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AN INDEFINITE CONVECTION-DIFFUSION OPERATOR

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Abstract

We give a mathematically rigorous analysis which confirms the surprising results in a recent paper of Benilov, O'Brien and Sazonov [*J. Fluid Mech.* 497 (2003) 201–224] about the spectrum of a highly singular non-self-adjoint operator that arises in a problem in fluid mechanics. We also show that the set of eigenvectors does not form a basis for the operator.

1. *Introduction*

In a recent paper [4] Benilov, O'Brien and Sazonov have shown that the equation

$$\frac{\partial f}{\partial t} = \varepsilon \frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{\partial f}{\partial \theta} \tag{1}$$

approximates the evolution of a liquid film inside a rotating horizontal cylinder. The variable θ is taken to lie in $[-\pi, \pi]$ and one assumes that the solutions f are sufficiently smooth and satisfy periodic boundary conditions.

The operator H on the RHS of (1) is highly non-self-adjoint (NSA) and it is not amenable to standard elliptic techniques because the second order coefficient is indefinite. For $\theta \in (0, \pi)$ the second order term has a diffusive effect on the evolution but for $\theta \in (-\pi, 0)$ its effect is anti-diffusive. Many of the calculations in [4] are based on an asymptotic or WKB analysis for small $\varepsilon > 0$, but this has dangers because infinite order approximate eigenvalues of NSA operators need not be close to true eigenvalues. Nor need eigenvalues computed by truncations of a highly non-self-adjoint operator to large finite-dimensional subspaces by standard methods be close to the eigenvalues of the original operator. Such issues are best understood by reference to the notion of pseudospectra (see [**5**, **10**, **19**]), as was made clear in [4].

Our goal in this paper is to re-derive some of the results in [4] for a fixed positive value of ε by a rigorous and non-asymptotic technique. We also provide strong numerical evidence that the eigenvectors do not form a basis. Theorem 16 provides precise information about the local regularity of the eigenfunctions of (2). In our numerical calculations we take $\varepsilon = 0.1$, as in [4].

Before proceeding we mention that essentially the same operator was discussed in [**19**, pp. 124–125, 406–408] and [**3**]. I also wish to thank D. Pelinovski for informing me about [**6**], in which closely related results are obtained for this operator by a different method.

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2. *A reformulation of the problem*

We focus attention on the spectral properties of the operator

$$(Hf)(\theta) := \varepsilon \frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f}{\partial\theta}\right) + \frac{\partial f}{\partial\theta} \tag{2}$$

initially defined on all C^2 periodic functions $f \in L^2(-\pi,\pi)$; the exact domain will be described below. We normally assume that $0 < \varepsilon < 2$ for reasons explained in Corollary 2. According to the WKB analysis of [4] the eigenvalue equation $-iHf = \lambda f$ has a sequence of real eigenvalues which converge to the integers as $\varepsilon \to 0$. Our goal is to prove that there are indeed real eigenvalues λ without depending on WKB analysis, and to provide a simple and rigorous method for computing them.

Before starting the spectral analysis, we point out that [4] provides two types of evidence that the Cauchy problem (1) is not well-posed. If (1) were associated with a one-parameter semigroup, then there would exist positive constants M and a such that

$$\|(zI - H)^{-1}\| \leqslant M(\mathrm{Re}(z) - a)^{-1}$$

for all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > a$; see [**10**, Theorem 8.2.1]. The pseudospectral portrait of H in [**4**, Figure 6] establishes that no such bound exists, to the extent that numerical data can. The techniques described in [**7, 8**] allow one to construct approximate eigenfunctions for H and can be used to prove rigorously that the resolvent norms do indeed behave as shown in [**4**, Figure 6]. Finally, the narrowly concentrated 'exploding' Gaussian wave-packets constructed in [**4**, Section 4.1] also show that the Cauchy problem is not well posed.

By expanding $f \in L^2(-\pi,\pi)$ in the form

$$f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} v_n \mathrm{e}^{in\theta},$$

one may rewrite the eigenvalue problem in the form $Av = \lambda v$, where $A = -iH$ is given by

$$(Av)_n = \frac{\varepsilon}{2} n(n-1) v_{n-1} - \frac{\varepsilon}{2} n(n+1) v_{n+1} + n v_n.$$

The (unbounded) tridiagonal matrix A is of the form

$$A = \begin{pmatrix} A_- & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_+ \end{pmatrix}$$

where A_- acts in $l^2(\mathbb{Z}_-)$, the central 0 acts in $\mathbb{C}$ and A_+ acts in $l^2(\mathbb{Z}_+)$. The matrix of A_+ is of the form

$$A_+ = \begin{pmatrix} 1 & -\varepsilon & & & & \\ \varepsilon & 2 & -3\varepsilon & & & \\ & 3\varepsilon & 3 & -6\varepsilon & & \\ & & 6\varepsilon & 4 & -10\varepsilon & \\ & & & 10\varepsilon & 5 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}. \tag{3}$$

It is similar to matrices called twisted Toeplitz matrices in [**18, 19**], and to an infinite matrix of Lenferink and Spijker [**14**] whose pseudospectra were analyzed in [**17**]. Its analysis is made technically harder by the fact that although the first few off-diagonal elements may be very small, they increase more rapidly than the diagonal elements as $n \to \infty$.

The coefficient map $n \to -n$ induces a unitary equivalence between A_- and $-A_+$, so we only need to study the spectrum of A_+. Since $A_+^* = DA_+D^{-1}$ where $D_{r,s} = \delta_{r,s}(-1)^r$, A_+ and A_+^* have the same spectrum. We assume that A_+ has its natural maximal domain

$$\mathcal{D} = \{v \in l^2(\mathbb{Z}_+) : A_+ v \in l^2(\mathbb{Z}_+)\}$$

and observe that A_+ is continuous with respect to the topology of pointwise convergence. This implies that it is closed as follows: if $\|f_n - f\| \to 0$ and $\|A_+ f_n - g\| \to 0$, then $f_n \to f$ pointwise so $A_+ f_n \to A_+ f$ pointwise; since norm convergence implies pointwise convergence $A_+ f = g$. We will see that the eigenvectors of A_+ decrease more rapidly as $n \to +\infty$ the smaller $\varepsilon > 0$ is. We will prove that the spectrum is discrete; that is, that it consists only of isolated eigenvalues of finite multiplicity, in Section 3. We give a different description of the domain of A_+ in Lemma 12 and Theorem 13.

Benilov *et al.* correctly state in [**4**] that one obtains very poor numerical results if one simply truncates A to produce a finite matrix whose eigenvalues are then computed. We study the matrix A_+ in a completely different manner.

The eigenvalue equation for A_+ may be written in the form

$$n(n-1)v_{n-1} - n(n+1)v_{n+1} + 2\frac{n-\lambda}{\varepsilon}v_n = 0. \tag{4}$$

We will assume familiarity with the theory of difference equations as developed, for example, in [**15**]. The reality of the coefficients of (4) implies that if λ is an eigenvalue then so is $\overline{\lambda}$. Although this does not imply that all the eigenvalues are real, [**3, 4, 6**] provide substantial numerical and asymptotic evidence that this is the case. However, if $\varepsilon = 0.1$ and λ is significantly bigger than 10, the eigenvalues are so ill conditioned that one must be cautious about relying on such results as a guide to the actual spectral behaviour of A_+. Without making any commitment on this issue, we focus attention on the real eigenvalues.

