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ON CHARACTERIZATION OF HYPERSURFACES OF DEGREES 2 AND 3 IN THE COMPLEX PROJECTIVE SPACE

BY

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RÉSUMÉ. Le but de cette note est de proposer une caractérisation des espaces projectifs complexes, des hyperquadriques et des hypersurfaces du troisième degré dans $P^n_{\mathbb{C}}$ à l'aide de leurs points d'intersection avec l'ensemble des zéros d'une section d'un fibré positif donné sur la variété ambiante. Ceci généralise et complète ainsi certains résultats présentés par Badescu et Itoh.

1. The aim of this note is to give a characterization of complex projective spaces, hyperquadrics and hypersurfaces of the third degree in $P^n_{\mathbb{C}}$ by their intersections with the zero set of a section of a positive bundle given on the ambient variety. This generalizes and completes some results of Badescu and Itoh ([1], [3]).

By a manifold we mean a smooth variety. A polarized variety is a (complex) variety V together with an ample line bundle F determined up to the numerical equivalence. We call (V_1, F_1) and (V_2, F_2) numerically equivalent if $\dim V_1 = \dim V_2$ and the self-intersection numbers of F_1 and F_2 coincide and $\dim H^0(V_1, F_1) = \dim H^0(V_2, F_2)$. A polarized manifold numerically equivalent to the projective space P^n, the complex hyperquadric $Q^n (n > 2)$ or to a cubic hypersurface $F^n (n > 2)$ naturally polarized, is biholomorfically isomorphic to P^n, Q^n or a hypersurface of the third degree (see e.g. [4] or [6] for references).

2. The main result of this note can be stated as follows.

THEOREM. *Let P be a projective algebraic complex manifold with a positive (in the sense of* [2]*) vector bundle E on it. Let V be a smooth set of zeros of a global section ξ of E. Let M be a closed submanifold in P, $\dim M = n$, such that $M \cap V$ is of dimension $r \geq 3$. Assume that $c_1(E|M \cap V) \geq n - r$ and that $M \cap V$ is isomorphic to a smooth hypersurface of degree d in P^{n+1}, where $d = 1, 2, 3$. Then M is isomorphic either to a smooth hypersurface of degree d in P^{n+1}, or to $\mathbb{P}^n$.*

3. To prove it, we need a lemma.

LEMMA. *Let (M, D) be an n-dimensional polarized complex manifold. Assume that $H^2(M, \mathbb{Z}) = \mathbb{Z}$ with ample bundles corresponding to positive integers. If*

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$$c_1(M) \geq (n+1)c_1(D)$$

or

$$c_1(M) = nc_1(D),$$

or

$$c_1(M) = (n-1)c_1(D) \text{ with } (c_1(D))^n = 3,$$

then, respectively, (M, D) is numerically equivalent to the projective space $P^n_{\mathbb{C}}$, to the hyperquadric Q^n or to a cubic hypersurface F^n in P^{n+1}.

PROOF. For the first two parts of the lemma see [4] (lemmas 5 and 8). We put

$$p(k) = \chi(M, D^{\otimes k}) = \sum_i (-1)^i \dim H^i(M, D^{\otimes k}),$$

$$q(k) = \chi(F^n, G^{\otimes k}) = \sum_i (-1)^i \dim H^i(F^n, G^{\otimes k}),$$

where G is the hyperplane line bundle on a cubic hypersurface F^n; p and q are then polynomials of degree n in k with the leading coefficients

$$a_n = \frac{1}{n!} c_1(D)^n = \frac{1}{n!} c_1(G)^n = b_n.$$

Since $c_1(M) > 0$, the Kodaira vanishing theorem implies

$$H^i(M, \mathcal{O}) = 0 \text{ for } i > 0.$$

In particular, the Picard variety of M is zero and $p(0) = \dim H^0(M, \mathcal{O}) = \dim H^0(F^n, \mathcal{O}) = q(0) = 1$. Since $c_1(D)$ and $c_1(G)$ are positive, the Kodaira vanishing theorem yields

$$H^i(M, (D^*)^{\otimes k}) = H^i(F^n, (G^*)^{\otimes k}) = 0 \text{ for } k > 0, \qquad 0 \leq i \leq n-1.$$

