma_i) + \left(\sum_{i=j}^k q_i \right) w_{n-1}(t - \tau), \tag{23}$$

and

$$\dot{w}_n(t) + p\dot{w}_n(t - \tau) + \sum_{i=1}^k q_i w_n(t - \sigma_i) = 0, \tag{24}$$

with the convention that when $j = 1$ the first sum in (23) is zero.

Set

$$\Lambda_n = \{ \lambda \geq 0: -\dot{w}_n(t) + \lambda w_n(t) \leq 0 \}.$$

As in Case 1, we will prove that Λ_n has the contradictory properties (P₁) and (P₂). We have,

$$w_n(t) < -pw_{n-1}(t - \tau) + \sum_{i=j}^k q_i (\sigma_i - \tau) w_{n-1}(t - \tau)$$

or

$$-w_n(t) / \left[p + \sum_{i=j}^k q_i(\tau - \sigma_i) \right] < w_{n-1}(t - \tau).$$

Hence, from (23)

$$\begin{aligned} 0 &\geq -\dot{w}_n(t) + \left(\sum_{i=j}^k q_i \right) w_{n-1}(t - \tau) \\ &> -\dot{w}_n(t) + \left[-\sum_{i=j}^k q_i \right] w_n(t) / \left[p + \sum_{i=j}^k q_i(\tau - \sigma_i) \right] \end{aligned}$$

which proves that

$$\lambda_1 \equiv -\sum_{i=j}^k q_i / \left[p + \sum_{i=j}^k q_i(\tau - \sigma_i) \right] \in \Lambda_n \quad \text{for } n = 1, 2, \dots$$

Next we will prove that Λ_n is bounded above. From (24) we find

$$\dot{w}_n(t) + p\dot{w}_n(t - \tau) < 0$$

and using (23) we obtain

$$\sum_{i=1}^{j-1} q_i w_{n-1}(t - \sigma_i) + \left(\sum_{i=j}^k q_i \right) w_{n-1}(t - \tau) + p\dot{w}_n(t - \tau) < 0$$

or

$$w_n(t) + \frac{1}{p} \sum_{i=1}^{j-1} q_i w_{n-1}(t + \tau - \sigma_i) + \frac{1}{p} \left(\sum_{i=j}^k q_i \right) w_{n-1}(t) > 0.$$

Integrating from $t - \tau$ to t we get

$$w_n(t) - w_n(t - \tau) + \frac{1}{p} \sum_{i=1}^{j-1} q_i \tau w_{n-1}(t - \sigma_i) + \frac{1}{p} \left(\sum_{i=j}^k q_i \right) \tau w_{n-1}(t - \tau) > 0$$

and so

$$\sum_{i=1}^{j-1} q_i w_{n-1}(t - \sigma_i) + \left(\sum_{i=j}^k q_i \right) w_{n-1}(t - \tau) < \frac{-p}{\tau} w_n(t).$$

Therefore (23) yields

$$-\dot{w}_n(t) + \frac{-p}{\tau} w_n(t) > 0$$

which proves that $\lambda_2 \equiv -p/\tau$ is an upper bound of Λ_n .

Let $\lambda \in \Lambda_n$ and set $\phi_n(t) = e^{-\lambda t} w_n(t)$. Then

$$\dot{\phi}_n(t) = -e^{-\lambda t} [-\dot{w}_n(t) + \lambda w_n(t)] \geq 0$$

and so $\phi_n(t)$ is an increasing function.

Now for $\lambda \in \Lambda_n$ with $\lambda \geq \lambda_1$ and with $\mu = 1/(-p + \frac{1}{\lambda_1} \sum_{i=1}^k q_i)$ we have,

$$\begin{aligned}
 & -\dot{w}_{n+1}(t) + (\lambda + \mu)w_{n+1}(t) \\
 &= -\sum_{i=1}^{j-1} q_i w_n(t - \sigma_i) - \left(\sum_{i=j}^k q_i \right) w_n(t - \tau) - (\lambda + \mu)w_n(t) \\
 &\quad - (\lambda + \mu)pw_n(t - \tau) + (\lambda + \mu) \sum_{i=j}^k q_i \int_{t-\sigma_i}^{t-\tau} w_n(s) ds \\
 &= e^{\lambda t} \left[-\sum_{i=1}^{j-1} q_i e^{-\lambda \sigma_i} \phi_n(t - \sigma_i) - \left(\sum_{i=j}^k q_i \right) e^{-\lambda \tau} \phi_n(t - \tau) - (\lambda + \mu) \phi_n(t) \right. \\
 &\quad \left. - (\lambda + \mu)pe^{-\lambda \tau} \phi_n(t - \tau) + (\lambda + \mu) \sum_{i=j}^k q_i e^{-\lambda \tau} \int_{t-\sigma_i}^{t-\tau} e^{\lambda s} \phi_n(s) ds \right] \\
 &\leq e^{\lambda t} \phi_n(t - \tau) \left[-\sum_{i=1}^{j-1} q_i e^{-\lambda \sigma_i} - \left(\sum_{i=j}^k q_i \right) e^{-\lambda \tau} - (\lambda + \mu) \right. \\
 &\quad \left. - (\lambda + \mu)pe^{-\lambda \tau} + (\lambda + \mu) \sum_{i=j}^k q_i (e^{-\lambda \tau} - e^{-\lambda \sigma_i})/\lambda \right] \\
 &= e^{\lambda t} \phi_n(t - \tau) \left[-\sum_{i=1}^k q_i e^{-\lambda \sigma_i} - \lambda - \lambda pe^{-\lambda \tau} - \mu - \mu pe^{-\lambda \tau} \right. \\
 &\quad \left. + \frac{\mu}{\lambda} \left(\sum_{i=j}^k q_i \right) e^{-\lambda \tau} - \frac{\mu}{\lambda} \sum_{i=1}^k q_i e^{-\lambda \sigma_i} \right] \\
 &\leq e^{\lambda t} \phi_n(t - \tau) \left[-m + \mu \left(-p + \frac{1}{\lambda_1} \sum_{i=1}^k q_i \right) \right] \\
 &= e^{\lambda t} \phi_n(t - \tau)(-m + m) = 0
 \end{aligned}$$

which completes the proof of the theorem in this subcase. The proof of the theorem is complete.

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