Although it is only applicable in finite dimensions, the following example shows that one should expect matrices such as (3) to have complex eigenvalues; if they do not, then a positive reason needs to be found.

EXAMPLE. If n is even and $a_1, a_2, ..., a_n, b_1, ..., b_n$ are generic real constants and M_t is the real $n \times n$ matrix

$$M_{t,r,s} = \begin{cases} a_r & \text{if } r = s, \\ tb_r & \text{if } s = r+1 \mod(n), \\ -tb_{r-1} & \text{if } s = r+1 \mod(n), \end{cases}$$

then the eigenvalues of M_t are all real for small enough real t and they are all complex for large enough real t. At certain critical values of t two real eigenvalues collide and are converted into a complex conjugate pair of complex eigenvalues.

The particular case $a_r = r/n$, $b_r = 1$ and $t = 1$ is sometimes called the *Scottish flag matrix*, and is described in detail in [**19**, pp. 80–81]. For this choice of a_r and

b_r and all real t the eigenvalues lie very close to one of five (or fewer for certain critical t) straight-line segments, four of which are at an angle of 45 degrees to the horizontal axis.

Highly NSA operators with real spectra are not common but include the PT-symmetric Hamiltonian

$$(Hf)(x) = -f''(x) + ix^3 f(x)$$

acting in $L^2(\mathbb{R})$, which has been intensively studied because of its possible importance in physics [**1, 2, 12**]. A very complete and rigorous analysis of PT-symmetric Hamiltonians with polynomial potentials is given in [**16**]. This includes a description of some cases in which the spectrum is real, and others in which there are complex eigenvalues. A new perspective on these examples may be found in [**13**].

We confine attention to the solutions of (4) with support in $\mathbb{Z}^+$, and regard the $n = 1$ case, namely $\varepsilon v_2 = (1 - \lambda)v_1$, as an initial condition; equivalently, one may impose the initial condition $v_0 = 0$. Since it is a second-order recurrence equation, the solution space of (4) is two-dimensional. We will see that one solution, often called the subordinate solution, lies in $l^2(\mathbb{Z}_+)$, but no others do so if $0 < \varepsilon < 2$. It follows that $\lambda > 0$ is an eigenvalue of A_+ if and only if the subordinate solution of the recurrence equation satisfies the initial condition.

If one assumes that (4) has a solution of the form $v_n = n^a(1 + b/n + O(1/n^2))$, then one finds that $a = -1+1/\varepsilon$ and $b = \lambda/\varepsilon$. This motivates our next two lemmas. In the following calculations we introduce constants $N^{(i)}_{\lambda,\varepsilon}$, and will use the fact that they can always be increased without affecting the results.

LEMMA 1. *If $\lambda \geqslant 0$, there exists $N = N^{(1)}_{\lambda,\varepsilon}$ such that if v_n is a solution of (4) satisfying $0 \leqslant v_{N-i} \leqslant (N - i)^a(1 - 1/(N - i))$ for $i = 1, 2$ where $a = -1 + 1/\varepsilon$, then $0 \leqslant v_n \leqslant n^a(1 - 1/n)$ for all $n \geqslant N$.*

Proof. Suppose that $n \geqslant \lambda + 3$ and $0 < v_{n-i} \leqslant (n - i)^a(1 - 1/(n - i))$ for $i = 1, 2$. Then

$$\begin{aligned} n^{-a}v_n &= n^{-a}\left(\frac{n-2}{n}v_{n-2} + 2\frac{n-1-\lambda}{\varepsilon n(n-1)}v_{n-1}\right) \\ &\leqslant \left(1-\frac{2}{n}\right)^{a+1}\left(1-\frac{1}{n-2}\right) + 2\frac{n-1-\lambda}{\varepsilon n(n-1)}\left(1-\frac{1}{n}\right)^a\left(1-\frac{1}{n-1}\right) \\ &= 1 - \frac{1}{n} - \left(2+\frac{2\lambda}{\varepsilon}\right)\frac{1}{n^2} + O(n^{-3}) \\ &\leqslant 1 - \frac{1}{n} \end{aligned}$$

for all $n \geqslant N = N^{(1)}_{\lambda,\varepsilon}$, where we also assume that $N^{(1)}_{\lambda,\varepsilon} \geqslant \lambda + 3$. The first line of the above equation also shows that $v_n \geqslant 0$. It follows inductively that $0 \leqslant v_n \leqslant n^a(1 - 1/n)$ for all $n \geqslant N$. □

COROLLARY 2. *If $\lambda \geqslant 0$ and $\varepsilon > 2$ then every solution of (4) lies in $l^2(\mathbb{Z}_+)$. In particular, every such λ is an eigenvalue of A_+.*

Proof. Let $N = N^{(1)}_{\lambda,\varepsilon}$. Let u be the solution of (4) such that $u_{N-2} = 0$ and $u_{N-1} = (N - 1)^a(1 - 1/(N - 1))$, and let v be the solution such that $v_{N-2} = (N - 2)^a(1 - 1/(N - 2))$ and $v_{N-1} = 0$. Since $a < -1/2$, Lemma 1 implies that both lie in $l^2(\mathbb{Z}_+)$. The space of all solutions is two-dimensional, so every solution lies in $l^2(\mathbb{Z}_+)$, and this applies in particular to the solution that satisfies the initial condition. □

It is possible that one could avoid the above conclusion by imposing boundary conditions at $+\infty$ if $\varepsilon > 2$, that is, by reducing the domain of A_+. We do not pursue this idea.

LEMMA 3. *For every $\lambda \in \mathbb{R}$ there exists $N = N^{(2)}_{\lambda,\varepsilon}$ such that if v_n is a solution of (4) satisfying*

$$v_n \geqslant \left(1 + \frac{k}{n}\right) n^a \tag{5}$$

for $n = N-1$ and $n = N-2$, where $a = -1 + 1/\varepsilon$ and $k = 1 + \lambda/\varepsilon$, then the same inequality holds for all $n \geqslant N$.