For any $k \leq n-2$ we have $c_1(M) - kc_1(D) > 0$ and $c_1(F^n) - kc_1(G) = (n-1) = kc_1(G) > 0$ and we may use the Kodaira vanishing theorem again. Combining it with the Serre duality we have

$$H^n(M, (D^*)^{\otimes k}) = H^0(M, D^{\otimes k} \otimes K_M) = 0 \text{ for } k \leq n-2,$$

$$H^n(F^n, (G^*)^{\otimes k}) = H^0(F^n, G^{\otimes k} \otimes K_F) = 0, \text{ for } k \leq n-2, \quad K_M \text{ and } K_F$$

being the canonical bundles.

Since $c_1(D^{\otimes(n-1)} \otimes K_M) = (n-1)c_1(D) + c_1(K_M) = 0$ by our assumption, we conclude that $D^{\otimes(n-1)} \otimes K_M = \mathcal{O}_M$ and similarly $G^{\otimes(n-1)} \otimes K_F = \mathcal{O}_F$. It follows that

$$\dim H^n(M, D^{*\otimes(n-1)}) = \dim H^0(M, \sigma) = p(0) = 1,$$

$$\dim H^n(F^n, G^{*\otimes(n-1)}) = \dim H^0(F^n, \sigma) = q(0) = 1.$$

Hence the values of two polynomials p and q of degree n coincide in n points $k =$

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$-n+1, -n+2, \ldots, -1, 0$. They also have the same leading coefficients, therefore $p(k) = q(k)$, (and $= (3k^2 + 3(n-1)k + n(n-1)) \prod_{j=1}^{n-2} (k+j)/n!$ as easy to see). Hence the Lemma follows.

4. **Proof of the Theorem**. Denote $M \cap V$ by M' and let $i: M' \to M$, $j: M \to P$ be the imbeddings. Let θ_M, $\theta_{M'}$ and N be the tangent bundle of M, the tangent bundle of M' and the normal bundle of M' in M, respectively. Then N is the pullback (under i) of the normal bundle of V in P and the latter is the pullback (under j) of E. Therefore we have an exact sequence on M'

$$0 \to \theta_{M'} \to i^*\theta_M \to i^*j^*(E) \to 0.$$

Hence for the Chern classes

$$c_1(i^*M) = c_1(M') + c_1(i^*j^*(E)).$$

Our assumptions on M' imply that $H^2(M', \mathbb{Z}) = \mathbb{Z}$ and we take the class of the Kähler form to be a positive element in this group. The generalized Lefschetz theorem (see e.g. [2]) says that the following sequences are exact:

$$0 \to H_2(M', \mathbb{Z}) \xrightarrow{i_*} H_2(M, \mathbb{Z}) \to 0,$$
$$0 \to H_1(M', \mathbb{Z}) \xrightarrow{i_*} H_1(M, \mathbb{Z}) \to 0.$$

But for our M' we have $H_1(M', \mathbb{Z}) = 0$. Then, by a simple homological algebra we infer that

$$i^*: H^2(M, \mathbb{Z}) \to H^2(M', \mathbb{Z}) \approx \mathbb{Z}$$

is also a group isomorphism. Since $H^1(M, \mathcal{O}) = 0$, we see that any element in $H^2(M, \mathbb{Z})$ is a class of a divisor. The restriction of an ample line bundle is again ample, i.e., i^* preserves the canonical "orientations" of these groups. The first Chern classes of E is invariant both under j^* and i^*. Hence we may write $i^*c_1(M) = c_1(M') + i^*j^*c_1(E)$. Since $c_1(M') = (r - d + 2)g'$, we see that $c_1(M) \geq (n - d + 2)g$, g, g' being the positive generators of $H^2(M, \mathbb{Z}) \approx H^2(M', \mathbb{Z})$. Since the latter isomorphism is induced by the embedding $M' \to M$ and d is a prime number or 1, we infer that the degree (i.e., the self-intersection number) of g is 1 or d. The first case gives $M = \mathbb{P}^n$, the second $M = F'^n$, a hypercubic.