Proof. Suppose that $n \geqslant \lambda + 3$, and that

$$v_{n-i} \geqslant \left(1 + \frac{k}{n-i}\right)(n-i)^a$$

for $i = 1$, 2. Then

$$\begin{aligned} n^{-a} v_n &= n^{-a}\left(\frac{n-2}{n} v_{n-2} + 2\frac{n-1-\lambda}{\varepsilon n(n-1)} v_{n-1}\right) \\ &\geqslant \left(1 + \frac{k}{n-2}\right)\left(1 - \frac{2}{n}\right)^{a+1} + 2\frac{n-1-\lambda}{\varepsilon n(n-1)}\left(1 + \frac{k}{n-1}\right)\left(1 - \frac{1}{n}\right)^{a} \\ &= 1 + \frac{k}{n} + \frac{2}{n^2} + O(n^{-3}) \\ &\geqslant 1 + \frac{k}{n}, \end{aligned}$$

provided that $n \geqslant N = N^{(2)}_{\lambda,\varepsilon}$. It follows inductively that (5) holds for all $n \geqslant N$. □

THEOREM 4. *If $0 < \varepsilon < 2$ and λ is a real eigenvalue of A_+, then $\lambda > 1$.*

Proof. Suppose that $A_+ v = \lambda v$ where $\lambda \leqslant 1$ and $v_1 = 1$. The initial condition $\varepsilon v_2 = (1-\lambda) v_1$ implies that $v_2 \geqslant 0$, and it then follows from the signs of the coefficients in (4) that $v_n > 0$ for all $n \geqslant 3$. Choosing N as in Lemma 3, the positivity of v_{N-1} and v_{N-1} imply that there exists a constant $c > 0$ such that

$$v_n / c \geqslant \left(1 + \frac{k}{n}\right) n^a$$

for $n = N-1$ and $n = N-2$. Lemma 3 now implies that

$$v_n / c \geqslant \left(1 + \frac{k}{n}\right) n^a$$

for all $n \geqslant N^{(2)}_{\lambda,\varepsilon}$. The lower bound $a > -1/2$ implies that $v \notin l^2(\mathbb{Z}_+)$, and hence that λ is not an eigenvalue of A_+. □

HYPOTHESIS. From this point onwards we assume that $0 < \varepsilon < 2$ and $\lambda \geqslant 0$.

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THEOREM 5. *For every $\delta > 0$ there exist $N = N_{\lambda,\varepsilon,\delta}$ and a solution v of* (4) *such that*

$$n^a \leqslant v_n \leqslant (1+\delta)n^a$$

for all $n \geqslant N$, where $a = -1 + 1/\varepsilon$.

Proof. Put

$$N = N_{\lambda,\varepsilon,\delta} = \max\left\{N^{(1)}_{\lambda,\varepsilon}, N^{(2)}_{\lambda,\varepsilon}, 2 + k/\delta\right\} \qquad \text{where } k = 1 + \lambda/\varepsilon,$$

and let v be the solution of (4) such that $v_{N-i} = (1+\delta)(N-i)^a$ for $i = 1, 2$. Lemma 1 implies that $0 < v_n \leqslant (1+\delta)n^a$ for all $n \geqslant N$. Since

$$v_n \geqslant \left(1 + \frac{k}{n}\right) n^a$$

for $n = N-1$ and $n = N-2$, we deduce by Lemma 3 that $v_n \geqslant (1 + k/n)n^a \geqslant n^a$ for all $n \geqslant N$. □

We will show that, up to a multiplicative constant, there is exactly one 'subordinate' solution v of (4) such that $\lim_{n\to+\infty} v_n = 0$. We identify this solution by solving the recurrence relation backwards from $n = M$ and then letting $M \to +\infty$.

LEMMA 6. *There exists $N = N^{(3)}_{\lambda,\varepsilon}$ such that if $M > N$ and $v_n = (-1)^n w_n$ is a solution of* (4) *satisfying $0 < w_{M+i} \leqslant (M+i)^{-c}$ for $i = 1, 2$ where $c = 1 + 1/\varepsilon$, then $0 < w_n \leqslant n^{-c}$ for all n satisfying $N \leqslant n \leqslant M$.*

Proof. The sequence w_n satisfies the recurrence relation

$$w_n = \frac{n+2}{n} w_{n+2} + \frac{2(n+1-\lambda)}{\varepsilon n(n+1)} w_{n+1}. \tag{6}$$

This has positive coefficients for $n \geqslant \lambda$ so the solution is positive if $\lambda < n \leqslant M$. Suppose inductively that $0 < w_{n+2} \leqslant (n+2)^{-c}$ and $0 < w_{n+1} \leqslant (n+1)^{-c}$ for such an n. Then

$$\begin{aligned} n^c w_n &\leqslant \left(1 + \frac{2}{n}\right)^{1-c} + \frac{2(n+1-\lambda)}{\varepsilon n(n+1)} \left(1 + \frac{1}{n}\right)^{-c} \\ &= 1 - \frac{2\lambda}{\varepsilon n^2} + O(n^{-3}) \\ &\leqslant 1 \end{aligned}$$

for all large enough n. By induction there exists $N = N_{\lambda,\varepsilon}$ such that $0 < w_n \leqslant n^{-c}$ provided that $N \leqslant n \leqslant M$. □

LEMMA 7. *There exists $N = N^{(4)}_{\lambda,\varepsilon}$ such that if $M > N$ and $v_n = (-1)^n w_n$ is a solution of* (4) *such that*

$$w_n \geqslant \left(1 - \frac{h}{n}\right) n^{-c} \tag{7}$$

for $n = M+1$ and $n = M+2$, where $c = 1 + 1/\varepsilon$ and $h = 1 + \lambda/\varepsilon$, then (7) *holds for all n satisfying $N \leqslant n \leqslant M$.*

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Proof. Suppose that $\max\{h, \lambda\} \leqslant n \leqslant M$ and (7) holds when n is replaced by $n+1$ or $n+2$. Then

$$\begin{aligned} n^c w_n &\geqslant \left(1 - \frac{h}{n+2}\right)\left(1 + \frac{2}{n}\right)^{1-c} + 2\frac{n+1-\lambda}{\varepsilon n(n+1)}\left(1 - \frac{h}{n+1}\right)\left(1 + \frac{1}{n}\right)^{-c} \\ &= 1 - \frac{h}{n} + \frac{2}{n^2} + O(n^{-3}) \\ &\geqslant 1 - \frac{h}{n}, \end{aligned}$$

provided that n is large enough. An induction now implies that there exists $N = N^{(4)}_{\lambda,\varepsilon}$ such that (7) holds for all n such that $N \leqslant n \leqslant M$. □

THEOREM 8. *There exist $N = N^{(5)}_{\lambda,\varepsilon}$ and a unique solution $v_n = (-1)^n w_n$ of (4) such that*

$$\left(1 - \frac{h}{n}\right) n^{-c} \leqslant w_n \leqslant n^{-c}$$

for all $n \geqslant N$, where $c = 1 + 1/\varepsilon$ and $h = 1 + \lambda/\varepsilon$. Hence

$$\lim_{n\to+\infty} w_n n^c = 1. \tag{8}$$

Proof. Let

$$M > N = N^{(5)}_{\lambda,\varepsilon} = \max\left\{N^{(3)}_{\lambda,\varepsilon}, N^{(4)}_{\lambda,\varepsilon}\right\}$$

and let $w^{(M)}$ denote the solution of (6) such that $w^{(M)}_n = n^{-c}$ for $n = M+1$ and $n = M+2$. Lemmas 6 and 7 imply that

$$\left(1 - \frac{h}{n}\right) n^{-c} \leqslant w^{(M)}_n \leqslant n^{-c}$$

for all n such that $N \leqslant n \leqslant M$. By choosing a sequence $M_r \to +\infty$ such that $w^{(M_r)}_N$ and $w^{(M_r)}_{N+1}$ converge as $r \to +\infty$, we see using (6) that $w^{(M_r)}_n$ converge for all $n \geqslant 1$. Denoting the limit by $w^{(\infty)}$, we deduce that

$$\left(1 - \frac{h}{n}\right) n^{-c} \leqslant w^{(\infty)}_n \leqslant n^{-c}$$

for all $n \geqslant N$. Putting

$$v^{(\infty)}_n = (-1)^n w^{(\infty)}_n,$$

the uniqueness of the solution $v^{(\infty)}$ subject to the normalization condition (8) follows from the fact that the solution space of (4) is two-dimensional and it contains a divergent sequence by Theorem 5. □

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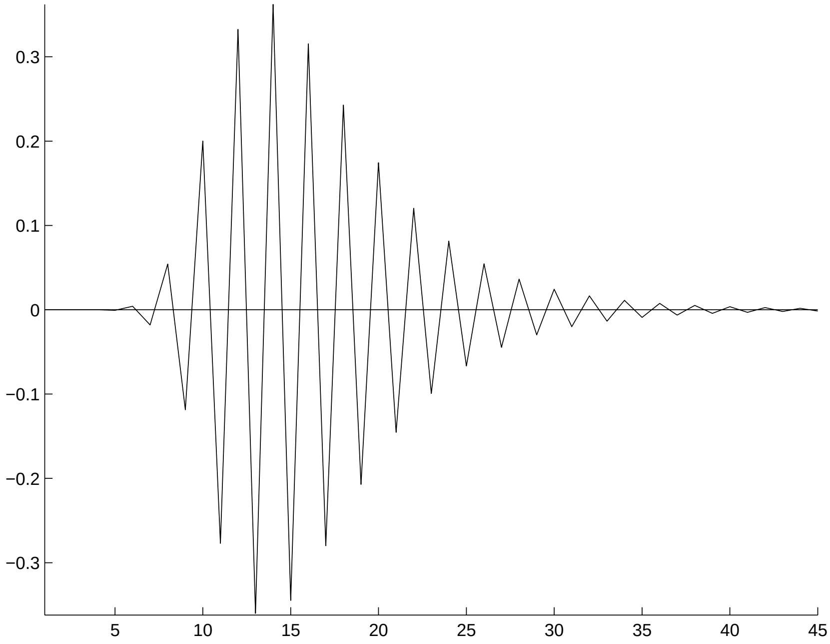

Figure 1: Eigenvector v for $\varepsilon = 0.1$ and $\lambda \sim 14.94784$

Numerical examples suggest that the following lemma is not the best possible and that w takes its maximum value very close to $n = \lambda$. Figure 1 portrays the eigenvector v of the operator A_+ for the eigenvalue $\lambda \sim 14.94784$ with $\varepsilon = 0.1$.

Lemma 9. *If $\lambda \geqslant 0$, then the unique subordinate solution w of (6) satisfies*

$$0 < w_{n+1} < w_n \qquad \text{for all } n \geqslant 2\lambda.$$

Proof. Let $w^{(M)}$ denote the solution of (6) constructed in the proof of Theorem 8. Then

$$\begin{aligned}
w_M^{(M)} &= \frac{M+2}{M}(M+2)^{-c} + \frac{2(M+1-\lambda)}{\varepsilon M(M+1)}(M+1)^{-c} \\
&= (M+1)^{-c}\left((1+2/M)^{1-c}(1+1/M)^{c} + \frac{2}{\varepsilon M}(1-\lambda/(M+1))\right) \\
&= (M+1)^{-c}\left(1 + c/M + O(M^{-2})\right) \\
&\geqslant (M+1)^{-c},
\end{aligned}$$

provided that M is large enough. Therefore

$$w_M^{(M)} \geqslant w_{M+1}^{(M)} \geqslant w_{M+2}^{(M)}.$$

We prove inductively that $w_n^{(M)} \geqslant w_{n+1}^{(M)}$ for all n such that $2\lambda \leqslant n \leqslant M$. If this holds with n replaced by $n+1$ or by $n+2$, then

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$$
\begin{aligned}
w_n^{(M)} - w_{n+1}^{(M)} &= \frac{n+2}{n} w_{n+2}^{(M)} + \frac{2(n+1-\lambda)}{\varepsilon n(n+1)} w_{n+1}^{(M)} \\
&\quad - \frac{n+3}{n+1} w_{n+3}^{(M)} - \frac{2(n+2-\lambda)}{\varepsilon(n+1)(n+2)} w_{n+2}^{(M)} \\
&= \frac{n+2}{n} w_{n+2}^{(M)} - \frac{n+3}{n+1} w_{n+3}^{(M)} \\
&\quad + \frac{2(n+1-\lambda)}{\varepsilon n(n+1)} w_{n+1}^{(M)} - \frac{2(n+2-\lambda)}{\varepsilon(n+1)(n+2)} w_{n+2}^{(M)} \\
&\geqslant \left(\frac{n+2}{n} - \frac{n+3}{n+1} \right) w_{n+3}^{(M)} \\
&\quad + \left(\frac{2(n+1-\lambda)}{\varepsilon n(n+1)} - \frac{2(n+2-\lambda)}{\varepsilon(n+1)(n+2)} \right) w_{n+2}^{(M)} \\
&\geqslant 0,
\end{aligned}
$$

provided that $n \geqslant 2\lambda$. This completes the induction. □

Finally, we take the same sequence M_r as in the proof of Theorem 8 to obtain

$$0 < w_{n+1}^{(\infty)} \leqslant w_n^{(\infty)} \qquad \text{for all } n \geqslant 2\lambda.$$

3. *Compactness of the resolvent*

In this section we prove that $0 \notin \mathrm{Spec}(A_+)$ and that A_+^{-1} is a Hilbert–Schmidt operator and hence is compact. This implies that the spectrum of A_+ is discrete and coincides with its set of eigenvalues. We cannot, however, prove that the spectrum is real. We define the Hilbert–Schmidt operator R on $l^2(\mathbb{Z}_+)$ by

$$(Rf)_m = \sum_{n=1}^{\infty} \rho_{m,n} f_n \tag{9}$$

where $\rho \in l^2(\mathbb{Z}_+ \times \mathbb{Z}_+)$ is given by (14). We then show directly that R is the inverse of A_+.

Let ϕ be the solution of

$$\frac{\varepsilon}{2} n(n-1)\phi_{n-1} - \frac{\varepsilon}{2} n(n+1)\phi_{n+1} + n\phi_n = 0 \tag{10}$$

that satisfies the initial conditions $\phi_0 = 0$ and $\phi_1 = 1$. One sees immediately that $\phi_n > 0$ for all $n \geqslant 1$. Theorem 5 implies that there exists a constant $c_1 > 0$ such that

$$c_1^{-1} n^a \leqslant \phi_n \leqslant c_1 n^a$$

for all $n \geqslant 1$.