Remarks and Examples.

(1) The assumption $c_1(D)^n = 3$ in the characterization of the hypercubic given above cannot be dropped. Namely, consider a smooth intersection $M = Q_1 \cap Q_2$ of two hyperquadrics in $\mathbb{P}^{n+2}$, $n \geq 3$. The exact sequence $0 \to \Theta_M \to \Theta_{\mathbb{P}^{n+2}|M} \to \mathcal{O}(2) \oplus \mathcal{O}(2) \to 0$ yields

$$c_1(\Theta_M) = (1+h)^{n+3}/(1+2h)^2 = 1 + (n-1)h + \ldots,$$

h being the class of a hyperplane, i.e., $c_1(M) = (n-1)c_1(\sigma(1))$.

(2) Recall the Veronese mapping

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$$v_m : P^n \to P^N, \qquad N = \binom{n+m}{m} - 1.$$

The homogeneous coordinates in P^N are $u_{i_0 i_1 \ldots i_n}$ with $i_0, \ldots, i_n$ being non-negative integers such that $i_0 + i_1 + \ldots + i_n = m$ and v_m defined by

$$u_{i_0 i_1 \ldots i_n} \circ v_m = z_0^{i_0} z_1^{i_1} \ldots z_n^{i_n},$$

where $z_0, z_1, \ldots, z_n$ are the homogeneous coordinates in P^n.

The section of $v_M(P^n)$ by the hyperplane $u_{m0\ldots 0} = 0$ gives a hyperplane $z_0 = 0$ in P^n. On the other hand, the section of $v_m(P^n)$ by the

$$u_{m0\ldots 0} + u_{0m\ldots 0} + \ldots + u_{0\ldots 0_m} = 0$$

corresponds to the Fermat hypersurface $z_0^m + z_1^m + \ldots + z_n^m = 0$. It shows that if the linear space intersection of M is a hypersurface of degree m, the variety M need not be such.

(3) Example 4.3 in [5] tells us how to construct a counter-example to our Theorem for hypersurfaces of a composite degree. To this effect consider the Veronese embedding of $\mathbb{P}^n$ into $\mathbb{P}^{\binom{n+1}{2}-1}$:

$$\mathbb{P}^n \ni (x_0, x_1, \ldots, x_n) \longmapsto (\ldots, x_i x_j, \ldots) \in \mathbb{P}^{\binom{n+1}{2}-1}.$$

Let $h_i : = x^2$, h_i's then correspond to hyperplanes in $\mathbb{P}^{\binom{n+1}{2}-1}$. Embed this projective space into $\mathbb{P}^{\binom{n+1}{2}}$ as the hyperplane $y = 0$ and define

$$V = (\Sigma h_i = 0) \cap (y^2 = \Sigma h^2 i) \cap (\text{image of } \mathbb{P}^n).$$

Then $H^2(V, \mathbb{Z}) = \mathbb{Z}$, $c_1 V = 4g$, but V is not a hyperquartic in $\mathbb{P}^{n+1}$. On the other hand, corollary 3.8 in [5] says that if X is a locally factorial scheme with an ample effective divisor Y, such that Y is, as a variety, a generalized complete intersection of type $(a_1, \ldots, a_c)$ of the weighted projective space $P(e_0, \ldots, e_n)$ then X is a weighted complete intersection of type $(a_1, \ldots, a_c)$ in $P(e_0, \ldots, e_n, a)$ — provided that $\dim Y \geq 3$. It means that (as in the theory of global deformations) the notion of weighted hypersurfaces behaves more regularly. The author does not know any results of that kind for hypersurfaces of prime degrees $d \geq 5$.

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