Let $\psi_n = (-1)^n w_n$ be the unique subordinate solution of

$$\frac{\varepsilon}{2} n(n-1)\psi_{n-1} - \frac{\varepsilon}{2} n(n+1)\psi_{n+1} + n\psi_n = 0 \tag{11}$$

such that w satisfies the normalization condition $\lim_{n\to+\infty} n^c w_n = 1$. Since

$$w_n = \frac{n+2}{n} w_{n+2} + \frac{2}{\varepsilon n} w_{n+1},$$

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we see that $w_n > 0$ for all $n \geqslant 1$, and indeed that there exists a constant $c_2 > 0$ such that

$$c_2^{-1} n^{-c} \leqslant w_n \leqslant c_2 n^{-c}$$

for all $n \geqslant 1$.

We finally define the discrete analogue of the Wronskian, namely

$$\sigma_n = \frac{\varepsilon}{2} n(n-1)\phi_{n-1} w_n + \frac{\varepsilon}{2} n(n+1)\phi_n w_{n+1} + n\phi_n w_n.$$

LEMMA 10. *The sequence σ_n is constant and positive.*

Proof. The positivity follows immediately from the positivity of ϕ_n and w_n. To prove the constancy, we use (11) to obtain

$$\begin{aligned} \tau_n &= (-1)^n \sigma_n \\ &= \frac{\varepsilon}{2} n(n-1)\phi_{n-1}\psi_n - \frac{\varepsilon}{2} n(n+1)\phi_n \psi_{n+1} + n\phi_n\psi_n \\ &= \frac{\varepsilon}{2} n(n-1)\left(\phi_{n-1}\psi_n - \phi_n \psi_{n-1}\right). \end{aligned} \tag{12}$$

If we perform the simplification using (10) instead, we obtain

$$\tau_n = \frac{\varepsilon}{2} n(n+1)\left(\phi_{n+1}\psi_n - \phi_n\psi_{n+1}\right). \tag{13}$$

By comparing (12) and (13) we deduce that $\tau_n = -\tau_{n+1}$. □

If $\varepsilon = 0.1$ and w_n, ϕ_n are normalized by $w_1 = \phi_1 = 1$, then $\sigma \sim 1.004928137$.

THEOREM 11. *If $0 < \varepsilon < 2$ and*

$$\rho_{m,n} = \begin{cases} (-1)^n \phi_m \psi_n / \sigma & \text{if } m \leqslant n, \\ (-1)^n \psi_m \phi_n / \sigma & \text{if } m > n, \end{cases} \tag{14}$$

then $\rho \in l^2(\mathbb{Z}_+ \times \mathbb{Z}_+)$. The Hilbert–Schmidt operator R defined by (9) satisfies $A_+ R f = f$ for all $f \in l^2(\mathbb{Z}_+)$. Indeed, $0 \notin \operatorname{Spec}(A_+)$ and $R = A_+^{-1}$.

Proof. The above bounds on ϕ, ψ, σ imply that

$$|\rho_{m,n}| \leqslant \begin{cases} c_4 m^a n^{-a-2} & \text{if } m \leqslant n, \\ c_4 m^{-c} n^{c-2} & \text{if } m > n. \end{cases} \tag{15}$$

It follows that

$$\sum_{m=1}^{\infty} |\rho_{m,n}|^2 \leqslant c_5 n^{-3}$$

and then that

$$\sum_{m,n=1}^{\infty} |\rho_{m,n}|^2 < \infty.$$

We conclude that R is a compact operator. If $\{e_n\}_{n=1}^{\infty}$ is the standard basis in $l^2(\mathbb{Z}_+)$, then a direct calculation shows that $A_+ R e_n = e_n$ for all n. By using the fact that A_+ is closed, one deduces that $\operatorname{Ran}(R) \subseteq \operatorname{Dom}(A_+)$ and that $A_+ R f = f$ for all $f \in l^2(\mathbb{Z}_+)$. We conclude from this that $\operatorname{Ran}(A_+) = l^2(\mathbb{Z}_+)$. The bound $0 < \varepsilon < 2$ implies that $\operatorname{Ker}(A_+) = \{0\}$ by Theorem 4, so we finally see that $0 \notin \operatorname{Spec}(A_+)$ and that $R = A_+^{-1}$. □

Because R is compact, its eigenvalues can in principle be computed as the limits of the eigenvalues of its truncations R_n to the subspaces of sequences with support in $\{1, 2, \dots, n\}$. However, the convergence is slow and R has a full matrix, so this is not computationally efficient.

We use the bounds on $\rho_{m,n}$ to prove that A_+ equals the minimal operator B_+ associated with the infinite matrix (3). This is defined as the closure of the restriction of A_+ to the subspace $\mathcal{D}$ consisting of those $v \in l^2(\mathbb{Z}_+)$ that have finite support.

LEMMA 12. *If $v \in \mathrm{Dom}(A_+)$ satisfies $|v_n| \leqslant \sigma n^{-s}$ for all $n \in \mathbb{Z}_+$ where $\sigma \geqslant 0$ and $s > 3/2$, then $v \in \mathrm{Dom}(B_+)$.*

Proof. We define the sequences $\phi_N, v_N \in \mathcal{D}$ by

$$\phi_{N,n} = \begin{cases} 1 & \text{if } n \leqslant N, \\ 2 - n/N & \text{if } N \leqslant n \leqslant 2N, \\ 0 & \text{if } n \geqslant 2N, \end{cases}$$

and $v_{N,n} = \phi_{N,n} v_n$. We then have

$$\begin{aligned} (A_+ v - B_+ v_N)_n &= \frac{\varepsilon}{2} n(n-1) v_{n-1} - \frac{\varepsilon}{2} n(n+1) v_{n+1} + n v_n \\ &\quad - \frac{\varepsilon}{2} n(n-1) \phi_{N,n-1} v_{n-1} + \frac{\varepsilon}{2} n(n+1) \phi_{N,n+1} v_{n+1} - n \phi_{N,n} v_n \\ &= \frac{\varepsilon}{2} n(n-1)(v_{n-1} - \phi_{N,n-1} v_{n-1}) \\ &\quad - \frac{\varepsilon}{2} n(n+1)(v_{n+1} - \phi_{N,n+1} v_{n+1}) + n(v_n - \phi_{N,n} v_n) \\ &= (1 - \phi_{N,n})(A_+ v)_n + \frac{\varepsilon}{2} n(n-1)(\phi_{N,n} - \phi_{N,n-1}) v_{n-1} \\ &\quad - \frac{\varepsilon}{2} n(n+1)(\phi_{N,n} - \phi_{N,n+1}) v_{n+1}. \end{aligned}$$

Therefore

$$A_+ v - B_+ v_N = (1 - \phi_N) A_+ v - w_N - x_N$$

where

$$w_N = \begin{cases} 0 & \text{if } n \leqslant N, \\ 0 & \text{if } n \geqslant 2N + 1, \\ \varepsilon n(n-1) v_{n-1}/2N & \text{if } N + 1 \leqslant n \leqslant 2N, \end{cases}$$

and

$$x_N = \begin{cases} 0 & \text{if } n \leqslant N - 1, \\ 0 & \text{if } n \geqslant 2N, \\ \varepsilon n(n+1) v_{n+1}/2N & \text{if } N \leqslant n \leqslant 2N - 1. \end{cases}$$

Both w_N and x_N vanish outside the interval $[N, 2N]$ and inside this interval they are dominated by $\varepsilon |n v_n| \leqslant \varepsilon \sigma n^{1-s}$. Therefore

$$\lim_{N \to \infty} \|w_N\| = \lim_{N \to \infty} \|x_N\| = 0.$$

Hence

$$\lim_{N \to \infty} \|A_+ v - B_+ v_N\| = 0.$$

We conclude that $v \in \mathrm{Dom}(B_+)$ and $B_+ v = A_+ v$. □

THEOREM 13. *The operators A_+ and B_+ are equal.*

Proof. Lemma 12 implies that $Re_n \in \mathrm{Dom}(B_+)$ for every e_n in the standard basis for $l^2(\mathbb{Z}_+)$. Hence $Rf \in \mathrm{Dom}(B_+)$ and $B_+Rf = f$ for all $f \in \mathcal{D}$. By using the boundedness of R and the closedness of B_+ we deduce that $Rf \in \mathrm{Dom}(B_+)$ for all $f \in l^2(\mathbb{Z}_+)$ and that $B_+Rf = f$ for all such f. Now let $f \in \mathrm{Dom}(A_+)$. If $g = RA_+f$, then $g \in \mathrm{Dom}(B_+)$ and $A_+g = B_+g = B_+RA_+f = A_+f$, so $A_+(f-g) = 0$. Since $\mathrm{Ker}(A_+) = \{0\}$ we deduce that $f = g$ and hence that $\mathrm{Dom}(B_+) = \mathrm{Dom}(A_+)$. □

COROLLARY 14. *If λ is a complex eigenvalue of A_+, then* $\mathrm{Re}(\lambda) \geqslant 1$.

Proof. If $w \in \mathcal{D}$, then an elementary calculation establishes that $\mathrm{Re}\langle B_+w, w\rangle \geqslant \langle w, w\rangle$. If v is the eigenvector associated with λ and $v_N \in \mathcal{D}$ converge to v in the graph norm of B_+, then $\|v_N - v\| \to 0$ and $\|B_+v_N - B_+v\| \to 0$. Therefore

$$\begin{aligned}\|v\|^2 = \lim_{N\to\infty} \|v_N\|^2 &\leqslant \lim_{N\to\infty} \mathrm{Re}\langle B_+v_N, v_N\rangle \\ &= \mathrm{Re}\langle B_+v, v\rangle = \mathrm{Re}(\lambda)\|v\|^2.\end{aligned}$$ □

The next corollary is a direct application of the Lumer–Phillips theorem; see [**10**, Theorem 8.3.5]. However, [**10**, Theorem 8.4.1] and the numerical plots of the pseudospectra of A_+ in [**4, 19**] together strongly suggest that T_t is not a holomorphic semigroup.

COROLLARY 15. *There exists a one-parameter contraction semigroup T_t on $l^2(\mathbb{Z}_+)$ with generator $Z = -A_+$.*

Our next theorem provides a precise bound on the decay of the Fourier coefficients of any eigenfunction of (2), and hence precise information about the degree of local regularity of the eigenfunction itself. The bound is expressed in terms of the norm

$$\|w\|_{\infty,\gamma} = \sup\{|w_n|n^\gamma : 1 \leqslant n < \infty\}.$$

In principle, the theorem also provides quantitative control on the rate of convergence of the spectrum of the truncation $A_{N,+}$ to A_+ as $N \to \infty$. However, the magnitudes of b and m may be too large for this to be numerically useful. If one already knows that all eigenvalues of H are real, then the theorem is not as sharp as Theorem 8, but (16) may still be useful.

THEOREM 16. *There exist constants b, m such that if $v \in \mathrm{Dom}(A_+)$, $\lambda \in \mathbb{C}$ and $A_+v = \lambda v$ then*

$$\|v\|_{\infty,c} \leqslant b|\lambda|^m \|v\|_2.$$

Proof. The theorem is an immediate corollary of the bound

$$\|R^m w\|_{\infty,c} \leqslant b\|w\|_2, \tag{16}$$

valid for all $w \in l^2(\mathbb{Z}_+)$. We prove this below by an inductive procedure using (15).

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If $w \in l^2(\mathbb{Z}_+)$, then

$$\begin{aligned} |(Rw)_m|^2 &= \left| \sum_{n=1}^{\infty} \rho_{m,n} w_n \right|^2 \\ &\leqslant \|w\|_2^2 \left(\sum_{n=1}^{m} |\rho_{m,n}|^2 + \sum_{n=m+1}^{\infty} |\rho_{m,n}|^2 \right) \\ &\leqslant b_1 m^{-3} \end{aligned}$$

by applying (15) together with the inequalities $c > 3/2$, $a > -1/2$. Therefore

$$\|Rw\|_{\infty,3/2} \leqslant b_2 \|w\|_2.$$

Next suppose that $\|w\|_{\infty,\gamma} < \infty$ where $1 \leqslant \gamma < c-1$. Since

$$\begin{aligned} |(Rw)_m| &= \left| \sum_{n=1}^{\infty} \rho_{m,n} w_n \right| \\ &\leqslant \|w\|_{\infty,\gamma} \left(\sum_{n=1}^{m} |\rho_{m,n}| n^{-\gamma} + \sum_{n=m+1}^{\infty} |\rho_{m,n}| n^{-\gamma} \right) \\ &\leqslant b_2 m^{-\gamma-1}, \end{aligned}$$

we deduce that

$$\|Rw\|_{\infty,\gamma+1} \leqslant b_3 \|w\|_{\infty,\gamma}.$$

If, however, $\gamma + 1 > c$ then a similar estimate only yields

$$\|Rw\|_{\infty,c} \leqslant b_3 \|w\|_{\infty,\gamma}.$$

Starting from one of $\gamma = 1$, $\gamma = 3/2$, we obtain (16) after a sufficient number of iterations. $\square$

4. λ-*dependence*

In this section we prove that the unique normalized subordinate solution $v_{\lambda,n} = (-1)^n w_{\lambda,n}$ of (4) provided by Theorem 8 depends continuously on λ.

We first observe that for any $\Lambda \geqslant 1$ the various constants $N_{\lambda,\varepsilon}^{(i)}$ defined above are uniformly bounded with respect to λ, provided that $0 \leqslant \lambda \leqslant \Lambda$. We use the notation $\tilde{N}_{\Lambda,\varepsilon}^{(i)}$ to refer to the relevant upper bounds.

LEMMA 17. *If* $0 \leqslant \lambda \leqslant \mu \leqslant \Lambda$, *then*

$$0 < w_{\Lambda,n} \leqslant w_{\mu,n} \leqslant w_{\lambda,n} \leqslant w_{0,n} < \infty$$

for all $n \geqslant \Lambda$.

Proof. The positivity of $w_{\lambda,n}$ for $n \geqslant \Lambda$ follows from the positivity of the coefficients of (6) for $n \geqslant \Lambda$ and the positivity of $w_{\lambda,n}$ for all $n \geqslant N = \tilde{N}_{\Lambda,\varepsilon}^{(5)}$. We need only prove the central inequality above, since the other two are special cases of it.

Theorem 8 implies that if $\delta > 0$, then

$$w_{\mu,n} \leqslant (1+\delta) w_{\lambda,n} \qquad \text{for all } n \geqslant N = \tilde{N}_{\Lambda,\varepsilon,\delta}^{(6)}. \tag{17}$$

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This inequality persists for all $n \in [\Lambda, N]$ by the monotonicity of the coefficients of (6). Since (17) holds for all $\delta > 0$ and all $n \geqslant \Lambda$, the required inequality follows by letting $\delta \to 0$. □

LEMMA 18. *If $0 \leqslant \lambda \leqslant \mu \leqslant \Lambda$ and $|\mu - \lambda| \leqslant \delta$ then*

$$0 < w_{\lambda,n} \leqslant p_{\Lambda,\varepsilon,n,\delta} w_{\mu,n} \tag{18}$$

for all $n \geqslant 2\Lambda$, where

$$p_{\Lambda,\varepsilon,n,\delta} = (1+\delta) \exp\left\{ 2\delta\varepsilon^{-1} \sum_{r=n}^{\infty} r^{-2} \right\}.$$

Proof. Since $1 + \delta \leqslant p_{\Lambda,\varepsilon,n,\delta}$, Theorem 8 implies that (18) holds for all $n \geqslant N = \tilde{N}^{(6)}_{\Lambda,\varepsilon,\delta}$. We prove inductively that the same inequality persists for $n \in [2\Lambda, N]$. If (18) holds with n replaced by $n+1$ and by $n+2$, then, using Lemma 9, we obtain

$$\begin{aligned}
w_{\lambda,n} &= \frac{n+2}{n} w_{\lambda,n+2} + \frac{2(n+1-\lambda)}{\varepsilon n(n+1)} w_{\lambda,n+1} \\
&\leqslant \frac{n+2}{n} p_{\Lambda,\varepsilon,n+2,\delta} w_{\mu,n+2} + \frac{2(n+1-\lambda)}{\varepsilon n(n+1)} p_{\Lambda,\varepsilon,n+1,\delta} w_{\mu,n+1} \\
&\leqslant p_{\Lambda,\varepsilon,n+1,\delta} \left(\frac{n+2}{n} w_{\mu,n+2} + \frac{2(n+1-\lambda)}{\varepsilon n(n+1)} w_{\mu,n+1} \right) \\
&\leqslant p_{\Lambda,\varepsilon,n+1,\delta} \left(\frac{n+2}{n} w_{\mu,n+2} + \frac{2(n+1-\mu)}{\varepsilon n(n+1)} w_{\mu,n+1} + \frac{2\delta}{\varepsilon n^2} w_{\mu,n+1} \right) \\
&\leqslant p_{\Lambda,\varepsilon,n+1,\delta} \left(w_{\mu,n} + \frac{2\delta}{\varepsilon n^2} w_{\mu,n+1} \right) \\
&\leqslant p_{\Lambda,\varepsilon,n+1,\delta} \left(1 + \frac{2\delta}{\varepsilon n^2} \right) w_{\mu,n} \\
&\leqslant p_{\Lambda,\varepsilon,n,\delta} w_{\mu,n}.
\end{aligned}$$

This completes the induction. □

The function f defined in (19) specifies the initial conditions, and vanishes if and only if λ is an eigenvalue.

THEOREM 19. *The unique normalized subordinate solution v_λ defined in Theorem 8 depends continuously on λ for $0 \leqslant \lambda < \infty$. Hence the function*

$$f(\lambda) := \varepsilon v_{\lambda,2} - (1-\lambda) v_{\lambda,1} \tag{19}$$

is continuous on $[0, \infty)$.

Proof. It is sufficient to prove that f is continuous on $[0, \Lambda]$ for every positive integer Λ. It follows directly from the estimates in Lemmas 17 and 18 that the map $\lambda \in [0, \Lambda] \to (w_{\lambda,2\Lambda}, w_{\lambda,2\Lambda+1})$ is continuous. Composing this with the continuous map $(w_{\lambda,2\Lambda}, w_{\lambda,2\Lambda+1}) \to \varepsilon v_{\lambda,2} - (1-\lambda) v_{\lambda,1}$ yields the second statement of the theorem; see [**15**]. □

The function f in (19) can be calculated as defined, but there are other formulae which are more stable numerically. These are based upon the following lemma.

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LEMMA 20. *Let v_λ be defined as in Theorem 8, and let ϕ_λ be the solution of*

$$n(n-1)\phi_{\lambda,n-1} - n(n+1)\phi_{\lambda,n+1} + 2\frac{n-\lambda}{\varepsilon}\phi_{\lambda,n} = 0$$

that satisfies the initial conditions $\phi_{\lambda,1} = \varepsilon$ and $\phi_{\lambda,2} = 1-\lambda$. Then the Wronskian

$$W(\lambda,n) = \frac{1}{2}(-1)^n n(n+1)(\phi_{\lambda,n+1}v_{\lambda,n} - \phi_{\lambda,n}v_{\lambda,n+1})$$

defined for $n \geqslant 1$ does not depend on n, and is given by

$$W(\lambda,n) = f(\lambda).$$

The proof closely follows that of Lemma 10.

5. *Numerical calculations*

Following the notation of Lemma 20, the eigenvalues λ_n are the solutions of $f(\lambda) = 0$, or equivalently of $W(\lambda, m_n) = 0$, where we may choose m_n to maximize the numerical stability of the calculation.

If λ is a real eigenvalue of A_+ and ϕ is the corresponding real, l^2 eigenvector then an elementary calculation yields

$$\lambda \sum_{n=1}^{\infty} \phi_n^2 = \langle A_+\phi, \phi\rangle = \sum_{n=1}^{\infty} n\phi_n^2.$$

This supports the numerical evidence that $|\phi_n|$ increases rapidly for $n \leqslant \lambda$ and decreases for $n > \lambda$, as in Figure 1. Assuming that this is the case, a good way of computing ϕ is to use forward iteration from $n = 1$ up to $n = [\lambda]$, backward iteration from ∞ for $n \geqslant [\lambda]$ and then match the two solutions at $n = [\lambda]$. One can therefore find the eigenvalues $\lambda \geqslant 1$ by solving $W(\lambda, [\lambda]) = 0$. Since $W(\lambda, n)$ is independent of n it is still a continuous function of λ even when n is chosen in this discontinuous manner.

Table 1: Eigenvalues and condition numbers for A_+ with $\varepsilon = 0.1$

n	λ_n	$\|P_n\|$
1	1.00968	1.0189
2	2.07334	1.1848
3	3.22978	1.8868
4	4.50134	4.3409
5	5.89993	13.341
6	7.43194	50.638
7	9.10097	226.20
8	10.9092	1152.9
9	12.8578	6561.3
10	14.9478	41018
15	27.5331	–
20	43.74	–

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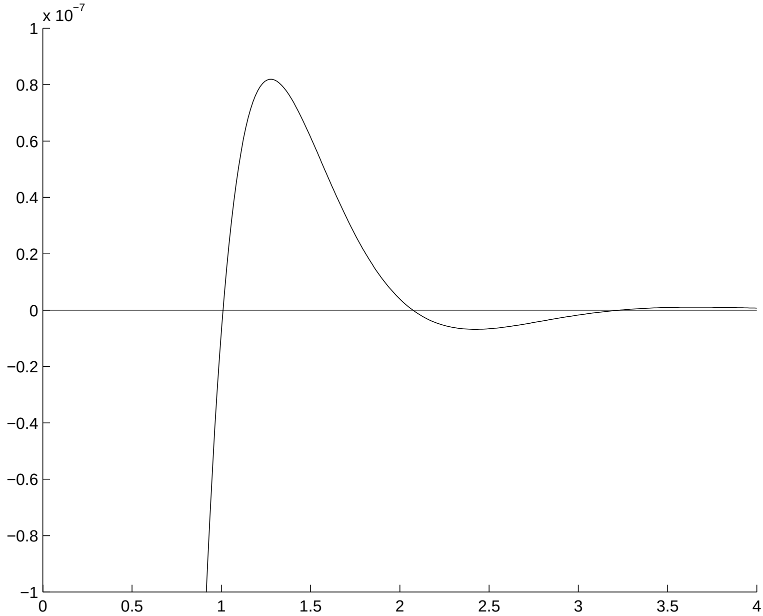

Figure 2: $f(\lambda)$ for $\varepsilon = 0.1$ and $0 \leqslant \lambda \leqslant 4$

Since f is continuous, one can compute the roots of $f(\lambda) = 0$ by evaluating $f(\lambda)$ numerically for a range of values of λ. We determined the subordinate solution by solving (6), starting from $M = 4000$ (and also $M = 8000$ to check consistency) with $w_{M+i} = (M+i)^{-c}$ for $i = 1$, 2. Figure 2 plots $f(\lambda)$ for $\varepsilon = 0.1$ and $0 \leqslant \lambda \leqslant 4$. The eigenvalues listed in Table 1 were obtained by solving $f(\lambda) = 0$ numerically, and are quite close to those obtained in [4]. We carried out a similar calculation using $W(\lambda, [\lambda])$ as described above, obtaining the same results.

The accuracy of the computations is limited by the rapidly increasing condition numbers of the spectral projections. Rounding errors makes it hard to compute more than 20 eigenvalues for $\varepsilon = 0.1$ *whatever* numerical algorithm is used. In our particular approach, the problem is that the function f gets rapidly smaller as λ increases, so the points at which it vanishes become harder to determine.

There is also cutoff error: we are forced to choose a value for M, and the convergence of the method is quite slow as M increases; on the other hand, the individual calculations are so simple that one can choose large values of M without making the implementation of the algorithm impractical. By choosing various values of M, one can be confident that the values in Table 1 are accurate to the stated accuracy.

For $\varepsilon = 1$, the Fourier coefficients decrease much more slowly, and the eigenvalue calculation is correspondingly more onerous. We computed the first five eigenvalues for $\varepsilon = 1$, determining the subordinate solution as before with M between 1000 and 32000. The apparent number of solutions of $f(\lambda) = 0$ increased from 7 to 11 as M increased in this range. For $M = 4000$ it appeared that the computation of the first five eigenvalues presented in Table 2 was reliable to the accuracy given.

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We conclude by providing strong evidence that the eigenvectors v_n of A_+ do not form a basis; this question was raised but not resolved in [**4**, **3**]. It seems quite plausible that they form a complete set in the sense that their linear span is dense.

If the eigenvectors form an unconditional basis, then there exists a bounded invertible operator S such that $\{Sv_n\}_{n=1}^{\infty}$ is a complete orthonormal sequence; see [**10**, Theorem 3.4.5]. This implies that $B = SA_+S^{-1}$ is self-adjoint and hence that

$$\|(zI - A_+)^{-1}\| \leqslant \|S\|\,\|S^{-1}\|\,|\,\mathrm{Im}(z)|^{-1}$$

for all $z \notin \mathbb{R}$. However, the pseudospectral portrait of [**4**, Figure 6], confirmed in [**19**, pp. 124–125, 406–408] for almost the same operator, shows, to the extent that numerical data can, that no such bound holds.

If the eigenfunctions of the differential operator H defined in (2) form a conditional (Schauder) basis, then the spectral projections

$$P_n f = \frac{\langle f, \phi_n^* \rangle}{\langle \phi_n, \phi_n^* \rangle} \phi_n$$

of H must be uniformly bounded in norm, where ϕ_n are the eigenfunctions of H and ϕ_n^* the corresponding eigenfunctions of H^*; see [**10**, Lemma 3.3.3]. However, it appears from [**4**, Figure 4] that the eigenfunctions ϕ_n concentrate more and more strongly around $\theta = \pi$ as n increases; the eigenfunctions ϕ_n^* should concentrate around $\theta = 0$ as $n \to \infty$ for similar reasons. If this is indeed the case, then the norms of the spectral projections (called the condition numbers of the eigenvalues in the numerical literature)

$$\|P_n\| = \frac{\|\phi_n\|\,\|\phi_n^*\|}{|\langle \phi_n, \phi_n^* \rangle|}$$

must diverge as $n \to \infty$ and the eigenfunctions do not form a basis.

In our reformulation of the problem, $\phi_{n,r}^* = (-1)^r \phi_{n,r}$ and

$$\|P_n\| = \frac{\sum_{r=1}^{\infty} |\phi_{n,r}|^2}{|\sum_{r=1}^{\infty} (-1)^r |\phi_{n,r}|^2|}.$$

The norms of the first ten spectral projections computed using this formula are presented in Table 1 and provide strong evidence that they diverge rapidly as n increases. It appears likely that the norms increase at an exponential rate as $n \to \infty$, which bodes badly for attempts to use the eigenvectors to expand general elements of $l^2(\mathbb{Z}_+)$, even with the help of resummation techniques.

Table 2: Eigenvalues of A_+ for $\varepsilon = 1$

n	λ_n
1	1.4485
2	4.3159
3	8.6219
4	14.3638
5	21.5414

The above conclusions are in line with those for the NSA harmonic oscillator,

$$(Hf)(x) = -f''(x) + cx^2 f(x)$$

acting in $L^2(\mathbb{R})$. If c is complex, then the eigenvalues and eigenfunctions can be written down in closed form, but it has been proved rigorously that the eigenfunctions do not form a basis; see [**7, 8, 9, 11**]. Once again, this may be understood in pseudospectral terms. The Cauchy problem for the NSA harmonic operator is well posed, but for the operator studied in this paper we have seen that it is not.

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